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# On the eventual absorption of a quantum particle

by

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ABSTRACT. — For a general system consisting of a quantum particle and an absorbing detector, the probability that the detector will eventually absorb the particle is calculated.

RÉSUMÉ. — Pour un système, qui est une particle quantique et un instrument absorbant, on obtient la probabilité que l'instrument absorbera la particle.

## 1. INTRODUCTION

Quantum stochastic processes [*1*] enable one to deal with quantum systems in which the time evolution is not unitary. In particular, one can consider a system consisting of a quantum particle with an absorbing detector, which will absorb the particle when it comes in contact with it, as for example in [*2*] [*3*]. The particle's space of states is $\mathcal{H}$, a separable Hilbert space, with a densely defined Hamiltonian H, describing its time evolution. We assume that the spectrum of H is only pure point, and its eigenvectors $e_i$, of eigenvalue $\mu_i$, form an orthonormal basis. For the combined system, the space of possible states is $\mathcal{F}(\mathcal{H})$, the symmetric Fock space built on $\mathcal{H}$, and let $\tilde{H}$ be the extension of H from $\mathcal{H}$ to $\mathcal{F}(\mathcal{H})$. In the terminology of quantum stochastic processes the state space (V, $\tau$) is taken as $V = \mathcal{I}_s(\mathcal{F}(\mathcal{H}))$, the Banach space of self adjoint trace class operators on $\mathcal{F}(\mathcal{H})$, with $\tau$ the trace. The detector is represented by the canonical annihilation operator $A(d)$, $d$ being an element of $\mathscr{H}$, which is supposed to describe the characteristics of the detector.

The quantum stochastic process for this system is a set of instrument $\mathscr{E}^t$, indexed by a time parameter $t \in [0, \infty)$, each of which maps the initial state of the system onto the state at time $t$, dependent upon what has happened up to time $t$. Formally $\mathscr{E}^t : V \times X_t \to V$, where $X_t$ is the sample space, consisting of all possible events that could have happened to the detector up to time $t$. Thus $X_t = \{ s : s \in [0, t] \} \cup z$, since either the particle has been absorbed at a time $s$ less than $t$, or no absorbing has yet occurred— the point $z$. The definition of such instruments was given in [*1*], where it was shown that $S_t v = \mathscr{E}^t(z)v$, $v \in V$, the states conditional on no hitting form a semigroup and if we write $S_t v = B_t v B_t^*$, then the no hitting time development is a perturbation of the usual time development, given by

$$B_t = \exp \left\{ - i\tilde{H}t - \frac{1}{2} A^*(d)A(d)t \right\}. \qquad 1.1$$

Let the initial state of the particle be $P_\Phi$, the projection operator onto the subspace of $\mathscr{F}(\mathscr{H})$ spanned by $\Phi = (0, h, 0, 0 \ldots)$. The system is then described by H, $h$ and $d$, and the probability that no absorption has occurred by time $t$ is

$$\frac{\langle \tau, \mathscr{E}^t(z)P_\Phi \rangle}{\langle \tau, P_\Phi \rangle} \qquad 1.2$$

while in [*1*], it is shown that the probability that the detector absorbs the particle in the interval $(t, t + \delta t)$ is

$$\frac{\langle \tau, A(d)(\mathscr{E}^t(z)P_\Phi)A(d)^* \rangle}{\langle \tau, P_\Phi \rangle} dt + o(dt). \qquad 1.3$$

We are interested in what is the relationship between H, $h$ and $d$, in order that with probability one, the detector will eventually absorb the particle. We will show that the particle will eventually be absorbed if and only if, when $d$ and $h$ are expanded in terms of the orthonormal basis consisting of eigenvectors of H, $d$ has a non-zero component of all the eigenvectors contained in the expansion of $h$. More formally

PROPOSITION 1. — If $d = \sum_{j=1}^{\infty} d_j e_j$ and $h = \sum_{j=1}^{\infty} h_j e_j$, then

$$\lim_{t \to \infty} \frac{\langle \tau, \mathscr{E}^t(z)P_\Phi \rangle}{\langle \tau, P_\Phi \rangle} = \sum_{\{j | \langle e_j, d \rangle \neq 0\}} | h_j |^2. \qquad 1.4$$

There is a slight ambiguity in the statement of the proposition since the eigenvectors for degenerate eigenvalues are not defined uniquely. However to avoid this ambiguity, if the initial one-particle state is $h$, we shall always assume that one of the eigenvectors in the eigenspace is the normalised projection of $h$ onto that eigenspace, and the others are orthogonal to this vector.

## 2. PROOF OF PROPOSITION

$$\langle \tau, \mathscr{E}^t(z)P_\Phi \rangle = \mathrm{tr}\,(B_t P_\Phi B_t^*) = \langle B_t\Phi, B_t\Phi \rangle . \qquad 2.1$$

Since $\Phi$ has only a 1-particle component and $B_t$ leaves this space invariant, it is sufficient to restrict attention to the 1-particle space, $\mathscr{H}$.

$$B_t\,|_{\mathscr{H}} = \exp\left(-\,iHt - \frac{1}{2}P_d t\right)$$

where $P_d$ is the projection onto the subspace spanned by $d$. Let $\mathscr{H}_1 = V\{\, e_i \,|\, \langle e_i, d \rangle \neq 0 \,\}$ and $\mathscr{H}_2 = V\{\, e_i \,|\, \langle e_i, d \rangle = 0 \,\}$ where $V\{\ \ \}$ implies the closed linear subspace spanned by the elements within $\{\ \ \}$. The proposition is proved if we can show that $B_t$ leaves the $\mathscr{H}_i$ invariant, is strongly contractive on $\mathscr{H}_1$, i. e. $\lim_{t\to\infty} B_t h = 0\ \forall h \in \mathscr{H}_1$, and is unitary on $\mathscr{H}_2$.

$$B_t\,|_{\mathscr{H}} = e^{-iHt} + \sum_{m=1}^{\infty} C_t^m , \qquad 2.2$$

where

$$C_t^m = \left(-\frac{1}{2}\right)^m \int_{t_1=0}^{t} \int_{t_m=0}^{t_{m-1}} e^{-iH(t-t_1)}P_d \ldots P_d e^{-iHt_m} dt_1 \ldots dt_m .$$

Since $P_d\,|_{\mathscr{H}_2} = 0$ and since $e^{-iHt}$ leaves $\mathscr{H}_2$ invariant $C_t^m\,|_{\mathscr{H}_2} = 0$. Thus $B_t\,|_{\mathscr{H}_2} = e^{-iHt}$ and so is unitary. It is now obvious that $B_t$ leaves $\mathscr{H}_2$ invariant. Let $h_1 \in \mathscr{H}_1$ and let $e_{i_2}$ be an eigenvector of H in $\mathscr{H}_2$, then

$$\langle B_t h_1, e_{i_2} \rangle = \langle h_1, B_t^* e_{i_2} \rangle = \langle h_1, e^{iHt} e_{i_2} \rangle = e^{i\mu_{i_2}t} \langle h_1, e_{i_2} \rangle = 0 . \qquad 2.3$$

This is true for all $e_{i_2}$ in $\mathscr{H}_2$ and so $B_t$ leaves $\mathscr{H}_1$ invariant.

Finally it is necessary to show that $B_t\,|_{\mathscr{H}_1}$ is strongly contracting. Davies [1] has shown that such a semigroup is a contraction semigroup, so we need only show that $-\,iH - \frac{1}{2}P_d$ has no pure imaginary eigenvalues. Consider the action of this operator on the eigenvectors $e_i$, which span $\mathscr{H}_1$

$$\left(-\,iH - \frac{1}{2}P_d\right)e_j = -\,i\mu_j e_j - \frac{1}{2}\sum_{k=1}^{\infty} \langle d, e_j \rangle \langle e_k, d \rangle e_k . \qquad 2.4$$

Choose the phase factors of the $e_j$, so that $\langle e_j, d \rangle$ is real and the $d_j$ 's are real with $\sum_j d_j^2 = 1$. The matrix for $\left(-iH - \frac{1}{2}P_d\right)\Big|_{\mathscr{H}_1}$ with this basis is of the form

$$\begin{pmatrix} -i\mu_1 - \frac{1}{2}d_1^2 & -\frac{1}{2}d_1d_2 & -\frac{1}{2}d_1d_3 & \dots \\ -\frac{1}{2}d_1d_2 & -i\mu_2 - \frac{1}{2}d_2^2 & -\frac{1}{2}d_2d_3 & \dots \\ \dots & \dots & \dots & \end{pmatrix} \qquad 2.5$$

and the eigenvalues of the operator are given by

$$\det\left(\lambda + \left(iH + \frac{1}{2}P_d\right)\Big|_{\mathscr{H}_1}\right) = 0 .$$

Put $\lambda = i\mu$, then

$$\begin{vmatrix} \mu + \mu_1 - \frac{1}{2}id_1^2 & -\frac{1}{2}id_1d_2 & -\frac{1}{2}id_1d_3 & \dots \\ -\frac{1}{2}id_1d_2 & \mu + \mu_2 - \frac{1}{2}id_2^2 & -\frac{1}{2}id_2d_3 & \dots \\ \dots & \dots & \dots & \end{vmatrix}$$

$$= \begin{vmatrix} \mu + \mu_1 - \frac{1}{2}id_1^2 & -\frac{1}{2}id_1d_2 & -\frac{1}{2}id_1d_3 & \dots \\ -\frac{d_2}{d_1}(\mu + \mu_1) & \mu + \mu_2 & 0 & \dots \\ -\frac{d_3}{d_1}(\mu + \mu_1) & 0 & \mu + \mu_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \prod_{j=1}^{\infty}(\mu + \mu_j) - \frac{1}{2}i\sum_{k=1}^{\infty} d_k^2 \prod_{j \neq k}(\mu + \mu_j) = f(\mu) . \qquad 2.6$$

$$\therefore |f(\mu)|^2 = \prod_{j=1}^{\infty}(\mu + \mu_j)^2 + \frac{1}{4}\left(\sum_{k=1}^{\infty} d_k^2 \prod_{j \neq k}(\mu + \mu_j)\right)^2 . \qquad 2.7$$

If the operator has a pure imaginary eigenvalue, then there exists a real value of $\mu$ for which $f(\mu)$, and therefore $|f(\mu)|^2$ is zero. However 2.7 is the sum of two squares which have non-coincident roots (since the $d_k^2$ must be non-zero). The result is thus proven.

## 3. ALTERNATIVE APPROACH

One is able to get more detailed information about the probability of absorption of the particle at any time, by using an expansion formula of the form 2.2 for $B_t$ on $\mathscr{F}(\mathscr{H})$,

$$B_t\Phi = e^{-i\tilde{H}t}\Phi + \sum_{m=1}^{\infty} C_t^m\Phi \qquad 3.1$$

where

$$C_t^m = \left(-\frac{1}{2}\right)^m \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m e^{-i\tilde{H}(t-t_1)} A^*(d)A(d) \ldots A^*(d)A(d)e^{-i\tilde{H}t_m}.$$

Since $\Phi$ has only a one-particle component $A(d)e^{-i\tilde{H}s}\Phi$ is only non-zero in the O-particle state, so

$$B_t\Phi = e^{-i\tilde{H}t}\Phi - \frac{1}{2}\int_0^t e^{-i\tilde{H}(t-s)}A^*(d)a(s)ds \qquad 3.2$$

where $a(s) = A(d)e^{-i\tilde{H}s}\Phi + A(d)\sum_{m=1}^{\infty} C_s^m\Phi$ is a function.

$$\therefore a(s) = A(d)B_s\Phi = A(d)e^{-i\tilde{H}s}\Phi - \frac{A(d)}{2}\int_0^s e^{-i\tilde{H}(s-x)}A^*(d)a(x)dx. \qquad 3.3$$

We assume for simplicity that $d$ is such that only finitely many $d_k$ 's are non-zero. Since $\Phi = (0, h, 0, \ldots)$ and using the expansions of $d$ and $h$ of Proposition 1,

$$a(s) = \sum_j h_j\bar{d}_j e^{-i\mu_j s} - \frac{1}{2}\int_0^s \sum_k |d_k|^2 e^{-i\mu_k(s-x)}a(x)dx. \qquad 3.4$$

Taking Laplace transforms, we get

$$\hat{a}(p) = \sum_j h_j\hat{a}_j(p) = \frac{\sum_j h_j\bar{d}_j \prod_{k\neq j}(p + i\mu_k)}{\prod_k (p + i\mu_k) + \frac{1}{2}\sum_k |d_k|^2 \prod_{j\neq k}(p + i\mu_j)}. \qquad 3.5$$

In [1], it was shown that the probability that the detector will absorb the particle in the interval $(t, t + \delta t)$ is

$$\frac{\langle \tau, A(d)B_tP_\Phi B_t^*A(d)^* \rangle}{\langle \tau, P_\Phi \rangle} dt + o(dt) = |A(d)B_t\Phi|^2 dt + o(dt) = |a(t)|^2 dt + o(dt). \qquad 3.6$$

Thus to calculate the density function of the absorbing time, one takes the inverse Laplace transform of $\hat{a}(p)$, to obtain $|a(s)|^2$. The probability that the detector will eventually absorb the particle is $\int_0^\infty |a(s)|^2 ds$. Plancherel's formula for Laplace transforms can be applied in this case, since $a(s)$ is obviously a $L^2$ functions. So

$$\int_0^\infty |a(s)|^2 ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{a}(ix)|^2 dx . \qquad 3.7$$

where $x$ is real. In section 2, we have shown that the probability particle is eventually absorbed is

$$1 - \sum_{\{j|d_j \neq 0\}} |h_j|^2 = \sum_{\{j|d_j = 0\}} |h_j|^2 . \qquad 3.8$$

Rewriting this result in terms of the $|a(s)|^2$ and using 3.7, it follows that the functions $\hat{a}_j(ix)$ of 3.5 must form an ortho-normal system, i. e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}_j(ix)\hat{a}_k(ix)dx = \delta_{jk} . \qquad 3.9$$

We again choose the $e_i$ so that $d_k$ is real for all $k$, and let $\lambda_k = d_k^2$ so $\Sigma\lambda_k = 1$. Proposition 1 thus implies that if we have a finite set of positive numbers $\lambda_k$, $k = 1 = N$ with $\sum_k \lambda_k = 1$, then the functions $g_n(x) = \frac{\sqrt{\lambda_n}}{x + \mu_n}$, with weight function

$$w(x) = \frac{\prod_{n=1}^{N}(x + \mu_n)^2}{\prod_{n=1}^{N}(x + \mu_n)^2 + \frac{1}{4}\left(\sum_{s=1}^{N}\lambda_s \prod_{r \neq s}(x + \mu_r)\right)^2} \qquad 3.10$$

form an orthonormal set, i. e.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g_n(x)g_m(x)w(x)dx = \delta_{nm} . \qquad 3.11$$

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