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Section A :
Physique théorique.

Geometry of dynamics in general relativity (*)

by

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ABSTRACT. — We discuss the geometrical structure of the configuration space for the classical field theories of physics in the framework of the theory of deformations.

RÉSUMÉ. — Nous discutons la structure géométrique de l'espace de configuration de la relativité générale. Le problème est d'associer une géométrie métrique à toute densité lagrangienne généralement covariante de telle sorte que les équations du champ apparaissent comme des équations géodésiques généralisées. Des généralisations de la théorie Finslerienne dans le Calcul des Variations de problèmes invariants par rapport au paramètre, et de la théorie de déformations d'Eells et Sampson fournissent deux nouvelles approches à ce problème. Restreignant notre attention à cette dernière approche, nous montrons que l'action des théories du champ classique de la physique peut être interprétée comme la fonctionnelle d'énergie d'Eells et Sampson, identifiant ainsi la géométrie appropriée pour l'espace de configuration.

The formulation of geometry of dynamics that is based on the Hamiltonian form of a theory is not particularly suited to general relativity because we are dealing with a degenerate Lagrangean system, and further

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the basic concepts of the canonical theory are non-relativistic. It is therefore of interest to ask what other framework we can utilize to define a configuration space for the gravitational field variables and endow it with structure as a consequence of the Einstein field equations. We shall here indicate that the appropriate generalizations of Finsler's theory of parameter-invariant problems in the calculus of variations [*1*] and the Eells-Sampson Theory of deformations, harmonic mappings of Riemannian manifolds [*2*] provide us with two fruitful approaches to this problem. The former is especially adapted to general relativity because of its particular gauge group, while the latter is a geometrical theory which can be used to discuss other classical field theories as well. In the framework of both of these approaches the notion of associating a metric geometry to each generally covariant Lagrangean density can be made precise and natural. We shall be interested in the geometry which results from the particular choice of Einstein's Lagrangean density in the variational principle for gravitational fields and as the solution of the field equations will be the geodesics in this geometry we shall speak of it as the geometry in the formulation of the Einstein equations. Finally when such a geometrization is effected an algorithm for constructing the appropriate canonical theory can also be given [*3*].

The Lagrangean density of the theory of general relativity is degenerate, that is the rank of the Hessean is less than its dimension. Hence the usual prescription for constructing the Hamiltonian fails and to achieve canonical form, following Dirac [*4*] and Arnowitt, Deser and Misner [*5*] we must first cast the theory into the form of a system subject to constraints. Based upon this approach De Witt [*6*] has given a formulation of geometry of dynamics. The origin of the degeneracy is not central to the canonical theory; however, in the following we shall propose an alternative formulation of the geometry of dynamics which consists of a systematic exploitation of this point.

The degeneracy of Einstein's Lagrangean density is of a type familiar from the calculus of variations since it arises from the requirements of general covariance that the action remain invariant under arbitrary analytic transformations of the coordinates. We are therefore confronted with an example of parameter-invariant, or homogeneous problems in the calculus of variations and we may utilize the techniques of the calculus of variations which have been designed to deal with this particular type of degeneracy. As a consequence of parameter invariance we obtain identities which restrict the Lagrangean. For systems with finite number of degrees of freedom it was shown by Finsler that such restrictions, which are in fact the source of the degeneracy, need not be regarded as a difficulty which must be overcome; but instead can be used to advantage to cast the action into the form of a path length which makes the Euler-Lagrange equations a geodesic equation. Thus the theory of parameter-invariant problems gives rise to generalized metric geometries and for particle mechanics the resulting geometry is Finsler geometry. When we consider the variational problem for systems described in terms of fields defined on some manifold, we find that the group of transformations required for parameter invariance coincides with the gauge group of arbitrary analytic coordinate transformation of general relativity. Then starting with the invariant action of Einstein the configuration space for the gravitational field variables will also have a Finsler-type geometry. To investigate it we must generalize Finsler's procedure to the case where the Lagrangean density is a functional of the Riemannian [7] metric tensor field on the space-time manifold. We note that while an arbitrary parametrization of Einstein's theory would enable us to use existing generalizations of Finsler's geometry [8] since the field variables would then be scalars, such an approach cannot be used as it will give rise to a degenerate metric. The details of the definition of a metric for the configuration space of the gravitational field variables, by the use of the Noether identities [9] which result from an invariant action principle involving vierbein fields will be published elsewhere.

We shall now turn to the subject of harmonic mappings of Riemannian manifolds which offers a direct geometrical insight into the structure of the Lagrangean density. Following Eells and Sampson [2] we start by defining for each point P in a Riemannian manifold (M, g) an inner product $\langle \ , \ \rangle_P$ on the space of 2-covariant tensors of the tangent space to M at P. If α^1 and α^2 are two such tensors then

$$\langle \alpha^1, \alpha^2 \rangle \equiv \alpha^1_{\mu\nu}\alpha^2_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}.$$

Next we consider a map f: $M \rightarrow M'$ where M, M′ are Riemannian manifolds with metrics g, g' and dimensionality n, n' respectively. Since g and the induced metric $f_* g'$ are 2-covariant tensors on M, we can define the invariant functional « energy »:

$$(1) \quad E(f) = \int \langle g, f_* g' \rangle *1 = \int g'_{ab}\frac{\partial f^a}{\partial x^\mu}\frac{\partial f^b}{\partial x^\nu}g^{\mu\nu}\sqrt{-g}d^nx \qquad (a = 1, \ldots, n', \ \mu = 1, \ldots, n)$$

of this mapping where *1 is the invariant volume element on M. Extremal maps, $\delta E(f) = 0$, are called harmonic. The reasons for this terminology become evident if we look at some of the familiar equations of physics which Eells and Sampson have thus brought together. When M′ is R^3 and M is a 1-dimensional manifold $E(f)$ becomes the Newtonian kinetic energy for a free particle and the statement of harmonic maps reduces to Newton's first law of motion. Other special cases of particular interest occur for M or M′ 1-dimensional and the harmonic maps are the geodesics of M′ or the Klein-Gordon field on the given curved background of M respectively. In the former case we can recognize Jacobi's formulation of the geometry of dynamics for particle mechanics if we remember that kinetic energy defines a Riemannian metric for the configuration space. Further there is an example of a restricted class of gravitational fields which fits into this scheme exactly. This is the problem of stationary axisymmetric exterior solution which is described by the line element

$$(2) \qquad ds^2 = e^{2\psi}(dt - \omega\, d\varphi)^2 - e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + e^{2\lambda}d\varphi^2]$$

of Weyl and Levi-Civita [*10*] where ψ, ω, γ, λ are functions of ρ, z only. Here it turns out [*11*] that if we consider a mapping from a Riemannian manifold M into another M′ with metrics (2) and

$$(3) \qquad ds'^2 = 4\, d\psi^2 - 4\, d\lambda\, d\gamma - e^{4\psi - 2\lambda} d\omega^2$$

respectively, then the extremals of the energy functional satisfy precisely the Einstein equations for (2). Indeed the metric in equation (3) is deduced from a comparison of Einstein's Lagrangean density formed in the usual way from equation (2) with the energy functional (1). From the viewpoint of regarding Einstein's variational principle as the extremizing of an invariant functional associated with the mapping of Riemannian manifolds the geometry of dynamics follows immediately. The Einstein field equations are recognized as a generalized goedesic equation with the connection defined in the usual way from equation (3) but where the paths range on M and are parametrized by the coordinates of spacetime.

For general gravitational fields the action cannot be cast into the form of the energy functional and therefore a metric such as the one in equation (3) cannot simply be read off. So a generalization of the energy functional and of the mappings which we must consider is necessary [*12*]. The novel feature of the general problem comes in the transformation properties of the fields which due to the symmetries imposed on equation (2) did not play a significant role earlier. The generalization of this example is most simply carried out in the language of fibred spaces [*13*]. Consider a bundle E over the spacetime manifold M and let ξ: E $\rightarrow$ M be a differentiable fibre on M with a Riemannian space as the typical fibre ([*14*] [*15*]). Following Trautman we shall call these Riemannian bundles. For stationary axi-symmetric gravitational fields we have [*16*] ξ: M $\times$ M′ $\rightarrow$ M and the Einstein action, energy, is the map I: H^1(M $\times$ M′) $\rightarrow$ R where H^1 stands for the space of section with derivations $\leq$ first order square integrable. Even as the metric g' was suggested by the appearance of the action in the form of the energy functional, in the classical field theories of physics the structure of the Lagrangean density enables us to introduce a natural metric along the fibres of E in addition to the Riemannian metric we had on M, i. e. along the fibres of TM. An example where we must consider mappings from less trivial bundles is provided by the Maxwell field. We take for ξ the fibre ξ: $T^*M \to M$ of the exterior differential forms on M and we have the Maxwell Lagrangean density

$$\mathscr{L}_M = \frac{1}{2} * \{ dA \wedge * dA + \delta A \wedge * \delta A \}, \tag{4}$$

where $A = A_\mu dx^\mu$ is the vector potential 1-form. Here d stands for exterior derivative and δ the co-differential, $\wedge$ denotes the wedge product and $*$ the dual operator. We note that the Lagrangean density (4) is again quadratic in each fibre. The action which is the Dirichlet integral for differential forms is in the form of an inner product for sections of T^*M with respect to a Riemannian metric introduced into the fibres and this metric is simply the contravariant metric. The dual problem of the natural Riemannian metric on the tangent bundle of a Riemannian space has been discussed by Sasaki [17]. The prominent role played by the co-tangent bundle suggests that we may formulate the Maxwell field using canonical, or sympletic structures as follows:

We first extend the notion of inner product to the space of 2-covariant tensors on the space of 2-forms on M, i. e. tensors with the symmetries of the Riemann tensor. If β^1, β^2 are two such tensors, we define at each point P on M:

$$\langle \beta^1, \beta^2 \rangle_P = \beta^1_{\mu_1\mu_2\mu_3\mu_4}\beta^2_{\nu_1\nu_2\nu_3\nu_4}g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}g^{\mu_4\nu_4} \tag{5}$$

At a point (x^μ, v_ν) in the co-tangent space M_P^*, where x^μ are the coordinates of P on M and v_ν are the components of co-tangent vectors at P, we have the canonical 2-form

$$\varpi = dx^\mu \wedge dv_\mu . \tag{6}$$

Now we shall consider maps $f: M \to T^*M$ where $\pi \circ f = A$, that is the composition of the projection $\pi: TM \to M$ with f gives us the vector potential. Then we find that

$$f * \varpi = dx^\mu \wedge dA_\mu = A_{\mu,\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}F$$

is the field tensor. Thus in equation (5) if we take β^1 to be

$$\beta^1 = g \perp g, \qquad (g \perp g)_{\alpha\beta\gamma\delta} \equiv g_{\alpha[\gamma}g_{\delta]\beta} \tag{7}$$

the metric for Λ_P^2, the space of 2-forms and choose

$$\beta^2 = (f * \varpi) \otimes (f * \varpi) = f * (\varpi \otimes \varpi) \tag{8}$$

which may be termed the « induced metric » on Λ_P^2, we can write equation (4) as

$$L_M = \langle g \perp g, f * (\varpi \otimes \varpi) \rangle . \tag{9}$$

This discussion of the Maxwell field can be put into a compact form which will also be suitable for the formulation of the theories of Yang-Mills, Einstein and all other compensating fields of Utiyama [18] in the framework of harmonic mappings.

We first generalise the concept of harmonic maps as $f: (M, g, \beta) \to (M', \beta')$, a mapping f from a Riemannian manifold M with metric g and a tensor field β of type (r, s) into some other manifold M′ and a tensor field β' which is again of type (r, s) for which the energy functional is extremized. It is evident that this definition includes the one given by Eells and Sampson. There we consider tensors of type $(0, 2)$ and identify β with g and let M′ be another Riemannian manifold with $\beta' = g'$. However, the advantage of the proposed generalization is that we can now regard M′ as a vector bundle over M itself and in fact for Utiyama fields M′ will be a principal fibre bundle with a structure group which is the same as the gauge group of the physical theory. It is well-known that Utiyama fields can be interpreted as the theory of an infinitesimal connection in such a fibre bundle. We shall show that this structure lends itself very naturally to the construction of tensors β, β' of type (0,4) whereby the content of the physical theory can be reduced to the statement of generalized harmonic mappings.

It is instructive to see first the precise role played by the potentials. Consider two vector bundles E, F over M such that $\pi_1 : E \to M, \pi_2 : F \to M$ and let σ be a map which takes fibres into fibres with $\pi_2(\sigma(x)) = \pi_1(x)$. The restriction of σ to a fibre over a point in M is a map between vector spaces and its derivative is a linear map. Now the fibre derivative is defined as $\tilde{\sigma} : E \to L(E, F)$ where L(E, F) is the space of linear maps between corresponding fibres. In classical mechanics the Lagrangean and the Legendre transformation are examples of σ and $\tilde{\sigma}$ respectively. For the Maxwell field we have the vector potential $A: TM \to M \times R$ but A is linear so it is the same as its fibre derivative. Hence

$$A : TM \to L(TM, M \times R) \cong T^*M$$

where $\cong$ denotes isomorphism and we identify T*M with M′. Finally since $f: M \to M'$ is defined and we have $\pi : TM \to M$ we must require $\pi \circ f = A$. We shall not repeat the process leading to the energy functional once again except to remark that the construction of β' for the Maxwell field is immediate because of the availábility of a symplectic structure on T*M.

Let us now turn to the Yang-Mills field which can be regarded as the generic example of this subject. We start with the space-time Manifold (M, g) and consider a vector bundle T^3M over M where a point in T^3M consists of a triad of vectors as well as a point in the base manifold. Such a structure is necessary in order to describe « isotopic spin ». The projection $\pi : T^3M \to M$ assigns to all the triad of vectors their origin. The Yang-

Mills potential A is a map $A : T^3M \to M \times R$ which is again a linear map and therefore the same as its fibre derivative. Thus

$$A : T^3M \to L(T^3M, M \times R)$$

and we identify the image space here with M′ which results in $\pi \circ f = A$. Following the general scheme for Utiyama fields, M′ will be a principal fibre bundle with the structure group SU(2) since the Yang-Mills field is invariant under gauge transformations of the second kind belonging to this Lie group. The connection on the principal fibre bundle M′ is described by the connection 1-forms and defines the horizontal subspaces. Using the local coordinates $(x, v^i_{\ \mu})$ on M′ we can write down the 2-forms

$$\varpi^i = dx^\mu \wedge Dv^i_{\ \mu}, \qquad (i = 1, 2, 3) \tag{10}$$

where

$$Dv^i_{\ \mu} = dv^i_{\ \mu} + f_{j\ k}^{\ i} v^j_{\ \mu} v^k_{\ \nu} dx^\nu \tag{11}$$

and $f_{j\ k}^{\ i}$ are the structure constants of SU(2). The Latin indices are labels which range over isotopic spin space. The tensor product of ϖ^i with itself is a matrix of tensors of type (0, 4) and using the metric

$$G_{ij} = f_{i\ n}^{\ m} f_{m\ j}^{\ n} \tag{12}$$

we form

$$\beta' = G_{ij} \varpi^i \otimes \varpi^j. \tag{13}$$

This amounts to taking the trace of $\varpi^i \otimes \varpi^i$. Applying the map f^* to this object we obtain a tensor of type (0, 4). For β we shall take once again the expression in eq. (7) and using the definition of the inner product (5) we construct the energy functional

$$E(f) = \int \langle g \perp g, f^* G_{ij}(\varpi^i \otimes \varpi^j) \rangle * 1 \tag{14}$$

which is readily seen to be the Yang-Mills action. Thus we have cast the Yang-Mills field into the framework of generalized harmonic mappings.

In the case of the gravitational field, with which we shall conclude our discussion, the formulation of Einstein's theory most suitable for our purposes will be the formalism of Newman and Penrose [*19*] since we shall start by endowing space-time with a spinor structure [*20*]. That is, over the space-time manifold M we have a second principal fibre bundle M′ with the structure group SL(2, C) and a map $\varphi : M' \to B$ which takes us from M′ to the principal fibre bundle B of oriented orthonormal tetrads on M such that φ maps each fibre of M′ into a single fibre of B and φ commutes with the group operations. This elaboration in the choice of M′ is necessary in order to define spinor fields without any sign ambiguities. SL(2, C) is a double valued representation of the proper homogeneous Lorentz group which is the structure group for B and a fibre of M′ is the universal covering space of a fibre of B. Now we can define a spinor field as a mapping from M′ into arrays of complex numbers with the appropriate transformation properties.

Hence we consider a mapping f: M $\rightarrow$ M′ where a typical fibre of M′ over a point P in M consists of the set of spin frames at P. A point in this fibre is given by four 2×2 complex traceless matrices and if in particular we choose the spin frame $(0^{\mathrm{A}}, \iota^{\mathrm{A}})$, $\mathrm{A} = 0,1$ of Newman and Penrose, the spin coefficients $\gamma_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{CD'}}$ become the local coordinates in the fibre. In analogy with the previous examples the spin coefficients play the role of potentials for this theory. We now consider a connection on M′. In terms of the local coordinates $(x^{\mu}, v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{CD'}} \leftrightarrow v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mu})$ on M′ we have

$$\varpi_{\mathrm{A}}{}^{\mathrm{B}} = dx^{\mu} \wedge \mathrm{D}v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mu}, \tag{15}$$

$$\begin{aligned} \mathrm{D}v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{CD'}} = dv_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{CD'}} &+ f_{\mathrm{N}}{}^{\mathrm{M}}{}_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{S}}{}^{\mathrm{R}} v_{\mathrm{M}}{}^{\mathrm{N}}{}_{\mathrm{CD'}} v_{\mathrm{R}}{}^{\mathrm{S}}{}_{\alpha} dx^{\alpha} \\ &- v_{\mathrm{C}}{}^{\mathrm{F}}{}_{\alpha} v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{FD'}} dx^{\alpha} - v^{\dagger\mathrm{F'}}{}_{\mathrm{D'}\alpha} v_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{CF'}} dx^{\alpha} \end{aligned} \tag{16}$$

where $f_{\mathrm{N}}{}^{\mathrm{M}}{}_{\mathrm{A}}{}^{\mathrm{B}}{}_{\mathrm{S}}{}^{\mathrm{R}}$ are the structure constants of SL(2, C) and $\dagger$ indicates Hermitian conjugate. In eq. (15) we have the definition of the field tensor and a straight-forward calculation shows them to be the various components of the Riemannian curvature tensor. We construct β' as in eq. (13) by forming the tensor product of $\varpi_{\mathrm{A}}{}^{\mathrm{B}}$ with itself and taking the inner product with respect to the invariant metric for SL(2, C) analogous to the expression in eq. (12). The choice of β remains the same as it was in eq. (7). The energy functional is thus identical to eq. (14) with the exception that the iso-spin indices (i) in that equation are now replaced by the pair (A, B) which are SL(2, C) labels. In this formulation the condition for extremizing the energy functional is equivalent to the requirement that the Bianchi identities be satisfied. The Einstein field equations themselves play the role of subsidiary conditions. For example the vacuum field equations of this theory reduce to the vanishing of the divergence of the Weyl spinor.

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[7] Here and in the following the adjective « Riemannian » will always be understood to occur with the prefix « pseudo », denoting that the signature is not necessarily positive definite.

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[9] With only one parameter the invariance of the action requires that the Lagrangean be homogeneous of degree one in the velocities and the Euler identities follow. Now the 4-dimensional space-time manifold is the range of the parameters and we have the Noether identities. They consist of the necessary and sufficient conditions for defining a globally invariant volume element for Riemannian geometry, Ricci's lemma, contracted Bianchi identities and the identically vanishing Hamiltonian tensor. It is this last identity that plays a fundamental role here.

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