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Invariant operators of inhomogeneous unitary group

by

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ABSTRACT. — Let $t_n \ltimes u(n)$ be the semi-direct sum of an abelian Lie algebra with the Lie algebra of unitary group $U(n)$. It is shown that an invariant of $u(n+1)$ is also an invariant of $t_n \ltimes u(n)$ if $\varphi: t_n \ltimes u \rightarrow u(n+1)$ is an expansion. Sixth-order invariants of semi-direct sum algebra are calculated by this method.

I. INTRODUCTION

In this note we discuss the problem of constructing invariants of inhomogeneous unitary group using a short-cut method which is specifically useful for the calculation of higher-order invariants and Casimir operators which are needed for the representation theory.

There are various methods of constructing invariants of the inhomogeneous unitary group. We use a method first given by Rosen [1] to compute some of the invariants of the inhomogeneous orthogonal group and which was later generalized by Nagel and Shah [2] to include all cases of inhomogeneous symplectic and orthogonal groups. We start from the Lie algebra of the group $T_n \ltimes U(n_1, n_2)$ which is the semidirect product of n -dimensional translation with $U(n_1, n_2)$ group. This is expanded to $u(n_1, n_2 + 1)$ such that the generators of $u(n_1, n_2 + 1)$ are functions of the generators of $t_n \ltimes u(n_1, n_2)$. Here t_n and $u(n_1, n_2)$ are the Lie algebras corresponding to the groups T_n and $U(n_1, n_2)$. From the Casimir operator of $u(n_1, n_2 + 1)$, which is easy to calculate, we extract invariants of $t_n \ltimes u(n_1, n_2)$ by using a lemma given below.

In Sec. II, we describe the construction of the generators of the algebra $u(n_1, n_2 + 1)$ and give the commutation relations. Sec. III describes the method of calculation. In Sec. IV we apply this method to calculate the sixth-order invariants of $t_n \ltimes u(n)$.

Following [4], we construct the generators of the Lie algebra $u(n_1, n_2 + 1)$ as an $Iu(n)$ realization via the following mapping φ (the generators of $\varphi u(n_1, n_2 + 1)$ are denoted as $L'_{\mu\nu}$, etc.):

$$\textcircled{A} \left\{ \begin{array}{l} L'_{\mu\nu} = L_{\mu\nu}; \quad Q'_{\mu\nu} = Q_{\mu\nu} \\ L'_{n+1,\nu} = R_\nu + \frac{\lambda}{\sqrt{-T^2}} \bar{R}_\nu \\ Q'_{n+1,\nu} = S_\nu + \frac{\lambda}{\sqrt{-T^2}} \bar{S}_\nu \\ Q'_{n+1,n+1} = \frac{-V}{T^2} \end{array} \right.$$

where

$$\begin{aligned} T^2 &= (g_{n+1,n+1})^{-1}(R_\mu R^\mu + S_\mu S^\mu) \\ \bar{R}_\nu &= \{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \} \\ \bar{S}_\nu &= \{ -R^\mu L_{\mu\nu} + S^\mu Q_{\mu\nu} \} \end{aligned}$$

and

$$V = \{ -2R^\mu S^\nu L_{\mu\nu} + R^\mu R^\nu Q_{\mu\nu} + S^\mu S^\nu Q_{\mu\nu} \}.$$

The braces $\{ , \}$ mean that each term is to be symmetrized in R_μ and S_μ with respect to $L_{\mu\nu}$, $Q_{\mu\nu}$, \bar{R}_ν and \bar{S}_ν but not relative to one another, and divided by the number of terms needed for symmetrization. An invariant and a Casimir operator are defined as follows:

Invariant: let $I(L_{\mu\nu}, Q_{\mu\nu}, S_\mu, R_\mu)$ be a homogeneous polynomial in $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ . Then I is said to be an invariant if it commutes with all $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ . The order of an invariant is the order of the polynomial.

Casimir operator: a Casimir operator is an invariant which cannot be expressed as a linear combination of lower-order invariants. In general, an m^{th} -order Casimir operator $C_m(G)$ for a group G is

$$C_m(G) = X_{\mu_1\mu_2} X_{\mu_2\mu_3} \dots X_{\mu_{m-1}\mu_m}$$

where $X_{\mu_i\mu_j}$ are the generators of the Lie algebra of G .

III. COMPUTATION OF INVARIANTS OF $t_n \in u(n_1, n_2)$

Let $g_i \in t_n \in u(n_1, n_2)$ which is expanded to $u(n_1, n_2 + 1)$ and let $h_\alpha \in u(n_1, n_2 + 1)$ such that $h_\alpha = h_\alpha(g_1, g_2, \dots, g_N)$ where $N = n(n + 1)/2$.

The Casimir operators of $u(n_1, n_2 + 1)$ are functions of g_1, \dots, g_N since

$$C_m(h_1, \dots, h_M) = C_m(g_1, \dots, g_N)$$

where C_m is the m^{th} -order Casimir operator of $u(n_1, n_2 + 1)$ and M is the

number of generators of $u(n_1, n_2 + 1)$. Let X_1 be an invariant of $t_n \oplus u(n_1, n_2)$, then

$$[X_1, L_{\mu\nu}] = 0 = [X_1, Q_{\mu\nu}]; \quad [X_1, R_\mu] = 0 = [X_1, S_\mu].$$

Let X_2 be an invariant of $u(n_1, n_2 + 1)$, then we have

$$\begin{aligned} [X_2, L'_{\mu\nu}] &= 0; & [X_2, Q'_{\mu\nu}] &= 0; & [X_2, L'_{\mu, n+1}] &= 0; \\ [X_2, Q'_{\mu, n+1}] &= 0; & [X_2, Q'_{n+1, n+1}] &= 0. \end{aligned}$$

LEMMA. — If $u(n_1, n_2 + 1)$ is realized in $\varepsilon_E(t_n \oplus u(n_1, n_2))$ through the expansion φ , then any operator X commuting with $h_\alpha \in u(n_1, n_2 + 1)$ also commutes with $g_i \in t_n \oplus u(n_1, n_2)$.

Proof. — The generators $L'_{\mu\nu}$, $Q'_{\mu\nu}$, $L'_{\mu, n+1}$ and $Q'_{\mu, n+1}$ are, by construction, functions of $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ . One can verify that any operator X which satisfies the relation satisfied by X_2 also satisfies the relation satisfied by X_1 . We show that $[R_\mu, X]$ and $[S_\mu, X]$ are zero as follows. Consider the commutator $[Q'_{n+1, n+1}, X]$ and substitute for $Q'_{n+1, n+1}$ to get

$$[Q'_{n+1, n+1}, X] = \frac{1}{T^2} [\{ -2R^\mu S^\nu L_{\mu\nu} + R^\mu R^\nu Q_{\mu\nu} + S^\mu S^\nu Q_{\mu\nu} \}, X]$$

which reduces to an expression involving commutators of the type $[R^\mu S^\nu, X]$, $[R^\mu R^\nu, X]$ and $[S^\mu S^\nu, X]$ because $[L_{\mu\nu}, X]$ and $[Q_{\mu\nu}, X]$ vanish due to (A). Now $[L_{n+1, \nu}, X] = 0$ implies that

$$[\lambda^{-1}R_\nu + \{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0$$

or

$$\lambda^{-1}[R_\nu, X] + [\{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0.$$

Notice that we have divided by λ the last three equations of (A). Now we may remark that if $X = X(\alpha) = \sum_i \alpha^i X_i$; $i = 0, 1 \dots$ with parameter α ,

and if I is another operator independent of α , then I commutes with X_i if and only if it commutes with $X(\alpha)$. This means that

$$[R_\nu, X] = 0 \quad \text{and} \quad [\{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0.$$

Using further the commutation relations of $L_{\mu\nu}$ and $Q_{\mu\nu}$ with X , we have also $[S^\mu, X] = 0$. So, if X is an invariant of $u(n_1, n_2 + 1)$ then it is also an invariant of $t_n \oplus u(n_1, n_2)$. Q. E. D.

IV. SIXTH-ORDER INVARIANTS OF $t_n \oplus u(n_1, n_2)$

Using the method developed in Sec. III, starting from the second-order Casimir operator of $u(n_1, n_2 + 1)$ we determine invariants of second,

fourth and sixth order for the Lie algebra $t_n \oplus u(n_1, n_2)$. The second-order Casimir operator of $u(n_1, n_2 + 1)$ is

$$C_2(u(n_1, n_2 + 1)) = C_2(u(n_1, n_2)) + 2Q'_{n+1, \nu} Q'_{\nu, n+1} - 2L'_{n+1, \nu} L'_{\nu, n+1} + (Q'_{n+1, n+1})^2.$$

In polynomial form one can have

$$C_2(u(n_1, n_2 + 1)) = \sum_i \mu^i A_i$$

where μ is some parameter related to λ . Substituting from equations (A) and using commutation relations we have the following invariants (all the indices are written as subscripts because for simplicity we take $g_{\mu\nu} = \delta_{\mu\nu}$):

$$A_1 = 2[S_\mu S_\mu - R_\mu R_\mu]$$

$$A_2 = 2[S_\mu S_\nu Q_{\mu\nu} - R_\mu R_\nu Q_{\mu\nu} + (2n + 1)R_\mu S_\mu - R_\mu S_\nu L_{\mu\nu}]$$

$$A_3 = 2[8S_\mu R_\lambda Q_{\mu\nu} L_{\lambda\nu} - S_\mu S_\lambda L_{\mu\nu} L_{\lambda\nu} - R_\mu R_\lambda Q_{\mu\nu} Q_{\lambda\nu} - 2R_\mu S_\lambda Q_{\mu\nu} L_{\lambda\nu} + R_\mu S_\mu Q_{\lambda\lambda} + (n + 2)R_\mu S_\nu Q_{\mu\nu} + (7n + 9)(n - 1)S_\mu S_\mu + n(n + 1)R_\mu R_\mu] + 4R_\mu R_\lambda S_\nu S_\sigma L_{\mu\nu} L_{\lambda\sigma} - 4R_\mu R_\lambda R_\sigma S_\nu L_{\mu\nu} Q_{\lambda\sigma} - 4R_\mu S_\nu S_\lambda S_\sigma L_{\mu\nu} Q_{\lambda\sigma} + R_\mu R_\nu R_\lambda R_\sigma Q_{\mu\nu} Q_{\lambda\sigma} + 2R_\mu R_\nu S_\lambda S_\sigma Q_{\mu\nu} Q_{\lambda\sigma} + S_\mu S_\nu S_\lambda S_\sigma Q_{\mu\nu} Q_{\lambda\sigma} + 4R_\mu R_\nu R_\sigma S_\nu Q_{\mu\sigma} - 4R_\mu R_\sigma R_\mu S_\nu Q_{\nu\sigma} + 4R_\mu S_\nu S_\sigma S_\nu Q_{\mu\sigma} - 4R_\mu S_\mu S_\nu S_\sigma Q_{\nu\sigma}.$$

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