

# ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 22, n° 2 (1975), p. 97-107

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## Absolute bounds on vertices and couplings

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ABSTRACT. — We prove absolute upper and/or lower bounds on  $\varphi^4$  and  $\varphi^6$  dimensionless vertices and physical coupling constants.

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### 1. DEFINITIONS AND ASSUMPTIONS

We derive absolute bounds on  $\varphi^4$  and  $\varphi^6$  dimensionless coupling constants  $g$ . For instance, in a pure  $\varphi^4$  model we prove

$$(1.1) \quad 0 \leq g \leq \text{const.},$$

in the single phase region (no symmetry breaking). We also obtain bounds on associated vertex functions and on connected parts. Our general methods presumably have other consequences for  $n$ -particle amplitudes,  $n \geq 8$ , which we do not pursue here.

DEFINITIONS. — Let

$$(1.2) \quad \langle 1 \dots n \rangle \equiv \langle \Phi(x_1) \dots \Phi(x_n) \rangle$$

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<sup>(1)</sup> Supported in part by the National Science Foundation under Grant MPS 74-13252.

<sup>(2)</sup> Supported in part by the National Science Foundation under Grant MPS 73-05037.

denote the  $n$ -point (Euclidean) Schwinger function, and let  $\langle 1 \dots n \rangle_T$  denote its connected part. We let

$$(1.3) \quad S_T^{(n)}(p) = S_T^{(n)}(p_1, \dots, p_{n-1}) \\ = \int \langle 1 \dots n \rangle_T \exp \left[ i \sum_{j=1}^{n-1} p_j x_j \right] dx_1 \dots dx_{n-1} \Big|_{x_n=0}$$

denote the Fourier transform of  $\langle 1 \dots n \rangle_T$ . We encounter  $S_T^{(2)}(p)$  often, and denote it  $\chi(p)$ . We use the standard spectral representation

$$(1.4) \quad \chi(p) = S_T^{(2)}(p) = \int \frac{d\rho(a)}{p^2 + a},$$

where  $d\rho(a)$  is the spectral measure. Furthermore, we let

$$(1.5) \quad \Gamma^{(n)}(p) = \Gamma^{(n)}(p_1, \dots, p_{n-1})$$

denote the  $n$ -point Euclidean vertex function (thus  $\Gamma^{(n)}$  is the amputated, one particle irreducible part of  $S_T^{(n)}(p)$ ). In an even theory, we have for example,

$$(1.6) \quad \Gamma^{(4)}(p) = S_T^{(4)}(p) \prod_{i=1}^4 \chi(p_i)^{-1},$$

where  $p_4 = -(p_1 + p_2 + p_3)$ . Also

$$(1.7) \quad \Gamma^{(6)}(p) = \left[ S_T^{(6)}(p) - \sum_X S_T^{(4)}(X) \chi(P_X)^{-1} S_T^{(4)}(\sim X) \right] \prod_{i=1}^6 \chi(p_i)^{-1},$$

where  $p_1 + \dots + p_6 = 0$ , where  $(X, \sim X)$  is a partition of  $(p_1, \dots, p_6)$  into two subsets of three elements, and where  $P_X$  is the sum of the momenta in  $X$ .

We let  $d \leq 4$  denote the space-time dimension and let  $m$  denote the mass gap. Note that  $\Gamma^{(n)}$  has the dimension of mass to the power  $d - \frac{1}{2}n(d - 2)$ .

We define the dimensionless amplitude  $g^{(n)}(p)$  by

$$(1.8) \quad g^{(n)}(p) = -m^{-d-n+\frac{1}{2}nd} \Gamma^{(n)}(p),$$

and the associated coupling constant  $g^{(n)}$  by

$$(1.9) \quad g^{(n)} = g^{(n)}(0).$$

For example,

$$(1.10) \quad g^{(4)} = -m^{d-4} \chi^{-4} \int \langle 1234 \rangle_T dx_1 dx_2 dx_3,$$

where  $\chi \equiv \chi(0)$ .

ASSUMPTIONS. — We make some or all of the following assumptions. In the case of weakly coupled  $\mathcal{P}(\varphi)_2$ , assumptions (a)-(f) have been proved, while for single phase, even  $\varphi_2^4$  models (h)-(i) have been proved. We expect (a)-(f) to hold for single phase  $\varphi_3^4$ ,  $\varphi_3^6$ , and  $\varphi_4^4$  models, while (h)-(i) are expected for all single phase even  $\varphi_d^4$  models,  $d \leq 4$ . The evidence for the validity of (1.4) in  $\varphi_4^4$  comes from perturbation theory. The proof of (i) in even, single phase  $\varphi_2^4$  has been established by Cartier (private communication).

(a) The Wightman axioms for a scalar boson field, or the Osterwalder-Schrader axioms for the corresponding Euclidean theory. The two point function  $\chi(p)$  satisfies (1.4). In particular we are assuming that  $\int \frac{d\rho(a)}{a}$  is finite.

(b) A positive mass gap,  $m > 0$ .

(c)  $\langle \varphi \rangle = 0$ .

(d) The second Griffiths inequality,

$$\langle A_1 A_2 \rangle \geq \langle A_1 \rangle \langle A_2 \rangle,$$

where  $(A_1, A_2)$  is a partition of  $(1, \dots, n)$ . For instance,

$$(1.11) \quad -\langle 1234 \rangle_T = -\{ \langle 1234 \rangle - \langle 14 \rangle \langle 23 \rangle - \langle 12 \rangle \langle 34 \rangle - \langle 13 \rangle \langle 24 \rangle \} \leq \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle.$$

(f) The measure  $d\rho(a)$  contains a delta function at  $a = m^2$ , of unit strength (This is a renormalization hypothesis).

(h) In the case of a  $\varphi^4$  interaction, it is appropriate to assume the Lebowitz inequality  $\langle 1234 \rangle_T \leq 0$ , or the set of Lebowitz inequalities,

$$(1.12) \quad \langle 12U \rangle \leq \langle 12 \rangle \langle U \rangle + \sum_W \langle W, 1 \rangle \langle \sim W, 2 \rangle$$

where  $W$  is an (odd) subset of  $U$ .

(i) In the case of a pure  $\varphi^4$  model, we assume

$$(1.13) \quad \langle 123456 \rangle_T \geq 0.$$

## 2. BOUNDS ON QUARTIC COUPLINGS

### THEOREM 2.1.

(i) Assume (a)-(d) above. Then

$$(2.1) \quad g^{(4)} \leq \text{const. } \chi^{-2} m^{-4}.$$

(ii) Assume (f) also, then  $\chi \geq m^{-2}$ , so

$$(2.2) \quad g^{(4)} \leq \text{const.}$$

(iii) Assume (h) also, then

$$(2.3) \quad 0 \leq g^{(4)} \leq \text{const. } \chi^{-2} m^{-4} \leq \text{const.}$$

*Remark 1.* — The constants are pure numbers, independent of all parameters. They can be determined explicitly by the proof.

*Remark 2.* — The upper bound on  $g^{(4)}$  is related to the picture of « critical point dominance » discussed in [I]. According to this picture, for  $d < 4$ ,  $g^{(4)}$  assumes its maximum value at the critical point, *i. e.* at the onset of phase transitions. According to standard ideas,  $m = 0$  at the critical point.

*Remark 3.* — For  $d < 4$ , the limit  $\lambda_0$  (= bare  $\varphi^4$  coupling constant)  $\rightarrow \infty$  with  $m$  fixed plays a role in the application of the Callan-Symanzik equations to the study of critical phenomena [2]. We use Theorem 2.1 below, combined with bounds from [3], to show that either this limit exists, or that in this limit  $g^{(4)} \rightarrow 0$  (see Corollary 2.4). If  $g^{(4)}$  is monotone in the dimensionless charge  $\lambda_0 m_0^{d-4}$ , then the alternative  $g^{(4)} \rightarrow 0$  is excluded. The possibility that  $g^{(4)}$  is monotone is discussed in [I], and is a consequence of conventional assumptions made in the study of the Callan-Symanzik equations.

*Remark 4.* — Other bounds on quartic couplings have been established by Lukaszuk and Martin [4], see also Healy [5]. Their bounds require different assumptions and methods, and yield different conclusions.

In the case of a pure  $\varphi^4$  model we also obtain bounds on  $g^{(4)}(p)$  for  $p \neq 0$ .

**THEOREM 2.2.** — Assume (a)-(h) above. There exists  $\delta > 0$ , such that if  $|p_i| < \delta m$ , then  $g^{(4)}(p)$  is analytic in  $p$  and bounded uniformly by

$$(2.4) \quad |g^{(4)}(p)| \leq \text{const. } \chi^{-2} m^{-4} \leq \text{const.}$$

*Proof of Theorem 2.1.* — By the positivity of  $d\rho$  (Wightman positivity condition)

$$(2.5) \quad \langle 12 \rangle \geq 0,$$

so the right side of (1.11) is positive. By symmetry we then have

$$(2.6) \quad -\langle 1234 \rangle_T \leq (\langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle)^\dagger (\langle 14 \rangle \langle 23 \rangle + \langle 12 \rangle \langle 34 \rangle)^\dagger (\langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle)^\dagger \\ = \left( \sum_{i=1}^8 A_i \right)^\dagger \leq 2 \sup A_i^\dagger.$$

Here each  $A_i$  is either

$$\langle 12 \rangle \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle \langle 34 \rangle = \prod_{1 \leq i < j \leq 4} (ij)$$

or else a permutation of

$$\langle 12 \rangle^2 \langle 34 \rangle^2 \langle 13 \rangle \langle 24 \rangle.$$

We bound each  $A_i$  using Proposition A2.1, of Appendix 2,

$$\langle x0 \rangle \leq O(1) |x|^{-d_\sigma - m(1-\varepsilon)|x|} \chi.$$

An elementary estimate shows that each  $A_i^{\frac{1}{2}}$  is integrable over  $dx_1 dx_2 dx_3$ . By homogeneity, and (A2.1),

$$\int A_i^{\frac{1}{2}} dx_1 dx_2 dx_3 \leq O(1) m^{-d} \chi^2,$$

from which we conclude

$$(2.7) \quad -S_T^{(4)}(0) \leq O(1) m^{-d} \chi^2.$$

By (1.8), with  $n = 4$ , and (1.6) we have (2.1). The theorem then follows.

*Proof of Theorem 2.2.* — Assuming (h) and (2.6),

$$(2.8) \quad |\langle 1234 \rangle_T| \leq 2 \sup_i A_i^{\frac{1}{2}}.$$

We bound the power series coefficients of  $S_T^{(4)}(p)$ , expanding about  $p = 0$ . The derivatives  $\partial/\partial p_j$  become multiplication by  $ix_j$  in the Fourier transform, and are dominated by the exponential decrease of (2.8), estimated by (A2.1). Thus the power series in momenta converges for  $|p_i| < m/3$ , and establishes the analyticity and boundedness of  $S_T^{(4)}(p)$ ,

$$(2.9) \quad |S_T^{(4)}(p)| \leq O(1) m^{-d} \chi^2.$$

Finally, the analyticity and boundedness of  $\chi(p)^{-1}$  follows from (1.4), for  $|p| < m$ , yielding

$$|\chi(p)|^{-1} \leq 2\chi(0)^{-1}.$$

Thus  $g^{(4)}(p)$  is analytic in  $p$  and

$$|g^{(4)}(p)| \leq O(1) m^{-d} \chi^{-2} \leq O(1) m^{-d+4},$$

with the constant independent of  $p, m, \lambda$  for  $|p_i| \leq \delta m$ . This completes the proof. We have also proved.

**COROLLARY 2.3.** — Assuming (a)-(h),  $S_T^{(4)}(p)$  is analytic for  $|p_i| \leq \delta m$  and bounded uniformly by (2.9).

We now let  $S_j^{(n)}$  be the  $n$ -point Schwinger function for a model satis-

fying (a)-(h) and labelled by the index  $j$  (for instance,  $S_{\lambda_0, m}^{(n)}$  may be the  $\varphi_2^4$  model, labelled by the bare coupling  $\lambda_0$  and mass  $m$ ).

**COROLLARY 2.4.** — Assume  $m_j$  is bounded away from zero. Then either there exists a convergent subsequence of Schwinger functions

$$(2.10) \quad S^{(n)} = \lim_i S_j^{(n)},$$

with  $S^{(n)}$  satisfying (a)-(h), or else  $g_T^{(4)}(p) \rightarrow 0$ .

*Proof.* — Assume for some  $p$ , with  $|p_i| \leq \delta m$ , there is a subsequence of  $g_r(p)$  bounded away from zero. Then by (2.4)

$$\chi_r^2 \leq \text{const. } m_r^{-4} (g_r^{(4)}(p))^{-1} \leq \text{const.}$$

This bound (uniform in  $r$ ) on

$$\int \langle 12 \rangle_{T, r} dx_1 = \int \langle 12 \rangle_r dx_1 \leq \text{const.}$$

proves a uniform bound

$$(2.11) \quad |\langle fg \rangle| \leq |f|_{\mathcal{S}} |g|_{\mathcal{S}}$$

for a suitable Schwartz space norm  $|\cdot|_{\mathcal{S}}$ . Thus by Theorem 1 of [3], a convergent subsequence of Schwinger functions  $S_{j_i}^{(n)}$  exists, and  $S^{(n)}$  is a tempered distribution. The properties (a)-(h) follow from the corresponding properties in the approximate theories.

Conversely, if  $g_j(p) \rightarrow 0$  for each  $p$  with  $|p_i| < \delta m$ , the uniform bound (2.4) ensures that the family of analytic functions  $g_j(p)$  converges to the analytic function zero.

### 3. BOUNDS ON $S_T^{(6)}(p)$

In this section we prove preliminary bounds on  $S_T^6$ .

**THEOREM 3.1.**

(i) Assume (a)-(h). Then there exists  $\delta > 0$  such that if  $|p_i| \leq m\delta$ , then  $S_T^{(6)}(p)$  is analytic and bounded by

$$(3.1) \quad m^{2d} \chi^{-3} |S_T^{(6)}(p)| \leq \text{const.}$$

(ii) Assume (a)-(i). Then

$$(3.2) \quad 0 \leq m^{2d} \chi^{-3} S_T^{(6)}(0) \leq \text{const.}$$

The constants are pure numbers, independent of the parameters.

*Proof.* — We consider  $p = 0$ . The extension to  $p \neq 0$  follows the proof of Theorem 2.2. Using (f), we have

$$(3.3) \quad \langle 123456 \rangle - \langle 12 \rangle \langle 3456 \rangle \geq 0.$$

The definition of the six point connected part is

$$(3.4) \quad \langle 123456 \rangle_T = \langle 123456 \rangle - \sum_Y \langle Y_1 \rangle \langle Y_2 \rangle + 2 \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle.$$

Here  $Y = (Y_1, Y_2)$  ranges over the fifteen partitions of  $(1, 2, \dots, 6)$  into two and four element subsets, while  $Z = (Z_1, Z_2, Z_3)$  ranges over the fifteen partitions of  $(1, \dots, 6)$  into three pairs. Thus

$$(3.5) \quad - \langle 123456 \rangle_T \leq \sum_{Y'} \langle Y_1 \rangle \langle Y_2 \rangle - 2 \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle.$$

where  $\Sigma'$  omits the partition  $\langle 12 \rangle \langle 3456 \rangle$ . We use (h) and the definition of  $\langle 1234 \rangle_T$  to obtain

$$(3.6) \quad - \langle 123456 \rangle_T \leq \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle,$$

where  $\Sigma'$  omits the three partitions which contain the factor  $\langle Z_1 \rangle = \langle 12 \rangle$ . By symmetry,

$$(3.7) \quad - \langle 123456 \rangle_T \leq 12 \sup_{j=1}^{15} \prod_{i=1}^3 \langle Z_i^{(j)} \rangle^{1/15},$$

where  $j = 1, 2, \dots, 15$  labels the two element subsets of  $(1, \dots, 6)$ , and  $(Z_1^{(j)}, Z_2^{(j)}, Z_3^{(j)})$  one of the twelve partitions of  $(1, \dots, 6)$  into three pairs, no one of which is  $j$ .

We now follow the proof of Theorem 2.1 to obtain

$$- S_T^{(6)}(0) \leq \text{const. } m^{-2d_\chi^3}.$$

proving one side of (3.1).

We next use (1.12) (see [3]) with  $U = (3456)$  to obtain

$$\langle 123456 \rangle \leq \langle 12 \rangle \langle 3456 \rangle + \sum_W \langle W, 1 \rangle \langle \sim W, 2 \rangle,$$

where  $W$  ranges over the eight odd subsets of  $U$ . Thus by (3.4),

$$\langle 123456 \rangle_T \leq - \sum_Y \langle Y_1 \rangle \langle Y_2 \rangle + 2 \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle$$



where  $\Sigma'$  ranges over the six partitions containing both 1 and 2 in the four element subset. By the definition of  $\langle 1234 \rangle_T$ ,

$$\langle 123456 \rangle_T \leq - \sum_Y \langle Y_1 \rangle_T \langle Y_2 \rangle_T + \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle.$$

By (1.11) we note that

$$\langle 123456 \rangle_T \leq 2 \sum_Z \langle Z_1 \rangle \langle Z_2 \rangle \langle Z_3 \rangle,$$

which is just a bound by twice the right hand side of (3.6). We now proceed as above to obtain the bound (3.2) for  $p = 0$ .

Finally, we remark that assuming (i) yields  $0 \leq S_T^{(6)}(0)$ , to complete the proof.

#### 4. BOUNDS ON $g^{(6)}$

We use the bounds of Chapter 3 to establish bounds on the six point vertex function in the  $\varphi^4$  model.

**THEOREM 4.1.** — Assuming (a)-(h), for  $|p_i| \leq m\delta$ ,  $g^{(6)}(p)$  is analytic and bounded by

$$|g^{(6)}(p)| \leq \text{const.}$$

where  $\delta$  is the smaller of the constants given by Theorems 2.2 and 3.1.

*Proof.* — By Theorem 3.1,  $m^{2d}\chi^{-3}S_T^{(6)}(p)$  is analytic and bounded. By Theorem 2.2  $g^{(4)}(p)$  is analytic and bounded. Note

$$g^{(4)}(p) = m^{d-4} \prod_{i=1}^4 \chi(p_i)^{-1} S_T^{(4)}(p).$$

The desired bound follows by

$$g^{(6)}(p) = -m^{2d-6} \Gamma^{(6)}(p)$$

and the representation (1.7).

## APPENDIX 1

## THE FREE EUCLIDEAN PROPAGATOR

We establish the elementary bounds on

$$c(m^2; x) = \frac{1}{\pi} \int \frac{e^{-ipx} dp}{p^2 + m^2} = m^{d-2} c(1; mx)$$

used below. By Euclidean invariance, we may evaluate  $c$  for  $x = x_t = (t, \vec{0})$ ,  $t > 0$ , and

$$f(t) = c(1; x_t) = \int \frac{e^{-t\mu} d\vec{p}}{\mu}$$

where  $\mu = \mu(\vec{p}) = (p^2 + 1)^{\frac{1}{2}}$ .

PROPOSITION A1.1. — There is a constant  $O(1)$  depending only on  $d$ , such that

$$f(t) \leq \begin{cases} O(1)t^{(1-d)/2}e^{-t} & t \geq \frac{1}{2} \\ O(1)t^{-(d-2)} & t \leq \frac{1}{2}, d \geq 3 \\ O(1)|\ln t| & t \leq \frac{1}{2}, d = 2 \end{cases}$$

REMARK. — The long and short range exponents of  $t$  differ.

*Proof.* — Let  $\mu_t(\vec{p}) = (p^2 + t^2)^{\frac{1}{2}}$ , so

$$f(t) = t^{-(d-2)}e^{-t}I(t)$$

where

$$I(t) = \int \mu_1^{-1} \exp[-(\mu_t - t)] d\vec{p}.$$

For  $t \geq \frac{1}{2}$ , and  $t \geq |\vec{p}|$ , we use

$$\mu_t - t \leq \text{const. } p^2 t^{-1}$$

to establish

$$I(t) \leq O(t^{(d-3)/2}).$$

Furthermore, for  $\frac{1}{2} \leq t \leq |\vec{p}|$ , we use  $\mu_t - t \geq \text{const. } |\vec{p}|$  to give

$$I(t) \leq O(t^{-1}).$$

Thus for  $t \geq \frac{1}{2}$ , we have established the proposition.

For  $t \leq \frac{1}{2}$ , and  $d \geq 3$ , we have

$$I(t) \leq \int |\vec{p}|^{-1} \exp\left[-|\vec{p}| + \frac{1}{2}\right] d\vec{p} \leq O(1).$$

To bound the case  $d = 2$ , with  $t \leq \frac{1}{2}$ , we use

$$I(t) \leq \text{const. } \int_0^\infty e^{-u} \frac{du}{(t^2 + u^2)^{\frac{1}{2}}} \leq O(|\ln t|).$$

## APPENDIX 2

## THE EUCLIDEAN PROPAGATOR

The Euclidean propagator  $\langle xy \rangle$  has the spectral representation given by (1.4),

$$\langle xy \rangle = \langle \Phi(x)\Phi(y) \rangle = \int d\rho(a)c(a; x-y).$$

PROPOSITION A2.1. — Assume (a)-(c) above, and let  $\varepsilon > 0$ . Then

$$(A2.1) \quad \langle x0 \rangle \leq O(1) |x|^{-d} e^{-m(1-\varepsilon)|x|} \chi,$$

where  $O(1)$  depends only on the dimension  $d$  and on  $\varepsilon$ .

PROOF. — We use proposition A1.1. Let  $d \geq 3$ . Then

$$\begin{aligned} \langle x0 \rangle &= \int d\rho(a) a^{(d-2)/2} c(1; \sqrt{ax}). \\ &\leq O(1) \int \frac{d\rho(a)}{a} a |x|^{-d+2} e^{-\sqrt{a}|x|(1-\varepsilon/2)} \\ &\leq O(1) e^{-m(1-\varepsilon)|x|} \left( \int \frac{d\rho(a)}{a} \right) |x|^{-d+2} \left( \sup_a a e^{-\sqrt{a}|x|\varepsilon/2} \right). \end{aligned}$$

The sup occurs for  $a = 4(\varepsilon x)^{-2}$ . Thus (A2.1) follows. For  $d = 2$ ,

$$\begin{aligned} \langle x0 \rangle &= \int d\rho(a) c(1; \sqrt{ax}) \\ &\leq O(1) \int_I \frac{d\rho(a)}{a} a |\ln \sqrt{ax}| + O(1) \int_{II} \frac{d\rho(a)}{a} a e^{-\sqrt{a}|x|}, \end{aligned}$$

where

$$I = \{ a : m \leq \sqrt{a} \leq |2x|^{-1} \} \quad \text{and} \quad II = \{ a : \sqrt{a} \geq |2x|^{-1} \}.$$

The integral over II is bounded as above, by (A2.1). The integral over I is bounded using

$$\sup_{a \in I} a |\ln \sqrt{a}| x| \leq |x|^{-2} \sup_{0 \leq u \leq 1} u^2 |\ln u| \leq O(1) |x|^{-2}.$$

This yields the bound  $O(1) |x|^{-2} \chi$ . Since  $|mx| \leq \frac{1}{2}$ , for  $a \in I$ , the integral over I is bounded by (A2.1).

REMARK. — The increase in the bound on the short range singularities (from  $x^{(2-d)}$  to  $x^{-d}$ ) is associated with an allowed divergence of the field strength renormalization integral

$$Z^{-1} = \int d\rho(a).$$

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(Manuscrit reçu le 9 août 1974)