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Matrix canonical realizations
of the Lie algebra $o(m, n)$
I. Basic formulæ and classification

by

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ABSTRACT. — The concept of matrix canonical realization of a Lie algebra is introduced. The generators of the Lie algebra of the pseudoorthogonal group $O(m, n)$ are recurrently expressed in terms of matrices with polynomial elements in a certain number of quantum-mechanical canonical variables p_i, q_i and they depend on a certain number of free real parameters. The realizations are, in the well-defined sense, skew-hermitean and Casimir operators are multiples of the identity element. Part of them are usual canonical realizations.

1. INTRODUCTION

In the previous paper we dealt with canonical realizations of the complexified Lie algebra of the orthogonal group in n-dimensional Euclidean space $O_C(n)$ [1]. By canonical realization we understood there an isomorphism mapping τ of a given Lie algebra G into the Weyl algebra W_{2N}, *i. e.* essentially into the algebra of polynomials in N pairs of quantum canonical variables p^i, q_i, $i = 1, 2, \ldots, N$. Among the other results we proved there that in any canonical realization of Lie algebra $o(m, n)$ of a pseudoortho-

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gonal group $O(m, n)$, $m \geqslant n > 0$, in W_{2N}, $N = m + n - 2$ (for the exception of the case $m + n = 6$), all Casimir operators are realized by constant multiples of the identity element (*) (we speak about Schur-realizations) and if $m + n > 6$ they depend on the quadratic ones in one of the two possible ways only (it is what we call « degeneration » of realization). It means that to remove partly or even fully the mentioned « degeneration » we must enlarge the number N of canonical pairs. In contrast to canonical realizations of $o(m, n)$ in W_{2N}, $N = m + n - 2$, the realizations with $N > m + n - 2$ need not be necessarily the Schur-realizations. So we come naturally to the question whether some realizations of $o(m, n)$ exist in which the « degeneration » is at least reduced and which are at the same time Schur-realizations.

In this paper we solve this question positively in the generalized framework of the so-called matrix canonical realizations in which the generators of considered Lie algebra are expressed by matrices with elements from W_{2N} (if the dimension of such matrices is M, we denote such a generalization of the Weyl algebra by the symbol $W_{2N,M}$).

The matrix canonical realizations represent one possible proper algebraical embedding of the Weyl algebra into a larger structure. It is known that another possibility is to embed the Weyl algebra into its quotient division ring. In both these cases the class of allowed functions, in terms of which the generators of a given Lie algebra can be expressed, is essentially enriched compared with the original Weyl algebra. Therefore the possibility of obtaining a wider class of realizations in these structures arises without necessity of changing the pure algebraical approach.

From the point of view of application to the representation theory we shall introduce further concept of skew-hermitean realization though the representation aspects are not discussed in this paper. By the skew-hermitean matrix canonical realization we shall essentially imply the one which after replacement of p^i and q_i by their Schrödinger representatives passes to a skew-symmetric representation on a suitable Hilbert space.

The main result of this paper lies in the formulæ describing recurrently two sets of matrix canonical skew-symmetric Schur-realizations of the Lie algebras $o(m, n)$, $n \geqslant 1$. In some special cases these formulæ coincide with earlier results of some authors (see e. g. Richard [3]). Every realization from the first set is uniquely determined by some finite-dimensional irreducible skew-hermitean representation of the compact Lie algebra $o(m - n)$ (**) and the finite sequence of n real numbers. If the dimension of representation of $o(m - n)$ is M then $o(m, n)$ is realized in $W_{2n(m-1),M}$.

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(*) As to the particular case $W_{2(m+n-3)} \subset W_{2(m+n-2)}$ see also [2].

(**) As to case $m - n = 0,1$ see the following footnote.

Realizations from the second set are usual canonical realizations in $W_{2d(m+n-d-1)}$, $d = 1, 2, \ldots, n - 1$; they are characterized by d-tuple of real constants.

We shall introduce further the concept of related and non-related realizations by means of which all realizations described above will be classified. We prove that any two realizations chosen from the both sets are non-related if they differ either in characterizing tuples of real numbers or in the case of realizations of the first type with the same characterizing tuples, if the irreducible representations of the algebra $o(m - n)$ are non-equivalent.

The exact formulation of all these statements is contained in Theorem 3. Its proof is based mainly on Theorem 1 where the basic recurrent formulæ are included. Theorem 2 shows that any skew-hermitean matrix canonical Schur-realization of a compact Lie algebra is usual matrix skew-hermitean representation, what generalizes the Joseph assertion [4].

Again, as in our previous paper, all considerations are purely algebraical. As to the problem of « degeneration », *i. e.* mutual dependence or independence of Casimir operators in the described realizations, we shall discuss these questions in the second part of this paper.

2. PRELIMINARIES

A. The (complex) Weyl algebra W_{2N} is the associative algebra with the identity element $\mathbf{1}$ over the field of complex numbers $\mathbf{C}$; its generating elements q_i, p^i, $i = 1, \ldots, N$, fulfill the usual canonical commutation relations

$$[p^i, p^j] = [q_i, q_j] = 0, \quad [p^i, q_j] = \delta^i_j \mathbf{1}.$$

As the consequence of the Poincaré-Birkhoff-Witt theorem the monomials

$$q^r(p)^s = q_1^{r_1} \ldots q_N^{r_N}(p^1)^{s_1} \ldots (p^N)^{s_N}$$

form the basis of W_{2N} (see [5], p. 178), *i. e.* every element $w \in W_{2N}$ can be uniquely written in the form

$$w = \sum_{r,s} a_{r,s} \cdot q^r(p)^s$$

where

$$a_{r,s} \equiv a_{r_1 \ldots r_N s_1 \ldots s_N} \in \mathbf{C}.$$

B. The symbol $o(m, n)$, $m \geqslant n \geqslant 0$, $m + n \geqslant 2$, denotes the Lie algebra of pseudoorthogonal group in $(m + n)$-dimensional pseudoeuclidean space with the metric tensor $g_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, m + n$. If $L_{\mu\nu} = - L_{\nu\mu}$ denotes $1/2 \cdot (m + n)(m + n - 1)$ elements of the basis of $o(m, n)$ then the commutation relations hold

$$[L_{\mu\nu}, L_{\rho\tau}] = g_{\nu\rho}L_{\mu\tau} - g_{\mu\rho}L_{\nu\tau} + g_{\nu\tau}L_{\rho\mu} - g_{\mu\tau}L_{\rho\nu} \tag{1}$$

$\rho, \tau = 1, 2, \ldots, m + n$. If $n \geqslant 1$ we can assume without the loss of generality the metric tensor having the form

$$(g_{\mu\nu}) = \text{diag}\,(g_{11}, \ldots, g_{m+n-2,m+n-2}, -1, 1).$$

In addition to the tensor basis $(L_{\mu\nu})$, the second one can be chosen

$$L_{ij}, \quad P_k = L_{k,m+n} + L_{k,m+n-1} \quad , \quad Q_k = L_{k,m+n} - L_{k,m+n-1},$$
$$R = L_{m+n-1,m+n} \quad ; \quad i, j, k, l = 1, 2, \ldots, m + n - 2,$$

in which the commutation relations (1) have the form:

$$
\begin{aligned}
&[L_{ij}, L_{kl}] = g_{jk}L_{il} - g_{ik}L_{jl} + g_{jl}L_{ki} - g_{il}L_{kj}, \\
&[L_{ij}, P_k] = g_{jk}P_i - g_{ik}P_j, \ [L_{ij}, Q_k] = g_{jk}Q_i - g_{ik}Q_j, \\
&[L_{ij}, R] = 0 \quad , \quad [R, P_\kappa] = P_k \quad , \quad [R, Q_k] = -Q_k, \\
&[P_i, P_j] = [Q_i, Q_j] = 0 \quad , \quad [P_i, Q_j] = -2(L_{ij} + g_{ij}R).
\end{aligned}
\tag{2}
$$

Note that the generators $P_1, \ldots, P_{m+n-2}$ and $Q_1, \ldots, Q_{m+n-2}$ form the bases of $(m + n - 2)$-dimensional Abelian subalgebras of $o(m, n)$.

C. Any irreducible finite-dimensional representation of the compact Lie algebra $o(m, 0) \equiv o(m), m \geqslant 2$, is, for the exception of $m = 2$, equivalent to a skew-hermitean one. Any such representation of $o(m)$ is uniquely determined by the so-called signature $\alpha = \left(\alpha_1, \ldots, \alpha_{\left[\frac{m}{2}\right]}\right)$, where the numbers $\alpha_1, \ldots, \alpha_{\left[\frac{m}{2}\right]}$ for $m > 2$ are either all integers or all half-integers such that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{\nu-1} \geqslant |\alpha_\nu|$ if $m = 2\nu$ and $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_\nu \geqslant 0$ if $m = 2\nu + 1$.

In the case of commutative Lie algebra $o(2)$ all irreducible skew-hermitean representations are one-dimensional and the generator L_{12} is represented as $ir E_1$, $r \in R$, where E_1 is identity operator. By the signature α of this representation we understand the one-point sequence $\alpha = \left(r \equiv \alpha_{\left[\frac{2}{2}\right]}\right)$.

Two equivalent irreducible representations have the same signatures and to different signatures there correspond non-equivalent irreducible representations ([6], p. 518-519).

3. BASIC CONCEPTS

Definition 1. — Let W_{2N} be the complex Weyl algebra in N canonical pairs and let Mat_M be the algebra of complex $M \times M$ − matrices. The tensor product $W_{2N,M} = W_{2N} \otimes \text{Mat}_M$ we shall call *matrix Weyl algebra*.

It is clear that $W_{2N,1} \simeq W_{2N} \simeq W_{2N} \otimes E_M \subset W_{2N,M}$ (E_M is the unit $M \times M$-matrix) and $\text{Mat}_M \simeq 1 \otimes \text{Mat}_M \subset W_{2N,M}$.

Definition 2. — A matrix canonical realization of the Lie algebra G is the homomorphism $\tau : G \rightarrow W_{2N,M}$. The homomorphism τ extends natu-

rally to the homomorphism mapping (denoted by the same symbol τ) of the enveloping algebra UG of G into $W_{2N,M}$. If G is simple then any homomorphism is either trivial or it is an isomorphism of G.

Remember that involution on an associative algebra (over **R** or **C**) is the mapping « $+$ » : $G \to G$ obeying the relations

$$(\alpha a + \beta b)^+ = \bar{\alpha} a^+ + \bar{\beta} b^+ \quad , \quad (ab)^+ = b^+ a^+, (a^+)^+ = a \quad , \quad a, b \in G.$$

DEFINITION 3. — Let an involution « $+$ » be defined on the algebra $W_{2N,M}$. A matrix canonical realization of a real Lie algebra G in $W_{2N,M}$ is called *skew-hermitean* if all the elements of G are realized by skew-hermitean expressions, *i. e.* if $\tau^+(a) = -\tau(a)$ for all $a \in G$.

NOTE 1. — An involution on W_{2N} together with the usual Hermitean adjoint in Mat_M defines involution in $W_{2N,M}$. In what follows, we consider such an involution with a special choice induced by

$$(p^i)^+ = -p^i \quad , \quad q_i^+ = q_i$$

on the algebra W_{2N}.

DEFINITION 4. — A matrix canonical realization $\tau : G \to W_{2N,M}$ is called *Schur-realization* if all the central elements of the enveloping algebra UG of G are realized by multiples of the identity element.

For the classification of matrix canonical realizations we introduce the following last concept:

DEFINITION 5. — Matrix canonical realizations τ and τ' of the Lie algebra G in $W_{2N,M}$ are called *related* if a conserving identity endomorphism ϑ of $W_{2N,M}$ exists so that either $\vartheta \circ \tau = \tau'$ or $\vartheta \circ \tau' = \tau$.

4. MATRIX CANONICAL REALIZATIONS OF THE LIE ALGEBRA $o(m, n)$

THEOREM 1. — Let a skew-hermitean Schur-realization of the algebra $o(m-1, n-1), m+n \geqslant 2$ (*), in $W_{2N,M}$ be given and let

$$M_{ij} = -M_{ji} \in W_{2N,M}$$

denote the realization of the basis elements of $o(m-1, n-1)$. Then

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(*) In order to reduce maximally the number of exceptional cases we consider also « Lie algebras » $o(0)$ and $o(1)$ when we define $M_{ij} = 0$. In the first case the formulæ (3) define the realization $\tau(R) = i\alpha 1$ of $o(1, 1)$ in $W_{0,1}$, while in the second case the realization of $o(2, 1)$ in $W_{2,1}$.

i) the following formulæ define the skew-hermitean Schur-realization of $o(m, n)$ in $W_{2(m+n-2+N),M}$:

$$\tau(P_i) = p_i,$$
$$\tau(L_{ij}) = q_i p_j - q_j p_i + M_{ij},$$
$$\tau(R) = -(qp) - \left[\frac{1}{2}(m + n - 2) - i\alpha\right]\mathbf{1} \quad , \quad \alpha \in \mathbf{R}, \tag{3}$$
$$\tau(Q_i) = -q^2.p_i - 2q_i\tau(R) - 2q^j M_{ji} \quad , \quad i, j = 1, 2, \ldots, m + n - 2,$$

where
$$(qp) = q^i p_i \quad , \quad q^2 = q^i q_i \quad , \quad p_i = g_{ij}p^j \quad , \quad q^i = g^{ij}q_j,$$

ii) two realizations (3) with different values of the parameter α are not related,

iii) two realizations (3) differing only in realization of $o(m - 1, n - 1)$ are related if and only if the realizations of $o(m - 1, n - 1)$ are related.

For the proof of this theorem we shall use the following two easily provable assertions:

(*a*) If an element $a \in W_{2N,M}$ commutes with canonical variable p^i (or q_i), then a does not depend on q_i (or p^i).

(*b*) If for $a \in W_{2N,M}$ and $(qp)_{N'} \equiv q_1 p^1 + \ldots + q_{N'}p^{N'}$, $N' \leqslant N$, the commutation relation

$$[(qp)_{N'}, a] = t \cdot a \quad , \quad t = 0, \pm 1, \pm 2, \ldots$$

is valid, then

$$a = \sum_{\substack{r,s \\ r-s=t}} a_{r,s} \cdot q^r(p)^s$$

$$(a_{r,s}q^r(p)^s \equiv a_{r_1 \ldots r_{N'}s_1 \ldots s_{N'}} \cdot q_1^{r_1} \ldots q_{N'}^{r_{N'}} \cdot (p^1)^{s_1} \ldots (p^{N'})^{s_{N'}},$$
$$r - s \equiv r_1 + r_2 + \ldots + r_{N'} - s_1 - \ldots - s_{N'})$$

and $a_{r,s}$ does not depend on $q_1, \ldots, q_{N'}, p^1, \ldots, p^{N'}$.

Proof. — One can by direct commutation verify that the expressions (3) form a realization τ of the basis of $o(m, n)$. With respect to involution we use (see note 1), all the expressions (3) are skew-hermitean and therefore they generate (through real linear combinations) the skew-hermitean realization τ of $o(m, n)$. We shall prove that τ is Schur-realization.

Let us take some arbitrary centre element z of $Uo(m, n)$. Its realization $\tau(z)$ is a polynomial

$$\tau(z) = \sum_{r,s} \beta_{r,s}(M_{ij})q^r(p)^s$$

$(r \equiv (r_1, \ldots, r_{m+n-2}), s = (s_1, \ldots, s_{m+n-2})$ which commutes with all the expressions (3). The coefficients $\beta_{r,s}(M_{ij})$ depend polynomially on the basis

elements M_{ij} of the realization of $o(m - 1, n - 1)$. Due to the commutations relations $[\tau(z), \tau(P_i)] = 0$, i. e. $[\tau(z), p_i] = 0$, in accord with the assertion (a), $\tau(z)$ does not depend on $q_1, \ldots, q_{m+n-2}$, i. e. $\tau(z) = \sum_s \beta_{0,s}(M_{ij})(p)^s$. As $\tau(z)$ commutes also with $\tau(R) = -(qp) + \text{const}$, the assertion (b) can be applied, which gives

$$\tau(z) = \beta_{0,0}(M_{ij}),$$

We shall use once more the fact that $\tau(z)$ realizes a centre element of $Uo(m, n)$. It implies that $\tau(z)$ commutes with all $\tau(L_{ij}) = q_i p_j - q_j p_i + M_{ij}$, and consequently

$$[\tau(z), M_{ij}] = 0.$$

Due to the assumption of the theorem the realization of $o(m - 1, n - 1)$ is Schur-realization, and therefore each polynomial in its basis elements commuting with all of them equals some multiple of the identity element, $\beta_{0,0}(M_{ij}) = \beta . \mathbf{1}$, $\beta \in \mathbb{C}$, what proves the first statement of the theorem.

$ii)$ and $iii)$ Let us consider two realizations τ and τ' of the type (3) and assume the existence of an endomorphism ϑ :

$$W_{2(m+n-2+N),M} \to W_{2(m+n-2+N),M}$$

such that $\vartheta(\mathbf{1}) = \mathbf{1}$ and $\vartheta \circ \tau = \tau'$ (i. e. τ and τ' are related). We shall show that $\alpha = \alpha'$ and that the corresponding realizations of $o(m - 1, n - 1)$ are also related. The equations $\vartheta \circ \tau(P_i) = \tau'(P_i)$ and $\vartheta \circ \tau(R) = \tau'(R)$ give immediatelly

$$\vartheta(p_i) = p_i \quad , \quad i = 1, 2, \ldots, m + n - 2 \tag{4}$$

and

$$\vartheta(qp) = (qp) + i(\alpha - \alpha')\mathbf{1}. \tag{5}$$

The element $\vartheta(q_i) \in W_{2(m+n-2+N),M}$ can be written in the form

$$\vartheta(q_i) = \sum_{r,s} \beta_{i,rs} q^r(p)^s$$

where $\beta_{i,rs} \equiv \beta_{i,r_1 \ldots r_{m+n-2}, s_1 \ldots s_{m+n-2}} \in W_{2N,M}$. The polynomial

$$\vartheta(qp) - (qp) = (\vartheta(q_i) - q_i) . p^i$$

is either zero or its lowest degree in the « variables » $p_1, \ldots, p_{m+n-2}$ is greater or equal to one. As this polynomial equals to $i(\alpha - \alpha')$, the second possibility is excluded and the first one implies $\alpha = \alpha'$, which proves the assertion ii.

For $m + n = 2, 3$ the theorem is proved completely because only trivial realization of the « Lie algebras » $o(0)$ and $o(1)$ exists; we shall assume therefore further $m + n > 3$.

The polynomial $\vartheta(q_i) - q_i$ commutes with all $p_j \equiv \vartheta(p_j)$ and therefore due to the assertion (a) does not depend on q_i, $i = 1, 2, \ldots, m + n - 2$, i. e.

$$\vartheta(q_i) = q_i + \sum_s \beta_{i,os}(p)^s$$

As ϑ is an endomorphism of $W_{2(m+n-2+N),M}$ and $\vartheta(qp) = (qp)$ (see eq. (5), $\alpha = \alpha'$), we know the commutation relations between $\vartheta(q_i)$ and (qp),

$$[(qp), \vartheta(q_i)] = [\vartheta(qp), \vartheta(q_i)] = \vartheta([(qp), q_i]) = \vartheta(q_i),$$

so that we can apply the assertion (b) to obtain:

$$\vartheta(q_i) = q_i \quad , \quad i = 1, 2, \ldots, m + n - 2. \tag{6}$$

The images $\vartheta(q_{m+n-1})$, $\vartheta(p_{m+n-1})$, ... of the other canonical variables commute with all $\vartheta(q_i)$; $\vartheta(p_i)$, $i = 1, 2, \ldots, m + n - 2$. From the assertion (a) the independence of $\vartheta(q_{m+n-1})$, $\vartheta(p_{m+n-1})$, ... of $q_i = \vartheta(q_i)$, $p_i = \vartheta(p_i)$, $i = 1, 2, \ldots, m + n - 2$, follows. Therefore the restriction of ϑ to $W_{2N,M} \subset W_{2(m+n-2+N),M}$ is the endomorphism of $W_{2N,M}$.

The equation $\vartheta \circ \tau(L_{ij}) = \tau'(L_{ij})$ together with eqs. (4), (6) lead immediately to $\vartheta(M_{ij}) = M'_{ij}$, which finishes the proof of the first part of the statement iii).

The proof of the last statement in the opposite direction is simple: If $\vartheta : W_{2N,M} \to W_{2N,M}$ is an endomorphism of $W_{2N,M}$ then we can easily extend it to $W_{2(m+n-2+N),M}$ putting

$$\vartheta(q_i) = q_i \quad , \quad \vartheta(p_i) = p_i \quad , \quad i = 1, 2, \ldots, m + n - 2.$$

If τ and τ' are two realizations of the type (3) (with $\alpha = \alpha'$) and ϑ is the endomorphism of $W_{2N,M}$ such that $\vartheta(M_{ij}) = M'_{ij}$, the mentioned extension gives $\vartheta \circ \tau = \tau'$ and completes the proof.

LEMMA 1. — Let $\gamma_{r,\mu}$, $r = 0, 1, \ldots, R$, $\mu = 1, 2, \ldots, n$, be elements of $W_{2N,M}$ obeying the following system of equations

$$\sum_{\substack{r,s=0 \\ r+s=t}}^{R} \sum_{\mu=1}^{n} (-1)^{r+1} \cdot \gamma_{r,\mu}^{+} \gamma_{s,\mu} = 0, \tag{7}$$

$t = 0, 1, \ldots, 2R$. Then $\gamma_{r,\mu} = 0$ for $r = 0, 1, \ldots, R$ and $\mu = 1, 2, \ldots, n$.

Proof. — By induction: a) For $N = 0$ the statement concerns $M \times M$ matrices; it can be proved easily that the matrix equation

$$\sum_{\mu=1}^{n} A_{\mu}^{+} A_{\mu} = 0$$

implies $A_\mu = 0$ for $\mu = 1, 2, \ldots, n$. As the first and the last equation of the system (7) are just of this form, we conclude $\gamma_{0,\mu} = \gamma_{R,\mu} = 0$. Substituting it into the remaining equations and repeating the procedure we obtain $\gamma_{1,\mu} = \gamma_{R-1,\mu} = 0$, etc. b) Let us assume that the statement is valid for $N - 1$ and not for N. Let us further assume that an index r_0 exists ($0 \leqslant r_0 \leqslant R$) such that $\gamma_{0,\mu} = \gamma_{1,\mu} = \ldots = \gamma_{r_0-1,\mu} = 0$ for all μ and $\gamma_{r_0,\mu} \neq 0$ for some μ. Then the first $2r_0$ equations of the system (7) is fulfilled identically while the $(2r_0 + 1)$-st looks as follows

$$(-1)^{r_0+1} \sum_{\mu=1}^{n} \gamma_{r_0,\mu}^+ \gamma_{r_0,\mu} = 0. \tag{8}$$

The elements $\gamma_{r_0,\mu}$ can be written in the form

$$\gamma_{r_0,\mu} = \sum_{k=0}^{K} \sigma_{k,\mu} \cdot q_N^{K-k} \cdot p_N^k + \gamma_{r_0,\mu}' \tag{9}$$

$\sigma_{k,\mu} \in W_{2(N-1),M}$, $K = 0, 1, \ldots$ is the highest degree of the polynomials $\gamma_{r_0,\mu}$, $\mu = 1, 2, \ldots, n$, in the « variables » q_N, p_N while the highest degree of the polynomial $\gamma_{r_0,\mu}'$ considered in the same way is less than K.

As the consequence of our assumption ($\gamma_{r_0,\mu} \neq 0$ for some r_0 and μ) some of the coefficients $\sigma_{k,\mu}$ differs from zero. Substituting now $\gamma_{r_0,\mu}$ from eq. (9) into eq. (8), we obtain up to the lower order terms

$$\sum_{\mu=1}^{n} \sum_{k,l=0}^{K} \sigma_{k,\mu}^+ \cdot \sigma_{l,\mu} (-1)^k \cdot q_N^{2K-l-k} p_N^{l+k} + \ldots = 0$$

from which we have

$$\sum_{\mu=1}^{n} \sum_{\substack{k,l=0 \\ k+l=s}}^{K} \sigma_{k,\mu}^+ \cdot \sigma_{l,\mu} (-1)^k = 0 \quad, \quad s = 0, 1, \ldots, 2K.$$

However, according to the statement of the lemma for $N - 1$, this equation implies $\sigma_{k,\mu} = 0$ for all k and μ and therefore contradicts our assumption.

As it is shown in ref. [4], no skew-hermitean Schur-realization of a compact Lie algebra could exist in W_{2N}. By means of the proved lemma we shall generalize now this result to realizations in $W_{2N,M}$, $M > 1$.

THEOREM 2. — i) Any skew-hermitean Schur-realization of the compact Lie algebra G in $W_{2N,M}$ does not depend on $q_i, p_i, i = 1, 2, \ldots, N$, i. e. τ is a usual matrix representation of G in $Mat_M \simeq W_{0,M} \subset W_{2N,M}$.

ii) Two such realizations τ and τ' are related if and only if they are equivalent in the usual matrix representation sense.

For $N = 0$ the assertion is trivially right, we assume therefore $N \geqslant 1$.

Proof. — As the algebra G is compact, a basis $X_1, \ldots, X_n$ can be chosen such that $I_2 = X_1^2 + \ldots + X_n^2$ belongs to a centre of UG. The realization τ is Schur-realization, and therefore $\tau(I_2) = \beta \mathbf{1}$, $\beta \in \mathbf{R}$. As the polynomial in q_N, p_N, every $\tau(X_\mu)$ can be written in the form

$$\tau(X_\mu) = \sum_{r=0}^{R} \gamma_{r,\mu} \cdot q_N^{R-r} \cdot p_N^r + \ldots,$$

$\gamma_{r,\mu} \in W_{2(N-1),M}$, where $R = 0, 1, \ldots$, denotes the highest degree in q_N and p_N of $\tau(X_1), \ldots, \tau(X_n)$ and the dots stand for the lower order terms. Due to skew-hermiticity of $\tau(X_\mu)$ we have

$$\beta \cdot \mathbf{1} = \tau(I_2) = - \sum_{\mu=1}^{n} \tau(X_\mu)^+ \cdot \tau(X_\mu).$$

Substituting here $\tau(X_\mu)$ and $\tau(X_\mu)^+$ from the above equation we obtain

$$\tau(I_2) = \sum_{\mu=1}^{n} \sum_{r,s=0}^{R} (-1)^{r+1} \gamma_{r,\mu}^+ \gamma_{s,\mu} \cdot q_N^{2R-r-s} \cdot p_N^{r+s} + \ldots$$

The assumption that R is the highest degree in « variables » q_N, p_N in the set of all elements $\tau(X_1), \ldots, \tau(X_n)$ means that at least one $\gamma_{r,\mu}$ differs from zero. If R would be greater than zero, the last equation implies eq. (7) and lemma 1 gives immediately $\gamma_{r,\mu} = 0$ for all μ and r. The only possibility is therefore $R = 0$, which proves the statement i).

ii) As the matrix algebra Mat_M has no non-trivial two-sided ideals, every nonzero endomorphism of Mat_M is an automorphism (see *e. g.* [7], p. 48). As any automorphism ϑ of Mat_M is the inner one, the regular matrix $S_\vartheta \in \mathrm{Mat}_M$ exists so that $\vartheta(A) = S_\vartheta^{-1} A S_\vartheta$ for every $A \in \mathrm{Mat}_M$ (see [7], p. 50), so the proof has been completed.

From this theorem it follows that with our involution on $W_{2N,M}$ (see Note 1) any skew-hermitean Schur-realization of compact Lie algebra is a usual matrix skew-hermitean representation in which all Casimir operators are multiples of the identity matrix. Such representations, however, are equivalent to a direct sum of irreducible mutually equivalent representations and without essential loss of generality we can limit ourselves to the irreducible ones only.

As we pointed out in Preliminaries, every irreducible skew-hermitean representation of the compact Lie algebra $o(m)$ is uniquely, up to equivalence, determined by its signature. Now we shall generalize this concept in the way suitable for our further use:

DEFINITION 6. — Let $m \geqslant n > 0$ be pair of natural numbers and $d = 1, 2, \ldots, n$. The finite sequence of real numbers

$$\alpha_{m,n} = \left(d; \alpha_1, \ldots, \alpha_{\left[\frac{m-n}{2}\right]}, \alpha_{\left[\frac{m-n}{2}\right]+1}, \ldots, \alpha_{\left[\frac{m+n}{2}\right]} \right)$$

is called *signature* if

i) for $d < n$, $\alpha_1 = \ldots = \alpha_{\left[\frac{m+n}{2}\right]-d} = 0$

ii) for $d = n$, $\left(\alpha_1, \ldots, \alpha_{\left[\frac{m-n}{2}\right]} \right)$ is a signature of irreducible skew-hermi-

tean representation of the compact Lie algebra $o(m - n)$.

Now we are in a position to formulate and prove our main theorem.

THEOREM 3. — i) To every signature $\alpha_{m,n} = \left(d; \alpha_1, \ldots, \alpha_{\left[\frac{m+n}{2}\right]} \right)$ the rela-

tions (3) define recurrently skew-hermitean Schur-realization $\tau \equiv \tau(\alpha_{m,n})$ of the Lie algebra $o(m, n)$ in $W_{2N(d),M(d)}$. Here the number $M(d)$ is for $d = n$ and $m - n \geqslant 2$ the dimension of the irreducible skew-hermitean representation of the Lie algebra $o(m - n)$ with the signature $\left(\alpha_1, \ldots, \alpha_{\left[\frac{m-n}{2}\right]} \right)$ and $M(d) = 1$ otherwise. The number $N(d)'$ is given as $N(d) = d(m + n - d - 1)$.

ii) Two such realizations are non-related if and only if their signatures are different.

Proof. — By induction : a) Firstly we shall prove the theorem for $o(m, 1)$. For $m = 1, 2$ the assertions are contained in theorem 1. Let us assume $m \geqslant 3$ and take a signature

$$\alpha_{m,1} = \left(1; \alpha_1, \ldots, \alpha_{\left[\frac{m-1}{2}\right]}, \alpha_{\left[\frac{m+1}{2}\right]} \right)$$

The sequence $\left(\alpha_1, \ldots, \alpha_{\left[\frac{m-1}{2}\right]} \right)$ determines the irreducible skew-hermitean representation of the compact Lie algebra $o(m - 1)$ (its dimension we denote as M), *i. e.* the skew-hermitean Schur-realization of $o(m - 1)$ in $W_{0,M}$. Using the formulæ (3) with $\alpha = \alpha_{\left[\frac{m+1}{2}\right]}$ we can define the skew-hermitean Schur-realization of the Lie algebra $o(m, 1)$ in $W_{2(m-1),M}$. Using further the assertions ii) and iii) of theorem 1, ii) of theorem 2 and the part C. of Preliminaries, we have the assertion ii) of the theorem for $o(m, 1)$.

b) Assume further that the statements of the theorem are valid for the algebra $o(m - 1, n - 1)$. Let us take the signature $\alpha_{m,n} = \left(d; \alpha_1, \ldots, \alpha_{\left[\frac{m+n}{2}\right]} \right)$. For $d > 1$ we shall use the realization of $o(m - 1, n - 1)$ in $W_{2N(d-1),M(d-1)}$ corresponding to the signature $\left(d - 1; a_1, \ldots, \alpha_{\left[\frac{m+n}{2}\right]-1} \right)$ to insert it in the formulæ (3) with $\alpha = \alpha_{\left[\frac{m+n}{2}\right]}$. If $d = 1$ we shall use for the same purpose

the trivial realization of $o(m - 1, n - 1)$ in $W_{0,1}$. Due to the assertion $i)$ of theorem 1 we obtain in this way the skew-hermitean Schur-realization $\tau \equiv \tau(\alpha_{m,n})$ of $o(m, n)$ in $W_{2N,M}$, where $M \equiv M(d - 1) = M(d)$ and $N = m + n - 2 + N(d - 1) \equiv N(d)$, what proves the assertion $i)$ of the theorem. The assertions $ii)$ $iii)$ of theorem 1 together with assumed validity of the theorem for $o(m - 1, n - 1)$ imply the assertion $ii)$.

5. CONCLUSION

All considered algebras $o(m, n)$, $m + n = N$, $N = $ const are different real form of their common complexification $o_C(N)$ (*). It is not difficult to see that all results contained in theorems 1 and 3 remain valid also for $o_C(N)$, if we ignore the skew-hermitean property and its consequences. As we are not forced now to respect theorem 2, relations (3) define reccurently a usual canonical Schur-realization of $o_C(N)$ in $W_{\frac{1}{2}(N-1)^2}$ for odd N or in $W_{\frac{1}{2}N(N-2)}$ for even N depending on the $\left[\dfrac{N}{2}\right]$ free parameters. As to the real forms, the same situation arises for algebras $o(n + 2, n)$, $o(n + 1, n)$ and $o(n, n)$ while in the remaining cases the described *canonical* realizations depend at most on $\left[\dfrac{m + n}{2}\right] - \left[\dfrac{m - n}{2}\right] = n$ free parameters only. To obtain in these cases the « full » number $\left[\dfrac{m + n}{2}\right]$ of the parameters in realizations described, we have to use the signature $\alpha_{m,n} = \left(n; \alpha_1, \ldots, \alpha_{\left[\frac{m+n}{2}\right]}\right)$ and realizations are right *matrix* canonical realizations.

It is also clear, that instead of realizations of the auxiliary Lie algebra $o(m - d, n - d)$, $d < n$ that we used in reduction of formulæ (3), any other realization of $o(m - d, n - d)$ can be taken. So, the possibility of deriving further, new, realizations of Lie algebra $o(m, n)$ may arise.

The concept of matrix canonical realization, especially the realizations of the Lie algebras $o(m, n)$ described in this paper, have the direct application in the representation theory. Replacing here p_i and q_i by their Schrödinger representatives we obtain immediately a skew-symmetric representation of $o(m, n)$ with « constant » Casimir operators. It has to be stressed that these representations were obtained purely in the algebraical way. As the second advantage of this approach, one can consider the fact that the analytical properties of representations can be investigated separately as the second step.

(*) Note that in the Cartan classification of semi-simple Lie algebras $o_C(2n) \simeq D_n$ and $o_C(2n + 1) \simeq B_n$.