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On the genericity of nonvanishing instability intervals in Hills equation

by

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ABSTRACT. — We prove that for (Baire) almost every $C^\infty$ periodic function $V$ on $\mathbb{R}$, $-d^2/dx^2 + V$ has all its instability intervals non-empty.

In the spectral theory of one dimensional Schrödinger operators [3] [10] with periodic potentials, a natural question occurs involving the presence of gaps in the spectrum. Let $H = -\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R}, dx)$ where $V(x + 1) = V(x)$ for all $x$. Let $A^p$ (resp. $A^\Lambda$) be the operator $-\frac{d^2}{dx^2} + V$ on $L^2([0, 1], dx)$ with the boundary condition $f'(1) = f'(0); f(1) = f(0)$ (resp. $f'(1) = -f'(0); f(1) = -f(0)$). Let $E_n^p$ (resp. $E_n^\Lambda$) be the $n$th eigenvalue, counting multiplicity, of $A^p$ (resp. $A^\Lambda$). Finally define

$$\alpha_n = \begin{cases} E_n^p & n = 1, 3, \ldots \\ E_n^\Lambda & n = 2, 4, \ldots \end{cases}$$

$$\beta_n = \begin{cases} E_n^\Lambda & n = 1, 3, \ldots \\ E_n^p & n = 2, 4, \ldots \end{cases}$$

$$\mu_n = \alpha_{n+1} - \beta_n$$

It is a fundamental result of Lyapunov that

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_n < \beta_n \leq \alpha_{n+1} \ldots$$

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and one can show [3] [10] that \( \sigma(H) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n] \). The numbers \( \mu_n \geq 0 \) enter naturally as the size of gaps in \( \sigma(H) \). In the older literature [9], the equation \(- f'' + Vf = Ef\) is called Hill's equation and the intervals \((\beta_n, \alpha_{n+1})\) (of length \( \mu_n \)) are called instability intervals.

One has the feeling that for most \( V \)'s the gap sizes \( \mu_n(V) \) are non-zero. This is suggested in part by a variety of deep theorems that show the vanishing of many \( \mu_n \)'s places strong restrictions on \( V \): for example, \( \mu_n(V) = 0 \) for all \( n \) implies that \( V \) is constant [1] [5]; \( \mu_n(V) = 0 \) for all odd \( n \) implies that

\[
V(x + \frac{1}{2}) = V(x) [1] [6]; \quad \text{and } \mu_n(V) = 0 \text{ for all but N values of } n \text{ forces } V
\]
to lie on a 2N-dimensional manifold [5] [4]. On the other hand, some argument is necessary to construct an explicit example of a \( V \) with each \( \mu_n(V) \neq 0 \) [7].

The situation is somewhat reminiscent of that concerning nowhere differential functions in \( C[0, 1] \). One's intuition is that somehow most functions in \( C[0, 1] \) are nowhere differentiable but some argument is needed to construct an explicit nowhere differentiable function. One's intuition in this case is established by a result that also settles the existence question: a dense \( G_\delta \) (« Baire almost every ») in \( C[0, 1] \) consists of nowhere differentiable functions [2].

In this note we wish to prove a similar result that asserts that, for most \( V \), \( \mu_n(V) \neq 0 \) for all \( n \). We do not claim that that result is of the depth of the above quoted results but we feel it is of some interest especially since it will be a simple exercise in the perturbation theory of eigenvalues [8] [10] [11].

**Theorem.** — Let \( X \) denote the vector space of real valued \( C^\infty \) functions on \( \mathbb{R} \) obeying \( V(x + 1) = V(x) \). Place the Frechet topology on \( X \) given by the seminorms

\[
||f||_n = \sup_x |D^nf(x)|.
\]

Then the set of \( V \) in \( X \) with \( \mu_n(V) \neq 0 \) for all \( n \) is a dense \( G_\delta \) in \( X \).

**Proof.** — Fix \( n \). We will show that \( \{ V \mid \mu_n(V) \neq 0 \} \) is a dense open set of \( X \). Thus \( \bigcap_n \{ V \mid \mu_n(V) \neq 0 \} \) is a \( G_\delta \) which is dense by the Baire category theorem.

Suppose that \( \mu_n(V) \neq 0 \). Suppose \( n \) is even (a similar argument works if \( n \) is odd). Thus \( E_{n+1}^p(V) \neq E_n^p(V) \). Now, the change of \( E_{n+1}^p(V + \lambda W) \) as \( \lambda \) changes can be bounded [8] by \( ||W||_\text{operator} \) and the \( W \)-independent data of the distance of \( E_{n+1}^p(V) \) from \( E_n(V) \) and \( E_{n+2}(V) \). As a result, there is a constant \( \varepsilon(V) \) so that \( \mu_n(V + W) \neq 0 \) if \( ||W||_\infty \leq \varepsilon(V) \). Since \( || - ||=\infty \) is a continuous seminorm, \( \{ V \mid \mu_n(V) \neq 0 \} \) is open.

Next suppose \( \mu_n(V) = 0 \) and again suppose that \( n \) is even. Since \( E_n = E_{n+1} \),
all solutions of \(- u'' + Vu = E_n u\) are periodic. Let \(u_1\) be the solution with \(u(0) = 0, u'(0) = 1\) and \(u_2\) the solution with \(u(0) = 1, u'(0) = 0\). Since \((u_1(x))^2 \neq (u_2(x))^2\) for \(x\) near 0, we can find \(W \in X\) with
\[
\int W(x) |u_1(x)|^2 dx \neq \int W(x)(u_2(x))^2 dx.
\]
It follows [8] that for \(\lambda\) small \(E_n(V + \lambda W) \neq E_{n+1}(V + \lambda W)\) and thus that \(\mu_n(V + \lambda W) \neq 0\). We conclude that \(\{ V \mid \mu_n(V) \neq 0 \}\) is dense.

We conclude by noting that the space \(X = C^\infty\) can be replaced by any topological vector space of continuous periodic functions which is a Baire space and which obeys:

a) || - ||\(_X\) is a continuous seminorm.

b) If \(\rho_1 \neq \rho_2\) as functions in \(L^1([0, 1])\), there is \(W\) in the space with
\[
\int \rho_1(x)W(x)dx \neq \int \rho_2(x)W(x)dx.
\]
In particular, we can take the \(C^p([0, 1])\) periodic functions with the \(C^p\) topology or the periodic entire analytic functions with the compact open topology.

REFERENCES


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