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# On the genericity of nonvanishing instability intervals in Hills equation

by

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ABSTRACT. — We prove that for (Baire) almost every $C^\infty$ periodic function V on $\mathbb{R}$, $-d^2/dx^2 + V$ has *all* its instability intervals non-empty.

In the spectral theory of one dimensional Schrödinger operators [*3*] [*10*] with periodic potentials, a natural question occurs involving the presence of gaps in the spectrum. Let $H = -\dfrac{d^2}{dx^2} + V$ on $L^2(\mathbb{R}, dx)$ where $V(x + 1) = V(x)$ for all $x$. Let $A^P$ (resp. $A^A$) be the operator $-\dfrac{d^2}{dx^2} + V$ on $L^2([0, 1], dx)$ with the boundary condition $f'(1) = f'(0)$ ; $f(1) = f(0)$ (resp. $f'(1) = -f'(0)$; $f(1) = -f(0)$). Let $E_n^P$ (resp. $E_n^A$) be the $n^{\text{th}}$ eigenvalue, counting multiplicity, of $A^P$ (resp. $A^A$). Finally define

$$\alpha_n = \begin{cases} E_n^P & n = 1, 3, \dots \\ E_n^A & n = 2, 4, \dots \end{cases}$$

$$\beta_n = \begin{cases} E_n^A & n = 1, 3, \dots \\ E_n^P & n = 2, 4, \dots \end{cases}$$

$$\mu_n = \alpha_{n+1} - \beta_n$$

It is a fundamental result of Lyapunov that

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq \alpha_n < \beta_n \leq \alpha_{n+1} \dots$$

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and one can show [3] [10] that $\sigma(H) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$. The numbers $\mu_n \geq 0$ enter naturally as the size of gaps in $\sigma(H)$. In the older literature [9], the equation $-f'' + Vf = Ef$ is called Hill's equation and the intervals $(\beta_n \, \alpha_{n+1})$ (of length $\mu_n$) are called instability intervals.

One has the feeling that for most V's the gap sizes $\mu_n(V)$ are non-zero. This is suggested in part by a variety of deep theorems that show the vanishing of many $\mu_n's$ places strong restrictions on V: for example, $\mu_n(V) = 0$ all $n$ implies that V is constant [1] [5]; $\mu_n(V) = 0$ all odd $n$ implies that $V\left(x + \frac{1}{2}\right) = V(x)$ [1] [6]; and $\mu_n(V) = 0$ for all but N values of $n$ forces V to lie on a 2N-dimensional manifold [5] [4]. On the other hand, some argument is necessary to construct an explicit example of a V with each $\mu_n(V) \neq 0$ [7].

The situation is somewhat reminiscent of that concerning nowhere differential functions in C[0, 1]. One's intuition is that somehow most functions in C[0, 1] are nowhere differentiable but some argument is needed to construct an explicit nowhere differentiable function. One's intuition in this case is established by a result that also settles the existence question: a dense $G_\delta$ (« Baire almost every ») in C[0, 1] consists of nowhere differentiable functions [2].

In this note we wish to prove a similar result that asserts that, for most V, $\mu_n(V) \neq 0$ for all $n$. We do not claim that that result is of the depth of the above quoted results but we feel it is of some interest especially since it will be a simple exercise in the perturbation theory of eigenvalues [8] [10] [11].

THEOREM. — Let X denote the vector space of real valued $C^\infty$ functions on $\mathbb{R}$ obeying $V(x + 1) = V(x)$. Place the Frechet topology on X given by the seminorms

$$\| f \|_n = \sup_x | D^n f(x) | .$$

Then the set of V in X with $\mu_n(V) \neq 0$ for *all* $n$ is a dense $G_\delta$ in X.

*Proof.* — Fix $n$. We will show that $\{ V \mid \mu_n(V) \neq 0 \}$ is a dense open set of X. Thus $\bigcap_n \{ V \mid \mu_n(V) \neq 0 \}$ is a $G_\delta$ which is dense by the Baire category theorem.

Suppose that $\mu_n(V) \neq 0$. Suppose $n$ is even (a similar argument works if $n$ is odd). Thus $E^P_{n+1}(V) \neq E^P_n(V)$. Now, the change of $E^P_{n+1}(V + \lambda W)$ as $\lambda$ changes can be bounded [8] by $\| W \|_{\text{operator}}$ and the W-independent data of the distance of $E^P_{n+1}(V)$ from $E_n(V)$ and $E_{n+2}(V)$. As a result, there is a constant $\varepsilon(V)$ so that $\mu_n(V + W) \neq 0$ if $\| W \|_\infty \leq \varepsilon(V)$. Since $\| - \|_\infty$ is a continuous seminorm, $\{ V \mid \mu_n(V) \neq 0 \}$ is open.

Next suppose $\mu_n(V) = 0$ and again suppose that $n$ is even. Since $E_n = E_{n+1}$, all solutions of $-u'' + Vu = E_n u$ are periodic. Let $u_1$ be the solution with $u(0) = 0$, $u'(0) = 1$ and $u_2$ the solution with $u(0) = 1$, $u'(0) = 0$. Since $(u_1(x))^2 \neq (u_2(x))^2$ for $x$ near 0, we can find $W \in X$ with

$$\int W(x) \mid u_1(x) \mid^2 dx \neq \int W(x)(u_2(x))^2 dx .$$

It follows [8] that for $\lambda$ small $E_n(V + \lambda W) \neq E_{n+1}(V + \lambda W)$ and thus that $\mu_n(V + \lambda W) \neq 0$. We conclude that $\{ V \mid \mu_n(V) \neq 0 \}$ is dense. ■

We conclude by noting that the space $X = C^\infty$ can be replaced by any topological vector space of continuous periodic functions which is a Baire space and which obeys:

*a*) $\| - \|_\infty$ is a continuous seminorm.

(*b*) If $\rho_1 \neq \rho_2$ as functions in $L^1([0, 1])$, there is W in the space with $\int \rho_1(x)W(x)dx \neq \int \rho_2(x)W(x)dx.$

In particular, we can take the $C^p([0, 1])$ periodic functions with the $C^p$ topology or the periodic entire analytic functions with the compact open topology.

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