T. H. Date

On relativistic magnetofluids


<http://www.numdam.org/item?id=AIHPA_1976__24_4_417_0>
On relativistic magnetofuids

by

T. H. DATE
Department of Mathematics, Shivaji University,
Kolhapur-416004 (M. S.). India

ABSTRACT. — The stress-energy-momentum tensor for thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability is constructed. Several consequences of the relativistic magnetohydrodynamic field equations are derived. « Maxwell-like » equations for the gravitational field in a magnetofluid are obtained and the expressions for the refractive index and the ray shear of a null gravitational field are computed.

INTRODUCTION

The classical magnetohydrodynamics has been applied with considerable success to the astronomical systems like magnetic variable stars, sun-spots, and spiral arms [1]-[3]. However, on astronomical scale the gravitational attractions far exceed electromagnetic attractions and repulsions. How to formulate a theory incorporating the intense gravitational fields which are inevitably present in the astronomical system? Precisely to meet this demand the theory of relativistic magnetohydrodynamics (RMHD) has come into existence.

The genesis of RMHD is in Minkowski’s electrodynamics of moving bodies. The significant contributions to RMHD are due to Coburn [4], Taub [5] and Greenberg [6]. On giving an elegant account of the RMHD field equations, Lichnerowicz [7] has established their existence and uniqueness of solutions. His RMHD field equations are used by Yodzis [8] to infer the magnetic effect in galactic cosmogony, gravitational collapse, and
pulsar theory and by Date [9] to study the local behaviour of congruences in self-gravitating magnetofluids. The solutions of Lichnerowicz's field equations are found by Date [10]-[11] and interpreted as a class of non-uniform cosmological models filled with irrotational and shearfree thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability. Bray [12] has obtained an exact solution of Lichnerowicz's RMHD field equations by assuming axial symmetry. Moreover, he [13]-[14] has obtained Godel type of universes filled with magnetofluid. Shaha [15] has extended definite material schemes to magnetohydrodynamics.

In this article, we propose to study the consequences of the RMHD field equations by modifying the stress-energy-momentum tensor in Ref. [7] for thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability. The field equations (Einstein equations, Maxwell equations, equations connecting thermodynamical variables) are used to study some consequences. Analogous to the derivation of Maxwell equations for propagation of electromagnetic field in matter, « Maxwell-like » equations for gravitational field in magnetofluid are derived by using the theory developed in Ref. [16]-[21]. The behaviour of gravitational radiation in the universe filled with magnetofluid is investigated and the propagation of a null gravitational field is studied.

The purpose of this article is two fold (i) it is of course of pure theoretical interest to extend the theory of relativistic hydrodynamics to that of RMHD (ii) it is observed that some astronomical objects like neutron stars possess very strong magnetic field of the order $10^{12} - 10^{13}$ G and very high electrical conductivity [22]. A thermally conducting, viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability befits theoretical considerations pertaining to such astronomical objects.

1. PRELIMINARIES

1.1. Geometry

Arbitrary co-ordinates $x^a$ are used in a four-dimensional Riemannian manifold $V_4$. The metric is

$$ds^2 = g_{ab}dx^adx^b, \quad (a, b = 1, 2, 3, 4)$$

where $g_{ab}$ are the gravitational potentials. Signature of the metric is $(-, -, -, +)$. The time-like curves are

$$x^a = x^a(a^i, s), \quad (i = 1, 2, 3)$$

where $a^i$ are Lagrangian co-ordinates of fluid element and $s$ is a parameter.
along the world line. The unit 4-velocity vector tangential to the world-line is

\[ u^i = \frac{dx^i}{ds} \quad (a^i \text{ fixed}), \]

with \( u^i u_i = +1 \).

The space-metric or projection operator is

\[ p_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha} u_{\beta}. \]

The covariant derivative of \( u^\alpha \) is decomposed as

\[ u_{;\beta} = \theta p_{\alpha\beta} + \dot{u}_{\beta} u_{\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}, \]

where the scalar \( \theta = 1/3 u^\alpha u_{;\alpha} \) is the expansion, \( \dot{u}^\alpha = u^\alpha_{;\beta} u^\beta \) is the acceleration vector, \( \sigma_{\alpha\beta} = u_{(\alpha;\beta)} - \dot{u}_{(\alpha} u_{\beta)} - \theta p_{\alpha\beta} \) is the shear tensor and \( \omega_{\alpha\beta} = u_{(\alpha;\beta)} - \dot{u}_{(\alpha} u_{\beta)} \) is the rotation tensor. The vorticity vector \( \omega^\alpha \) is defined as

\[ \omega^\alpha = \frac{1}{2} \eta^{\alpha\beta\epsilon} u_{\beta} u_{;\epsilon}, \]

where \( \eta^{\alpha\beta\epsilon} \) is the Levi-Civita permutation tensor. The magnitudes of \( \sigma_{\alpha\beta} \) and \( \omega_{\alpha\beta} \) are given by

\[ \sigma^2 = \frac{1}{2} \sigma^\alpha_\beta \sigma^\beta_\alpha, \quad \omega^2 = \frac{1}{2} \omega^\alpha_\beta \omega^\beta_\alpha. \]

Here semicolon indicates covariant differentiation, round brackets around suffixes denote symmetrization and square brackets around suffixes denote anti-symmetrization. Units are such that \( k \), the gravitational constant and \( c \), the velocity of light are 1.

The equation of a space-like curve is

\[ x^\alpha = x^\alpha(\eta^i, \zeta), \quad (i = 1, 2, 3) \]

where \( \eta^i \) takes constant values for a particular curve and \( \zeta \) is the parameter along the space-like curve. The unit vector tangent to the space-like curve is given by

\[ n^\alpha = dx^\alpha/d\zeta, \quad (\eta^i \text{ fixed}) \]

with \( n^\alpha n_\alpha = -1 \).

The expansion parameter \( \Theta \) associated with space-like curve is defined as

\[ \Theta = \frac{1}{2} (n^\alpha_{;\alpha} - n_{\alpha\beta} u^\alpha u^\beta). \]

1.2. The stress-energy-momentum tensor of the fluid

A stress-energy-momentum tensor for a thermally conducting, viscous, compressible fluid has a general form [21]

\[ T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta} + \nu \sigma_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha, \]

Vol. XXIV, n° 4-1976.
where $\rho$ is the matter energy density of the fluid, $\rho$ is the isotropic pressure of the fluid, $\nu (\geq 0)$ is the coefficient of shear viscosity and $q^2$ is the heat flux vector.

The matter energy density $\rho$ is connected with the proper matter density $\rho_o$ and the internal energy density $\varepsilon$ by [7]:

$$\rho = \rho_o(1 + \varepsilon).$$

The equations connecting thermodynamical variables are

$$TdS = d\varepsilon + \rho d(1/\rho_o),$$

$$S^a = \rho_o S u^a + q^a/T,$$

$$q^a = K(T_{\beta} - T_{\beta}^\alpha) p^\alpha,$$

where $T$ is the rest temperature, $S$ is the specific entropy, $S^a$ is the entropy flux vector and $K$ is the heat conduction coefficient.

### 1.3. Stress-energy-momentum tensor of the magnetofluid

An asymmetric stress-energy-momentum tensor of electromagnetic field was given by Minkowski [7]:

$$T_{\alpha\beta}^{(em)} = \frac{1}{4} g_{\alpha\beta} \mathcal{H}^{\gamma\delta} - \mathcal{H}^{\alpha\gamma} \mathcal{H}_\beta^{\gamma},$$

where $\mathcal{H}_{\alpha\beta}$ is the skew symmetric electric field-magnetic induction tensor and $\mathcal{G}_{\alpha\beta}$ is the skew symmetric magnetic field-electric induction tensor. The tensor $T_{\alpha\beta}^{(em)}$ in terms of the electric field vector $e^a$, the magnetic field vector $h^a$, the electric induction vector $d^a$, the magnetic induction vector $b^a$ and the vectors $v^a, w^a$ corresponding to energy flux and momentum flux of the electromagnetic field respectively, is of the form

$$T_{\alpha\beta}^{(em)} = \left( \frac{1}{2} g_{\alpha\beta} - u^\alpha u_\beta \right) (e^\gamma d^\gamma + h^\gamma b^\gamma) - (e^\gamma d_\beta + h^\gamma b_\beta) - (u^\gamma v_\beta + u^\gamma w_\beta).$$

Under the assumptions of infinite electrical conductivity and constant magnetic permeability $T_{\alpha\beta}^{(em)}$ reduces to [7].

$$M_{\alpha\beta} = \mu \left[ \left( \frac{1}{2} g_{\alpha\beta} - u^\alpha u_\beta \right) |\vec{h}|^2 - h_\beta h^\beta \right],$$

where $\mu$ is the constant magnetic permeability,

$$|\vec{h}|^2 = -h^2 h_x \quad \text{and} \quad u^2 h_x = 0.$$

The total stress-energy-momentum tensor for thermally conducting,
viscous, compressible fluid with infinite electrical conductivity and constant magnetic permeability is the sum of $T_{\mu\nu}$ and $M_{\mu\nu}$:

\begin{equation}
(1.17) \quad T_{\mu\nu} = \left(\rho + p + \mu |\vec{H}|^2\right)u_\mu u_\nu - \left(p + \frac{1}{2\gamma} \mu |\vec{H}|^2\right)g_{\mu\nu} + \nu \sigma_{\mu\nu} + q_x u_\nu + q_\mu u_\mu - \mu h_\mu h_\nu.
\end{equation}

For thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability ($\nu = q_x^2 = 0$) we get Lichnerowicz’s [7] tensor

\begin{equation}
(1.18) \quad T_{\mu\nu}^{(L)} = \left(\rho + p + \mu |\vec{H}|^2\right)u_\mu u_\nu - \left(p + \frac{1}{2\gamma} \mu |\vec{H}|^2\right)g_{\mu\nu} - \mu h_\mu h_\nu.
\end{equation}

1.4. The field equations

The field equations of relativistic magnetohydrodynamics are the Einstein equations

\begin{equation}
(1.19) \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -T_{\mu\nu}.
\end{equation}

where $T_{\mu\nu}$ is given by (1.17) and the Maxwell equations

\begin{equation}
(1.20) \quad (u^\mu h_\nu - u^\nu h_\mu)_{;\nu} = 0.
\end{equation}

Remark. — Taub [5] has derived the field equations for self-gravitating charged fluid with constant electric permittivity and constant magnetic permeability by using variational principle. His equations reduce to (1.19) and (1.20) under the assumption of infinite electrical conductivity.

1.5. Free gravitational field

The Riemann curvature tensor $R_{\mu\nu\rho\delta}$ can be algebraically separated into the Ricci tensor $R_{\mu\nu}$ and the Weyl tensor $C_{\mu\nu\rho\delta}$ as

\begin{equation}
(1.21) \quad R_{\mu\nu\rho\delta} = C_{\mu\nu\rho\delta} - g_{\mu[\rho} R_{\nu]\delta} - g_{\rho[\nu} R_{\mu]\delta} - 1/3 R g_{\mu[\rho} g_{\nu]\delta}.
\end{equation}

The Weyl tensor represents the free gravitational field. It can be decomposed into « electric » and « magnetic » components [1/6]:

\begin{equation}
(1.22) \quad E_{\mu\nu} = C_{\mu\nu},
\end{equation}

\begin{equation}
(1.23) \quad H_{\mu\nu} = \frac{1}{2} C_{\mu}^{\nu\delta\eta} \epsilon_{\nu\delta\rho\sigma} u_\rho u_\sigma.
\end{equation}

Vol. XXIV, n° 4 - 1976.
satisfying the properties
\[ E_{\alpha\beta} = E_{\alpha(\beta)}, \quad H_{\alpha\beta} = H_{\alpha(\beta)}, \]
\[ E_\gamma = H_\gamma = E_{\alpha\beta}u^\alpha = H_{\alpha\beta}u^\alpha = 0. \]
The matter current \( J_{\alpha\beta\gamma} \) is defined as \([19]\).
\[
(1.24) \quad J_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}^\delta. \]

2. THE MAXWELL EQUATIONS

We now study some consequences of the Maxwell equations \((1.20)\). It is shown by Khade \([23]\) that the equations \((1.20)\) can be identified as material transport laws. Magnetic field vector \( h^\alpha \) being space-like, it can be written as
\[
(2.1) \quad h^\alpha = |\vec{h}|n^\alpha, \quad (n^\alpha n_\alpha = -1) \]
where \( n^\alpha \) is a unit space-like vector.

**Theorem 2.1.** — Congruences of magnetic lines are expansion-free if and only if the magnitude of the magnetic field conserves along the lines of force.

From Maxwell equations \((1.20)\), we get
\[
(2.2) \quad u^\alpha h^\beta + u^\beta h^\alpha,\beta - u^\beta h^\alpha_{\;\alpha} - u^\alpha h^\alpha_{\;\alpha} = 0.
\]
Contracting \((2.2)\) with \( u_\alpha \), we have
\[
(2.3) \quad h^\alpha_{\;\alpha} - h_{\alpha\beta}u^\alpha u^\beta = 0.
\]
On substituting \((2.1)\) and using \((1.8)\), equation \((2.3)\) reduces to
\[
(2.4) \quad \frac{\frac{\partial}{\partial x} + \frac{1}{2}(\log |\vec{h}|)_{,\alpha}}{n^\alpha = 0. \quad (\log |\vec{h}|)_{,\alpha}n^\alpha = 0.
\]
Therefore, we must have
\[
(2.5) \quad \frac{\frac{\partial}{\partial x}}{= 0 \Leftrightarrow (|\vec{h}|)_{,\alpha}n^\alpha = 0. \quad Thus, the proof of the theorem is complete.

**Theorem 2.2.** — For a Born-rigid flow of the magnetofluid, the magnitude of the magnetic field is conserved along the world-line.

On using the kinematical parameters associated with time-like congruences, Maxwell equation \((2.2)\) can be written in the form
\[
(2.6) \quad (\sigma_{\alpha\beta} + \omega_{\alpha\beta})h^\beta + h^\alpha_{;\beta}u_\alpha - 2\theta h_\alpha - h_{\alpha\beta}u^\beta = 0.
\]
Transvecting with \( h^\alpha \), \((2.6)\) produces
\[
(2.7) \quad \sigma_{\alpha\beta}h^\alpha h^\beta + 2\theta |\vec{h}|^2 + \frac{1}{2}(|\vec{h}|^2)_{,\beta}u^\beta.
\]

Annales de l'Institut Henri Poincaré - Section A
When the flow is Born-rigid ($\sigma_{a\beta} = \theta = 0$), we get

\[(2.8) \quad (|\vec{h}|)_\beta u^\beta = 0.\]

Thus, the magnitude of the magnetic field vector is conserved along the world line when $\theta = \sigma_{a\beta} = 0$.

**Theorem 2.3.** — The magnetic field in the magnetofluid is divergence-free if and only if the acceleration and the magnetic field vector are orthogonal to each other.

Since $u^x$ and $h^x$ are orthogonal, equation (2.3) yields

\[(2.9) \quad h^x, x + h^x u_x = 0.\]

Hence, we have

\[(2.10) \quad h^x, x = 0 \Leftrightarrow h^x u_x = 0,\]

and the proof is complete.

### 3. Heat Transfer Equation

From the equation of conservation $T^\alpha, _\beta = 0$, we shall develop some differential identities which govern the behaviour of the magnetofluid. For magnetofluid characterized by (1.17), equation of conservation is

\[(3.1) \quad (\rho + p + \mu |\vec{h}|^2) u^x + (\rho + p + \mu |\vec{h}|^2)(\dot{u}^x + 3\theta u^x) \]

\[= \left( p + \frac{1}{2} \mu |\vec{h}|^2 \right) g^{x\beta} + v \sigma^{x\beta} + 4\theta q^x + q_x u^x + \dot{q}^x \]

\[+ (\omega^{x\beta} + \sigma^{x\beta}) q_y - \mu (|\vec{h}|^2) n^\beta n^\beta n^x - \mu |\vec{h}|^2 (\hat{n}^x + 3\phi n^x) = 0,\]

where $\hat{n}^x = n^x, \mu u^\mu$ and $\phi = 1/3n^\beta n^\beta$.

Contracting (3.1) by $u_x$ and $p_{\beta x}$ and using Maxwell equations (1.20), we get

\[(3.2) \quad \dot{\rho} + 30(\rho + p) + 2v\sigma^2 + q_x, x + q_x u_x = 0,\]

\[(3.3) \quad (\rho + p + \mu |\vec{h}|^2) u_x - \left( p + \frac{1}{2} \mu |\vec{h}|^2 \right) p_x + \frac{1}{2} (|\vec{h}|^2) n^\beta n^\beta n_x \]

\[+ v(\sigma^{\beta x} + 2\sigma^{x\beta} u_x) + 4\theta q_x + q^x p_x + (\omega^{\beta x} + \sigma^{\beta x}) q^x = 0.\]

It is interesting to observe that the same equation (3.2) holds true even for thermally conducting, viscous, compressible fluid [21]. While in equation (3.3) magnetic field is explicitly present. On substituting equations (1.10) and (1.11) in equation (3.2) we get the heat transfer equation:

\[(3.4) \quad (\rho_0 u^x) , x + T \rho_0 S_x, x u^x = 2v\sigma^2 + \dot{u}^x q_x - q_x, x.\]
If we assume that the matter density is conserved i.e. \((\rho_0 u^2)_{,\alpha} = 0\), and use (1.12) the heat transfer equation reduces to

\[
S_{,\alpha} = 1/T(2\nu\sigma^2 + 1/K T |\bar{q}|^2),
\]

where \(|\bar{q}|^2 = -q^2q_\alpha\).

Thus, for thermally conducting, viscous, compressible magnetofluid the entropy generation is always positive.

**Remark (1).** — It is to be noted that the magnetic field is not explicitly present in the entropy generation equation. Moreover, the entropy generation is not only due to heat flux but also due to viscosity.

**Remark (2).** — It is shown by Date [24] that for variable magnetic permeability, magnetic field is explicitly present in the heat transfer equation.

**Remark (3).** — For thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability, Lichnerowicz's [7] results can be recovered.

### 4. GRAVITATIONAL FIELD IN PERFECT MAGNETOFUID

For perfect magnetofluid \((v = q^2 = 0)\) equations (3.2) and (3.3) reduce to

\[
\frac{\rho}{\rho} + 3a(p + p) = 0,
\]

\[
(\rho + p + \mu |h|^2)u_x - \perp p \left( p + \frac{1}{2} \mu |h|^2 \right) - \mu |h|^2 n_{\alpha\beta} u^\theta = 0,
\]

where \(\perp\) indicates projection by \(\rho_{\alpha\beta}\) orthogonal to \(u_x\). Only kinematical parameters associated with time-like congruences appear in equations (4.1) and (4.2) are expansion and acceleration. In Szekeres' [19] sense, these equations represent the "inert part" of the gravitational field in the perfect magnetofluid. The "active part" of the gravitational field in the perfect magnetofluid can be found by observing the propagation of the free gravitational field. This part occurs in the matter current \(J_{\alpha\beta\gamma}\) given by

\[
J_{\alpha\beta\gamma} = \rho_{\alpha\beta\gamma} u_{\gamma} + \frac{1}{3} \perp \rho_{\alpha\beta\gamma} \gamma + (\rho + p + \mu |h|^2) \times (\omega_{\alpha\beta\gamma} + \sigma_{\gamma\alpha\beta} + \omega_{\gamma\alpha\beta}),
\]

\[
+ \frac{1}{2} \mu |h|^2 g_{\alpha\beta} u_{\gamma} + \mu |h|^2 g_{\alpha\beta} u_{\gamma} + \frac{1}{2} \mu |h|^2 g_{\alpha\beta} u_{\gamma}
\]

\[
+ \frac{1}{2} \mu \left( h_{\alpha\beta} u_\gamma - h_{\alpha\gamma} u_\beta \right) h_{\alpha\beta} + \mu h_{\alpha\beta} h_{\alpha\beta} u_{\gamma} + \mu h_{\alpha\beta} h_{\alpha\beta} + \mu h_{\alpha\beta} h_{\alpha\beta}.
\]

Using the decompositions in (1.21)-(1.23), the field equations analo-
ON RELATIVISTIC MAGNETOFLUIDS

425

gous to Maxwell equations for the free gravitational field in perfect magneto-
fluid are as follows:

(4.4) \[ 3H_{\beta}\omega^\beta + p_\beta^\gamma p_\gamma p^\delta E_{\beta\gamma} - \eta_{\alpha\beta}\delta E_{\beta\gamma} \]
\[ = - \frac{1}{3} \rho_{\beta\gamma} p^\beta + \frac{1}{2} \mu_{h_{\beta\gamma}} h_{\gamma} - \frac{1}{2} \mu_{h_{\beta\gamma}} h_{\gamma} u^\gamma u^\gamma \cdot \]

(4.5) \[ \dot{H}_{\beta\gamma} + p_\beta^\gamma (\eta_{\beta\gamma} + \delta \epsilon_{\beta\gamma} u^\gamma) = 0 \]
\[ - H_{\beta\gamma} + H_{\gamma\beta} - \eta_{\alpha\beta\gamma} + \eta_{\beta\alpha\gamma} u^\alpha u^\beta u^\gamma = - \mu (h^\gamma h^\delta) u^\gamma \eta_{\beta\gamma} \cdot \]

(4.6) \[ p_\beta^\gamma H_{\beta\gamma} - 3E_{\beta}^\gamma - \eta_{\beta\gamma} \delta E_{\beta}^\gamma \]
\[ = - (\rho + p + \mu |\vec{h}|^2) \omega_{\beta} + \mu_{h_{\beta\gamma} h_{\gamma\delta} \delta^\gamma u^\delta u^\gamma \cdot \]

(4.7) \[ \dot{\omega}_\beta + \dot{H}_{\beta\gamma} + p_\beta^\gamma (\eta_{\beta\gamma} + \delta \epsilon_{\beta\gamma} u^\gamma) = 0 \]
\[ - \eta_{\beta\gamma} + \eta_{\beta\alpha\gamma} u^\alpha u^\beta u^\gamma \cdot \]
\[ - \frac{1}{4} \mu |\vec{h}|^2 \eta_{\beta\gamma} - \frac{1}{2} \mu_{h_{\beta\gamma} h_{\beta\gamma} \delta^\gamma u^\gamma u^\gamma \cdot \}
\[ - \frac{1}{2} \mu (h_{\beta\gamma} h_{\beta\gamma} + \delta^\gamma u^\gamma u^\gamma \cdot \]

In absence of the magnetic field, equations (4.4) to (4.7) reduce to Hawking’s [18] equations for perfect fluid. Comparison with his equations shows that the magnetic field on the right of the equations (4.4) to (4.7) produces disturbance in the gravitational radiation. If the undisturbed state is conformally flat (C_{\alpha\beta\gamma} = 0) then the equations (4.4) to (4.7) become

(4.8) \[ \frac{1}{3} \rho_{\beta\gamma} p^\beta + \frac{1}{2} \mu_{h_{\beta\gamma}} u^\gamma - \frac{1}{2} \mu_{h_{\beta\gamma}} u^\gamma h_{\gamma} = 0 \cdot \]

(4.9) \[ (h^\gamma h^\delta) u^\gamma \eta_{\beta\gamma} = 0 \cdot \]

(4.10) \[ (\rho + p + \mu |\vec{h}|^2) \omega_{\beta} - \mu_{h_{\beta\gamma} h_{\beta\gamma} \delta^\gamma u^\gamma u^\gamma = 0 \cdot \]

(4.11) \[ \frac{1}{2} (\rho + p + \mu |\vec{h}|^2) \omega_{\beta} + \frac{1}{4} \mu (|\vec{h}|^2)^2 \omega_{\beta} + \frac{1}{6} \mu |\vec{h}|^2 \delta^\gamma u^\gamma \cdot \]
\[ + \frac{1}{2} \mu_{h_{\beta\gamma} h_{\beta\gamma} \delta^\gamma u^\gamma u^\gamma \cdot \}

For uniform magnetic field (h_{\beta\gamma} = 0), these equations produce

(4.12) \[ \omega_{\beta} = \sigma_{\beta} = 0 \cdot \]

(4.13) \[ \rho_{\beta} p^\beta = \sigma_{\beta} p^\beta = 0 \cdot \]

When the equation of state is p = p(\rho), we get

(4.14) \[ \rho_{\beta} p^\beta = u^\beta = 0 \cdot \]

Thus, for uniform magnetic field, the universe filled with perfect magneto-
fluid is spatially homogeneous and isotropic.

Vol. XXIV, n° 4 - 1976.
5. PROPAGATION OF A NULL GRAVITATIONAL FIELD IN A PERFECT MAGNETOFLUID

Following Szekeres [19], we write the tetrad in terms of a Vierbein $(u^a, s^a, g^a, f^a)$

$$k^a = 1/\sqrt{2}(u^2 + s^2), \quad t^a = 1/\sqrt{2}(g^2 + if^2),$$

$$m^a = 1/\sqrt{2}(u^2 - s^2), \quad \bar{t}^a = 1/\sqrt{2}(g^2 - if^2).$$

In a null gravitational field, we choose $k^a$ pointing along the ray propagation. The Weyl tensor takes the form

$$C_{\alpha\beta\gamma\delta} = 2C(k_{[\alpha \beta}]k_{[\gamma \delta]} - k_{[\alpha \delta]}k_{[\gamma \beta]}),$$

where $C$ is a real constant. The ray shear $\Sigma$ and refraction $\mathcal{R}$ of the gravitational field are of the forms

$$\Sigma = k_{x;\beta}t^\beta,$$

$$\mathcal{R} = k_{x;\beta}t^\beta = \mathcal{R}_1 + i\mathcal{R}_2.$$

From equation (5.1) we have

$$k^\gamma C_{x\beta\gamma\delta} + k^\gamma C_{x\beta\gamma\delta} = 0,$$

$$k^\gamma C_{x\beta\gamma\delta} + k^\gamma C_{x\beta\gamma\delta} = 0.$$

On substituting equations (4.3), (5.1) to (5.3) in equations (5.4), (5.5) we get

$$\frac{1}{2} \rho_{\beta\gamma}k^\gamma u_{\beta} - \frac{1}{2} \rho_{\beta}k^\beta + \frac{1}{6} \rho_{\beta}u_{\beta} - \frac{1}{6} \rho_{\beta}k^\beta g_{a\beta}$$

$$+ (p + p + \mu |\bar{h}|^2)(\omega_{\beta\gamma}k^\gamma u_{\beta} + \frac{1}{2} \sigma_{\beta\gamma}u_{\beta}k^\gamma - \frac{1}{2} \sigma_{\beta\gamma} + \frac{1}{2} \omega_{\beta\gamma}u_{\beta}k^\gamma - \frac{1}{2} \omega_{\beta\gamma})$$

$$+ \frac{1}{4} \mu(\bar{h}^2)_{,\omega}u_{\beta} - \frac{1}{4} \mu(\bar{h}^2)_{,\omega}u_{\beta} + \frac{1}{2} \mu |\bar{h}|^2 \theta u_{\beta}k^\beta - \frac{1}{2} \mu |\bar{h}|^2 \theta g_{a\beta}$$

$$+ \frac{1}{2} \mu(\bar{h}^2)_{,\omega}k^\beta - \frac{1}{2} \mu(\bar{h}^2)_{,\omega}k^\beta g_{a\beta} + \frac{1}{2} \mu \{ h_{\alpha\beta} - h_{\beta\alpha}k^\beta \} h^\beta u_{\beta}$$

$$+ \frac{1}{2} \mu k_{,\beta} \{ h_{\alpha} - h_{\beta\alpha}k^\beta \} u_{\beta} + \frac{1}{2} \mu(h_{\beta\alpha}h_{\beta}k^\gamma - h_{\beta\gamma}h_{\gamma}k^\beta) + \mu h_{(\gamma\delta)}h_{\beta}k^\gamma$$

$$= C\Sigma k_{\beta} - \frac{C\mathcal{R}_1}{\sqrt{2}}k_{\beta} - \frac{C\mathcal{R}_2}{\sqrt{2}}k_{\beta} = 0.$$
In absence of the magnetic field, from equations (5.6) and (5.7) we get the following equations, as found by Kundt and Trumper [17]:

\[(\rho + p)\sigma = \sqrt{3}C\Sigma,\]

\[(\rho + p)\omega = \sqrt{2}C|\mathcal{H}|.\]

Moreover, we observe that due to magnetic field, the expressions for \(\Sigma\) and \(\mathcal{H}\) are not explicit as in equations (5.8) and (5.9) and a pure null wave does not propagate along shear-free null geodesics even if the magneto-fluid is non-rotating and non-shearing.

ACKNOWLEDGMENT

The author is grateful to Dr. L. Radhakrishna for valuable discussion and to Professor J. V. Narlikar for remarks on an earlier version of this paper.

REFERENCES


(Manuscrit reçu le 3 octobre 1975)