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by

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ABSTRACT. — It is shown that a natural generalization of Kostant's results concerning prequantization yields a characterization of quantizing A-bundles over \((A, \lambda)\)-quantizable symplectic manifolds. Furthermore, proof is given for the statement that any quantizing bundle over a Hamilton G-space can be considered as a G-quantizing bundle.

1. INTRODUCTION

Geometric quantization ([2] [5]) gives a procedure for the construction of representations of the Poisson algebra \(\mathfrak{g}(M)\). The basic step consists of a quantizing bundle over a symplectic manifold \((M, \omega)\). Generalizing the prequantization technique of Kostant, we introduced a more general definition of a quantizing bundle [7] which includes (up to association) Kostant's Hermitian line bundle and Souriau's espace fibré quantifiant.

In the following a mathematical description of these generalized quantizing bundles over \((A, \lambda)\)-quantizable symplectic manifolds is given. A theorem of Milnor [3] determines a bijection between the group \(\pi_1^A(M, m_0)\) of group homomorphisms \(\pi_1(M, m_0) \to A\) and the group \(F(A, M)\) of equivalence classes of flat principal bundles over \(M\) with abelian structure group \(A\). It then follows that the set \(Q(A, \lambda, M, \omega)\) of equivalence classes of quantizing bundles over \((A, \lambda)\)-quantizable \((M, \omega)\) is characterized by a free and transitive action

\[
Q(A, \lambda, M, \omega) \times \pi^A_1(M, m_0) \to Q(A, \lambda, M, \omega)
\]

of \(\pi_1^A(M, m_0)\) on \(Q(A, \lambda, M, \omega)\).
The infinitesimal action of the Poisson algebra $\mathfrak{g}(M)$ on $M$ can be lifted to the total space $P$ of a quantizing bundle over $(M, \omega)$. This is most useful when applied to representations by complete vector fields. Especially, Theorem 3 says that any quantizing bundle over a Hamilton $G$-space $(G/K, \omega, \Phi)$ appears (up to equivalence) as a $G$-quantizing bundle. Moreover, Theorem 3 establishes a natural one-one correspondence between $Q(A, \lambda, G/K, \omega)$ and a set of Lie group homomorphisms $K \to A$. This is used to generalize a theorem of Kostant [2] characterizing Hermitian line bundles over Hamilton $G$-spaces.

2. QUANTIZING BUNDLES

Throughout $(P, A, M)$ will denote a smooth principal bundle over a connected manifold $M$ with abelian structure group $A$. $\pi$ will denote the projection $P \to M$. Let $\alpha$ be a connection form on $P$ and let $\omega$ be a symplectic structure on $M$. Given a linear map $\lambda : R \to A$ from the real numbers into the Lie algebra of $A$, we say that

$$(P, \alpha, A, \lambda, M, \omega)$$

is an $(A, \lambda, M, \omega)$-bundle if

$$d\alpha = \lambda \pi^* \omega.$$ 

It is called a quantizing $A$-bundle (or simply quantizing bundle) if $\lambda$ is injective. In this case, we will say that $(M, \omega)$ is $(A, \lambda)$-quantizable. Otherwise, if $\lambda = 0$, then an $(A, \lambda, M, \omega)$-bundle is said to be a flat principal bundle.

Let $U = \{ U_i ; i \in I \}$ be a simple covering of $M$. If we suppose $\{ f_{ij} ; i, j \in I \}$ to be the transition functions of $(P, A, M)$ corresponding to a trivialization $\{ U_i, \phi_i ; i \in I \}$, a formal computation shows that an $(A, \lambda, M, \omega)$-bundle is characterized by a system $\{ f_{ij}, \alpha_i ; i, j \in I \}$ of $(A, \lambda, U, \omega)$-functions; that is, there exist $\alpha_{ij} \in \mathfrak{g}(U_i \cap U_j)$ with

$$f_{ij} = \exp \alpha_{ij}, \quad d\alpha_{ij} = \alpha_j - \alpha_i, \quad d\alpha_i = \lambda \omega.$$ 

Here $\exp$ denotes the exponential map $a \to A$.

Furthermore, two $(A, \lambda, M, \omega)$-bundles $(P, \alpha, A, \lambda, M, \omega)$ and $(P', \alpha', A, \lambda, M, \omega)$ are equivalent iff the associated systems $\{ f_{ij}, \alpha_i ; i, j \in I \}$ and $\{ f'_{ij}, \alpha'_i ; i, j \in I \}$ of $(A, \lambda, U, \omega)$-functions are equivalent, i.e. iff there are $\beta_i \in \mathfrak{g}(U_i)$ such that

$$f_{ij} = \exp - \beta_i f_{ij} \exp \beta_j, \quad \alpha'_i = \alpha_i + d\beta_i.$$ 

For the special case where $\lambda$ is injective, the proof can be found in [7]; exactly the same proof gives the corresponding result for the general case.

Denote by $P(A, \lambda, U, \omega)$ the set of equivalence classes of systems of
(A, λ, U, ω)-functions. The equivalence class of \{ f_{ij}, \alpha_i ; i, j \in I \} is denoted by \([f_{ij}, \alpha_i ; i, j \in I]\). If λ is injective we put

\[ P(A, \lambda, U, \omega) = Q(A, \lambda, U, \omega) ; \]

otherwise

\[ P(A, O, U, \omega) = F(A, U) . \]

It is not hard to conclude that we may define a map

\[ \varphi^H_\lambda : P(A, \lambda, U, \omega) \times F(A, U) \to P(A, \lambda, U, \omega) \]

by

\[ ([f_{ij}, \alpha_i ; i, j \in I], [f_{ij}', \alpha'_i ; i, j \in I]) \to [f_{ij}f_{ij}', \alpha_i + \alpha'_i ; i, j \in I] . \]

The proof of the following result is a straightforward calculation.

**Proposition 1.** —

1. \( \varphi^H_0 \) makes \( F(A, U) \) into an abelian group.
2. Suppose that \( Q(A, \lambda, U, \omega) \) is not empty. Then \( \varphi^H_0 : Q(A, \lambda, U, \omega) \times F(A, U) \to Q(A, \lambda, U, \omega) \)

is a free and transitive action of \( F(A, U) \) on \( Q(A, \lambda, U, \omega) \).

Now consider a refinement \( \mathcal{B} = \{ V_j ; j \in J \} \) of the open covering \( U = \{ U_i ; i \in I \} \). Choose a map \( \sigma : J \to I \) such that \( V_j \subset U_{\sigma_j} \) for \( j \in J \).

This defines a map

\[ r^H_\mathcal{B} : P(A, \lambda, U, \omega) \to P(A, \lambda, \mathcal{B}, \omega) \]

by the equation

\[ r^H_\mathcal{B}[f_{ij}, \alpha_i ; i, j \in I] = [f_{\alpha_k, \sigma_k} \alpha_{\sigma_k} ; k, l \in J] . \]

Let \( \tau : J \to I \) be another map with \( V_j \subset U_{\tau_j} \). Suppose \( \alpha_j \in \mathcal{S}(U_j \cap U_{\tau_j}) \) such that \( f_{ij} = \exp \alpha_{ij} \) \( d\alpha_{ij} = \alpha_j - \alpha_i \). Then the \( \beta_k = \alpha_{\sigma_k, \tau_k} \in \mathcal{S}(V_k) \), \( k \in J \), define an equivalence between \( \{ f_{\alpha_k, \sigma_k} \alpha_{\sigma_k} ; k, l \in J \} \) and \( \{ f_{\alpha_k, \tau_k} \alpha_{\tau_k} ; k, l \in J \} \).

Therefore \( r^H_\mathcal{B} \) does not depend on the choice of refinement map \( \sigma : J \to I \).

Notice that \( r^H_\mathcal{B} \) is the identity, and if \( \mathcal{B} \) is a refinement of \( \mathcal{B} \) then \( r^H_\mathcal{B} = r^H_\mathcal{B} \).

Hence \( \{ P(A, \lambda, U, \omega), r^H_\mathcal{B} \} \) forms a direct system over the directed set of open coverings of \( M \). We call the direct limit

\[ P(A, \lambda, M, \omega) . \]

The elements of \( P(A, \lambda, M, \omega) \) will be denoted by \( [P, \alpha, A, \lambda, M, \omega] \).

Since the equivalence classes of principal bundles over \( M \) with abelian structure group \( A \) are in a natural one-one correspondence with the elements of the cohomology group \( H^1(M, A) \), the above discussion gives the following theorem.

**Theorem 1.** — There is a natural one-one correspondence between the elements of \( P(A, \lambda, M, \omega) \) and the equivalence classes of \( (A, \lambda, M, \omega) \)-bundles.

If \( \lambda \) is injective then we write

\[ P(A, \lambda, M, \omega) = Q(A, \lambda, M, \omega) ; \]

otherwise

\[ P(A, \Omega, M, \omega) = F(A, M). \]

It is easy to check that the \( \phi^M_\lambda \) define a map

\[ \phi^M_\lambda : P(A, \lambda, M, \omega) \times F(A, M) \to P(A, \lambda, M, \omega) \]

in a natural way. By Proposition 1 we have

**Proposition 2.**

(1) \( \phi^0_\lambda \) makes \( F(A, M) \) into an abelian group.

(2) Assume that \((M, \omega)\) is \((A, \lambda)\)-quantizable. Then

\[ \phi^M_\lambda : Q(A, \lambda, M, \omega) \times F(A, M) \to Q(A, \lambda, M, \omega) \]

is a free and transitive action of \( F(A, M) \) on \( Q(A, \lambda, M, \omega) \). In other words, the group of equivalence classes of flat principal bundles over \( M \) with structure group \( A \) acts freely and transitively on the set of equivalence classes of quantizing bundles over \((A, \lambda)\)-quantizable \((M, \omega)\).

Suppose that \((P, G, M)\) is a principal bundle with structure group \( G \). Let \( \rho : G \to A \) be a group homomorphism. Then the \( \rho \)-bundle associated with \((P, G, M)\) is the principal bundle

\[ (P \times_\rho A, A, M), \]

where \( P \times_\rho A \) is the orbit space of the right \( G \)-action on \( P \times A \) given by letting \( g \in G \) take \((p, a)\) to \((pg, a\rho(g))\). The equivalence class of \((p, a)\) is denoted by \([p, a]\). Note that the structure group \( A \) acts on \( P \times_\rho A \) by \([p, a]a' = [p, a^{-1}a]\) for \( a' \in A\).

Now let \( \tilde{M} \) be the universal covering manifold of the connected manifold \( M \) and let \((\tilde{M}, \pi_1(M, m_0), M)\) stand for the principal bundle with structure group \( \pi_1(M, m_0) \) and covering projection \( \rho : \tilde{M} \to M \). Next, consider the trivial principal bundle \((\tilde{M} \times A, A, \tilde{M})\) with the canonical flat connection. Since \( \rho \) is a local diffeomorphism, the \( A \)-equivariant principal bundle homomorphism

\[ (\varphi, \rho) : (\tilde{M} \times A, A, \tilde{M}) \to (\tilde{M} \times_\rho A, A, M) \]

given by \( \varphi(\tilde{m}, a) = [\tilde{m}, a] \) induces a flat connection form \( \alpha_\rho \) on \( \tilde{M} \times_\rho A \). Here \( \rho \) is an element in the group \( \pi_1(M, m_0) \) of group homomorphisms \( \pi_1(M, m_0) \to A \).

In the case when \( A \) is abelian, the following fact can be derived from a result of Milnor [3].

**Proposition 3.** — The association

\[ \rho \to (\tilde{M} \times_\rho A, \alpha_\rho, A, M) \]

induces a group isomorphism

\[ \pi_1(A, m_0) \to F(A, M). \]
An immediate application of Propositions 2 and 3 is the generalization of theorems of Kostant ([2], p. 135 and 142) to $(A, \lambda)$-quantizable manifolds.

**Theorem 2.** Assume that $(M, \omega)$ is $(A, \lambda)$-quantizable. Then there is a canonical free and transitive action of $\pi_1^\lambda(M, m_0)$ on $Q(A, \lambda, M, \omega)$.

**Corollary.** Assume that $(M, \omega)$ is simply connected and $(A, \lambda)$-quantizable. Then $Q(A, \lambda, M, \omega)$ has exactly one element.

### 3. Lie Group Actions

Given a symplectic manifold $(M, \omega)$, let $\{\varphi, \psi\}$ be the Lie algebra structure on $\mathfrak{g}(M)$ defined by

$$\{\varphi, \psi\} = \xi_\varphi \psi = \omega(\xi_\varphi, \xi_\psi).$$

Here $\xi_\varphi$ is the Hamiltonian vector field corresponding to $\varphi \in \mathfrak{g}(M)$. For any quantizing bundle $(P, \alpha, A, \lambda, M, \omega)$ over $(M, \omega)$ the Lie algebra homomorphism

$$\varphi \in \mathfrak{g}(M) \rightarrow \xi_\varphi \in \mathfrak{g}(M)$$

can be lifted to an injective homomorphism

$$\delta : \mathfrak{g}(M) \rightarrow \mathfrak{g}(P)$$

by setting

$$(\delta \varphi)_p = (\bar{\xi}_\varphi)_p^* - (\lambda \varphi(\pi p))_p^*,$$

$p \in P ([2] [7])$. Here $\bar{\xi}_\varphi^*$ is the horizontal lift of $\xi_\varphi$ and $x^*$ is the vector field on $P$ induced by $x \in \mathfrak{a}$. The map $\delta$ is called prequantization.

We shall need the following fact.

**Lemma 1.** $\alpha$ is an invariant 1-form of $\delta \varphi$ for $\varphi \in \mathfrak{g}(M)$; that is,

$$L_{\delta \varphi} \alpha = 0.$$

**Proof.** We have

$$i(\delta \varphi)\alpha_p = - \alpha_p((\lambda \varphi(\pi p))^*) = - \lambda \varphi(\pi p),$$

i. e.

$$d(i(\delta \varphi)\alpha) = - \lambda d(\varphi \pi) = - \lambda \pi^* d \varphi.$$

On the other hand

$$i(\delta \varphi)dx = i(\delta \varphi)\lambda \pi^* \omega = \lambda \pi^* i(\xi_\varphi)\omega = \lambda \pi^* d \varphi.$$

Consequently

$$L_{\delta \varphi} \alpha = i(\delta \varphi)dx + d(i(\delta \varphi)\alpha) = 0.$$

Next suppose $G$ is a connected and simply connected Lie group. Let $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}(M)$ be a Lie algebra homomorphism from the algebra $\mathfrak{g}$ of left invariant vector fields on $G$ into the Poisson algebra $\mathfrak{g}(M)$. We assume that

$$x \in \mathfrak{g} \rightarrow \xi_{\Phi(x)} \in \mathfrak{g}(M)$$

is an infinitesimal action by complete vector fields. Then it is not hard to see that each \( (\delta \Phi)(x) \in \mathfrak{V}(\mathcal{P}) \) generates a global flow

\[
F_{(\delta \Phi)(x)} : \mathcal{P} \times \mathbb{R} \to \mathcal{P}
\]

by

\[
F_{(\delta \Phi)(x)}(p, t) = F_{\Phi(x)}(p, t) \exp - t\lambda(\Phi(x))(\pi p).
\]

Hence, in view of a result of Palais [4], we have

**Proposition 4.** In the above situation, there exists a \( G \)-action

\[
P \times G \to P,
\]

written \((p, g) \to pg\), such that

1. \( p \exp x = F_{(\delta \Phi)(x)}(p, 1) \) for \( x \in g \);
2. \( \alpha \) is \( G \)-invariant.

Here \( \exp \) means the exponential map \( g \to G \).

We come now to Hamilton \( G \)-spaces. Let \((G/K, \omega)\) be a homogeneous symplectic manifold and let

\[
\theta : G/K \times G \to G/K,
\]

written \( ([g], g') \to [g]g' \), be the natural right \( G \)-action given by \([g]g' = [g^{-1}g] \) for \([g] \in G/K, g' \in G\). The infinitesimal action \( g \to \mathfrak{V}(G/K) \) associated to \( \theta \) will be denoted by \( \theta \), too.

Given a Lie algebra homomorphism \( \Phi : \mathfrak{g} \to \mathfrak{N}(G/K) \), we call

\[
(G/K, \omega, \Phi)
\]
a *Hamilton \( G \)-space* if

1. \( G \) is connected and simply connected;
2. \( \omega(x) \in \mathfrak{v}(X) \) for \( x \in \mathfrak{g} \).

**Lemma 2.** With the notation above,

\[
(\Phi(\text{ad } g^\prime x))[g] = (\Phi(x))[g^{-1}g]
\]

for \( x \in \mathfrak{g}, g, g' \in G \).

**Proof.** Let \( \eta_{[\mathfrak{g}]} \in T_{[\mathfrak{g}]}(G/K) \), then

\[
\eta_{[\mathfrak{g}]}(\Phi(\text{ad } g^\prime x)) = \omega_{[\mathfrak{g}]}(z_{\Phi(\text{ad } g^\prime x)}, \eta)
= \omega_{[\mathfrak{g}]}(\theta(\text{ad } g^\prime x), \eta) = \omega_{[\mathfrak{g}]}((R_{g^{-1}})^* \theta_{[\mathfrak{g}^-]}'(x), \eta_{[\mathfrak{g}]}).
\]

Since \( \omega \) is \( G \)-invariant, it follows that

\[
\eta_{[\mathfrak{g}]}(\Phi(\text{ad } g^\prime x)) = \omega_{[\mathfrak{g}^-]}'(\theta(x), (R_{g})^* \eta_{[\mathfrak{g}]})
= ((R_{g})^* \eta_{[\mathfrak{g}^-]'}(\Phi(x))) = \eta_{[\mathfrak{g}]}(\Phi(x)R_{g}).
\]

Thus

\[
\Phi(\text{ad } g^\prime x) = \Phi(x)R_{g}^{-1}.
\]

Let \( \rho : K \to A \) be a Lie group homomorphism. The \( \rho \)-bundle
(G x ρ A, A, G/K) associated with (G, K, G/K) can be regarded as a right G-bundle by [g, d]g' = [g'⁻¹g, a]. Observe that the actions of G and A on G x ρ A commute.

A quantizing bundle (G x ρ A, α, A, λ, G/K, ω) over a Hamilton G-space (G/K, ω, Φ) is called a G-quantizing bundle if

\[(δΦ)(x) = x^+\]

for \(x \in \mathfrak{g}\). Here \(x^+\) denotes the vector field on \(G \times ρ A\) induced by \(x \in \mathfrak{g}\).

We shall prove that each quantizing bundle over a Hamilton G-space is equivalent to a G-quantizing bundle. First, we need some material concerning invariant connections [6].

**Proposition 5.** There is a one-one correspondence between the set of G-invariant connections on \((G \times ρ A, A, G/K)\) and the set of linear maps \(Λ : \mathfrak{g} \to A\) with

(i) \(Λ(γ) = ρ(y)\) for \(y \in \mathfrak{t}\);

(ii) \(Λ(ad kx) = Λ(x)\) for \(k \in K, x \in \mathfrak{g}\),

where \(\mathfrak{t}\) denotes the Lie algebra of \(K\); the correspondence is given by

\[Λ(x) = -α_{[e,e]}(x^+)\]

for \(x \in \mathfrak{g}\).

For the proof of Proposition 5 see also [1].

Now let \(K^Φ_λ\) be the set of Lie group homomorphisms \(ρ : K \to A\) such that

\[ρ(y) = λ(Φ(y))[e]\]

for \(y \in \mathfrak{t}\). As a consequence of Lemma 2 and Proposition 5 we get

**Proposition 6.** Let \((G/K, ω, Φ)\) be a Hamilton G-space. Then, for any \(ρ \in K^Φ_λ\), there is exactly one G-invariant connection (say \(α^ρ\)) on \((G \times ρ A, A, G/K)\) such that

\[λ(Φ(x))[e] = -α^ρ_{[e,e]}(x^+)\]

for \(x \in \mathfrak{g}\).

The following result generalizes a theorem of Kostant ([2], p. 203).

**Theorem 3.** Suppose that \((G/K, ω, Φ)\) is a Hamilton G-space. Then, for any \(ρ \in K^Φ_λ\),

\[(G \times ρ A, α^ρ, A, λ, G/K, ω)\]

is a G-quantizing bundle. Moreover, this association induces a natural one-one correspondence between \(K^Φ_λ\) and \(Q(A, λ, G/K, ω)\).

Thus each element in \(Q(A, λ, G/K, ω)\) is represented by exactly one G-quantizing bundle. Observe that \((G/K, ω, Φ)\) is \((A, λ)\)-quantizable iff \(K^Φ_λ\) is not empty.

4. PROOF OF THEOREM 3

We first prove that each \( \rho \in K^\Phi_\lambda \) induces a G-quantizing bundle \((G \times \rho A, \alpha', A, \lambda, G/K, \omega)\). It is sufficient to show that

(a) \( d\alpha' = \lambda^* \omega \);

(b) \( (\delta \Phi)(x) = x^+ \) for \( x \in \mathfrak{g} \).

(a) Choose \( \xi_i^{\pm} = (x_i)^{\pm} \in T_{[g,a]}(G \times \rho A) \) with \( x_i \in \mathfrak{g}, y_i \in \mathfrak{a}, i = 1, 2 \). Then

\[
\xi^i = x^+_i + y^+_i.
\]

By the very definition of \( \alpha' \) (compare [7], p. 107) we have

\[
\alpha'(\xi^1, \xi^2) = \alpha'(x^+_1, x^+_2).
\]

Next, observe that

\[
[R_g] \cdot x^+_i = (\text{ad } g^{-1} x^+_i)_{[g,a]}
\]

for \( x \in \mathfrak{g}, g, g' \in G \). Hence, by Proposition 6,

\[
\alpha'_\mathfrak{g}(x^+_i) = -\lambda(\Phi(\text{ad } g^{-1} x_2))[e].
\]

Differentiation yields

\[
\alpha'(\xi^1, \xi^2) = -\lambda(\Phi(\text{ad } g^{-1}[x_1, x_2]))[e].
\]

Since the actions of \( G \) and \( A \) on \( G \times \rho A \) commute, we get

\[
\alpha'([\xi^1, \xi^2]) = -\lambda(\Phi(\text{ad } g^{-1}[x_1, x_2]))[e].
\]

(1), (4) and (5) imply

\[
d\alpha'(\xi^1, \xi^2) = -\lambda(\Phi(\text{ad } g^{-1}[x_1, x_2]))[e].
\]

On the other hand

\[
\pi^* \omega(\xi^1_{[g,a]}, \xi^2_{[g,a]}) = \omega(\theta(x_1), \theta(x_2)) = -\lambda(\Phi(x_1, x_2))[g].
\]

If we combine (6), (7) and Lemma 2, the assertion (a) follows easily.

(b) To prove (b) we use the G-invariance of \( \alpha' \). Given \( x \in \mathfrak{g} \), we can write (see [7], p. 104)

\[
x^+_i = (\theta(x))^a_i + (z'(x^+_i))_{[g,a]},
\]

It follows from (3) and Lemma 2 that

\[
\alpha'(x^+_i) = -\lambda(\Phi(x))[g].
\]

This proves (b).

Thus, given \( \rho \in K^\Phi_\lambda \), we have shown how to construct a G-quantizing bundle. Conversely, any quantizing bundle \((P, z, A, \lambda, G/K, \omega)\) over a
Hamilton G-space \((G/K, \omega, \Phi)\) generates an element \(\rho \in K_{\Phi}^\omega\). To prove this, observe that
\[
\theta : x \in g \rightarrow \xi_{\Phi(x)} \in \mathfrak{B}(G/K)
\]
defines an infinitesimal action of \(G\) on \(G/K\) by complete vector fields. Hence, by Proposition 4, there exists a right \(G\)-action of \(P\) such that
\[
p \exp tx = F_{(\Phi_{(x)})}(p, t)
\]
for \(t \in \mathbb{R}, x \in g\). Now define \(\rho : K \rightarrow A\) by
\[
(*) \quad p_0k = p_0\rho^{-1}(k), \quad p_0 \in \pi^{-1}[e]
\]
for \(k \in K\). Then \(\rho \in K_{\Phi}^\omega\) since
\[
p_0 \exp ty = p_0 \exp - t\lambda(\Phi(y))[e]
\]
and
\[
p_0\rho^{-1} (\exp ty) = p_0 \exp - t\rho(y)
\]
for \(y \in \mathfrak{f}\). Observe that for a \(G\)-quantizing bundle \((G \times \rho A, \alpha^\rho, A, \lambda, G/K, \omega)\) one obtains
\[
[e, e]k = [e, e]\rho^{-1}(k), \quad k \in K.
\]
Thus we have reduced the proof of Theorem 3 to the following proposition.

**PROPOSITION 7.** — Let \((P, \alpha, A, \lambda, G/K, \omega)\) be a quantizing bundle over the Hamilton \(G\)-space \((G/K, \omega, \Phi)\). Define \(\rho \in K_{\Phi}^\omega\) by \((*)\). Then \((P, \alpha, A, \lambda, G/K, \omega)\) and \((G \times \rho A, \alpha^\rho, A, \lambda, G/K, \omega)\) are equivalent.

**Proof.** — It is easy to check that the assignment
\[
[g, a] \in G \times \rho A \rightarrow p_0g^{-1}a^{-1} \in P
\]
defines a \(G, A\)-equivariant principal bundle isomorphism
\[
\varphi : (G \times \rho A, A, G/K) \rightarrow (P, A, G/K).
\]
We show that \(\varphi^*\alpha = \alpha^\rho\). Clearly
\[
(\varphi^*\alpha)_{e,e}(x^+) = \alpha_{p_0}((\delta\Phi)(x))
\]
for \(x \in g\). By Lemma 1 and an argument used above we obtain
\[
(\varphi^*\alpha)_{e,e}(x^+) = - \lambda(\Phi(x))[e],
\]
for \(x \in g\). Since \(\varphi^*\alpha\) is a \(G\)-invariant connection form, Proposition 6 gives
\[
\varphi^*\alpha = \alpha^\rho.
\]
The result now follows.

### 5. CHARACTERIZATION OF \(K_{\Phi}^\omega\)

In conclusion, we compute the action
\[
K_{\Phi}^\omega \times \pi_1^\lambda(G/K, [e]) \rightarrow K_{\Phi}^\omega
\]
given by Theorems 2 and 3 more explicitly. For this purpose, consider the principal bundle \((G, K, G/K)\) with structure group \(K\). Let \(K_0 \subset K\) be the identity component of \(K\). Define an action

\[
G/K_0 \times K/K_0 \to G/K_0
\]

of \(K/K_0\) on \(G/K_0\) by setting \(([g]_0, [k]_0) \to [gk]_0\). Since \(G\) is simply connected, \((G/K_0, K/K_0, G/K)\) is a principal bundle with structure group \(K/K_0 \cong \pi_1(G/K, [e])\). Therefore, for \(\rho \in K^\Phi_\omega, \sigma \in \pi_1^\Phi(G/K, [e])\), the association

\[
(\rho, \sigma) \to \rho \sigma,
\]

\((\rho \sigma)(k) = \rho(k)\sigma([k]_0)\), defines an action of \(\pi_1^\Phi(G/K, [e])\) on \(K^\Phi_\omega\). The following result proves that, in view of Theorem 3, this action can be identified with the action of \(\pi_1^\Phi(G/K, [e])\) on \(Q(A, \lambda, G/K, \omega)\).

**Proposition 8.** — Let \((G/K, \omega, \Phi)\) be an \((A, \lambda)\)-quantizable Hamilton \(G\)-space. Then the bijection

\[
\rho \in K^\Phi_\omega \to [G \times_\rho A, A^{\rho}, A, \lambda, G/K, \omega] \in Q(A, \lambda, G/K, \omega)
\]

is a \(\pi_1^\Phi(G/K, [e])\)-equivariant map.

**Proof.** — If \((G, K, G/K)\) is characterized by transition functions \(\{g_{ij}; i, j \in I\}\), then the system \(\{g^0_{ij}; i, j \in I\}\) defined by \(g^0_{ij}(x) = [g_{ij}(x)]_0\), \(x \in U_i \cap U_j \subset G/K\), represents \((G/K_0, K/K_0, G/K)\). Now

\[
\{ (\rho \sigma)g_{ij}; i, j \in I \}
\]

are transition functions associated with \([G \times_\rho A, A^{\rho}, A, \lambda, G/K, \omega]\), whereas \([G \times_\rho A, A^{\rho}, A, \lambda, G/K, \omega]\) is described by

\[
\{ (\rho g_{ij})(\sigma g^0_{ij}); i, j \in I \}.
\]

Since, for \(x \in U_i \cap U_j\),

\[
((\rho \sigma)g_{ij})(x) = ((\rho g_{ij})(\sigma g^0_{ij}))(x)
\]

we conclude that

\[
[G \times_\rho A, A^{\rho}, A, \lambda, G/K, \omega] = [G \times_\rho A, A^{\rho}, A, \lambda, G/K, \omega]_{\sigma}
\]

This proves the assertion.

**References**


(Manuscrit reçu le 5 juin 1975)