ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 27, n° 1 (1977), p. 61-71
<http://www.numdam.org/item?id=AIHPA_1977__27_1_61_0>

Ann. Inst. Henri Poincaré,
Vol. XXVII, n° 1, 1977, p. 61.

Section A :
Physique théorique.

Boundary conditions at past null infinity for zero-rest-mass fields including gravitation

by

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ABSTRACT. — Boundary conditions at past null infinity are formulated for scalar and electromagnetic fields in flat space-time, and for the gravitational field in General Relativity, which ensure absence of incoming radiation. Restrictions on the behaviour of sources in the infinite past, which are required in order to guarantee that retarded solutions for bounded sources in flat-spacetime obey these radiation conditions, are discussed. Finally, some difficulties concerned with these conditions in the gravitational case are pointed out.

1. INTRODUCTION

In order that a physical system be described as being isolated, the possibility of matter or radiation falling onto the system from infinity must be excluded: an isolated system should evolve according to its own internal dynamics alone. In a scattering process, the system itself comes in from infinity, so the question as to whether such a system is isolated or not is unclear. For convenience, here, only systems whose material sources are uniformly spatially bounded for all time earlier than some given moment are considered. With this restriction, only incoming radiation can disturb the system from outside.

The systems to be considered are classical scalar (spin zero) and electromagnetic (spin one) zero-rest-mass (hereafter: zrm) fields in flat space-time, and gravitational (spin two) fields in General Relativity. The conditions ensuring freedom from incoming radiation for these systems have often been given the misnomer « outgoing radiation conditions » and usually they have been imposed at spacelike infinity [*1*], [*2*], [*3*] (in one case, at future null infinity [*6*]). The radiation conditions discussed here will be formulated either asymptotically along incoming null geodesics as in [*4*], or on the hypersurface $\mathscr{I}^-$ at past null infinity [*5*].

In all three cases, radiation conditions will be derived by means of an energy argument. Since the classical scalar and electromagnetic fields have well-defined energy-momentum tensors, whereas the gravitational field does not, the treatment of the latter field is postponed until section 5. The intervening sections treat the two former cases in flat space-time.

For the special relativistic scalar and electromagnetic fields, it is convenient to introduce a certain coordinate system and frame field based on a timelike straight line in Minkowski space (Battelle conventions [*14*] for tensor and spinor indices will be used). Having chosen such a line, which may be thought of as the world-line of the centre of mass of the system, for example, introduce the corresponding time t, and spherical polar coordinates r, θ, ϕ. The complex stereographic coordinate $\zeta = e^{i\phi} \cot \theta/2$ will also be used. Label the incoming past light cones with vertices on the chosen line by the advanced time $v = t + r$. The function r is an affine parameter on generators of these past light cones. The functions $\{ t, r, \zeta \}$ or $\{ v, r, \zeta \}$ will be used as coordinates in space-time; $\{ v, \zeta \}$ are convenient coordinates on past null infinity $\mathscr{I}^-$. The complex function ζ ($\zeta = \infty$ being permitted) labels generators of the past null cones.

In addition to the coordinates, a tetrad of null vectors at each point not on the chosen timelike straight line is defined by the following directional derivatives:

$$
\begin{aligned}
l^a\partial/\partial x^a &= \partial/\partial v + \frac{1}{2}\partial/\partial r \\
n^a\partial/\partial x^a &= -\,\partial/r \\
m^a\partial/\partial x^a &= (\sqrt{2}/r)(\partial/\partial\theta + i/\sin\theta\,\partial/\partial\phi) \\
&= -\,(\sqrt{2}e^{i\phi}/r)(1 + \zeta\bar{\zeta})\partial/\partial\zeta.
\end{aligned}
\tag{1}
$$

Here, l^a and n^a are real, future-pointing null vectors tangent to the future, outgoing and past, incoming light cones with vertices on the chosen line, respectively. The real and imaginary parts of the complex null vector m^a are tangent to the spherical sections $r = \text{const} > 0$ of the past light cones labelled by v. The situation is sketched in fig. 1. Let $\{ o^A, \iota^A \}$ be one of the two spin frames for SL(2, $\mathbb{C}$)-spinors which is associated to the above frame by $o^A\bar{o}^{A'} = l^a$, $o^A\bar{\iota}^{A'} = m^a$, and $\iota^A\bar{\iota}^{A'} = n^a$. The signature of space-time is taken to be $(+---)$, so the above vectors are related to the space-time metric η_{ab} by

$$\eta_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}. \tag{2}$$

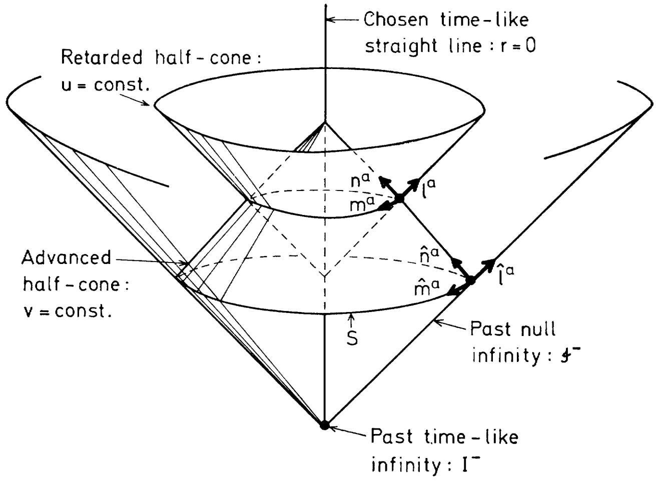

FIG. 1. — Hypersurfaces and frame vectors used to discuss radiation conditions in flat space-time.

Covariant derivatives, in flat or curved space-time, will be denoted by ∇_a and the covariant derivative in the direction of l^a will be denoted $l^c\nabla_c = \mathrm{D}$.

In General Relativity, coordinates and a frame having some of the properties of those chosen above may be constructed; this will be discussed in section 5.

Finally, the notion of weak asymptotic smoothness will be required. The order symbols o and O have their usual meaning, while weak asymptotic smoothness of some order, denoted $^*o$ or $^*\mathrm{O}$, means that formal differentiation of an order relation leaves the order unchanged. This notion is weaker than uniform smoothness [7]; the function $(\sin r)/r$ for example is weakly asymptotically, but not uniformly, smooth of order $1/r$.

2. THE ENERGY ARGUMENT IN FLAT SPACE-TIME

The radiation concept for zrm fields is intimately connected with the notion of energy transport to and from infinity. Indeed, the radiation field may be defined as that part of the total field which contributes to an energy flux at infinity. It is tempting, therefore to attempt to derive a radiation condition from a more physical energy condition. Now radiation propagates along null straight lines in flat space-time, so the condition that no energy flow onto the system from infinity must be formulated either asymptotically along generators of past null cones, or at past null infinity $\mathscr{I}^-$ itself. At the same time, an operational condition would require that no observer measure a net inflow of energy. If we consider a sequence of timelike curves, possible world lines of observers, each of which is further from the system than his predecessor, then the limiting curve will coincide, at infinity, with some generator of past null infinity and with some generator of future null infinity. Idealized, limiting observers residing on a generator of future null infinity measure outgoing radiation, while incoming radiation would be measured by idealized, limiting observers residing on a generator of past null infinity.

Each observer may be imagined to avail himself of an ergometer to measure energy. An ergometer is a device which measures the net energy flow (see (3), below) in a prescribed direction, during some time interval, across a small two-dimensional surface. Suppose an observer has normalized future-pointing four-velocity u^a, and that his ergometer has area A and spacelike unit normal vector q^a. Then if the field he is measuring has stress-energy tensor T^{ab}, the net energy measured by his ergometer in the interval τ_1 to τ_2 of his proper time is

$$\int_{\tau_1}^{\tau_2} \int_A T^{ab} u_a q_b dA d\tau. \tag{3}$$

Due to our choice of signature of the space-time metric, energy whose flux is in the direction $-q^a$ is counted positively, and *vice versa.*

Consider now a sequence of ergometers momentarily arrayed along a null straight line coming in from past null infinity. The normal vectors q^a of their measuring surfaces are to be directed outward, so that incoming energy flux is counted positively. Since the energy-momentum tensor of a field which falls off asymptotically as $1/r$ can itself be expected to fall off as $1/r^2$ along this line, the area of the measuring surfaces of the ergometers must be scaled up by a factor r^2 as one recedes into the past, in order that one has the possibility of measuring a nonvanishing flux in the limit. The condition that no ergometer register incoming energy flux at infinity is therefore

$$\limsup_{\substack{r \to \infty \\ v,\zeta \text{ const.}}} \int_{\tau_1}^{\tau_2} \int_A T_{ab} u^a q^b r^2 dA d\tau \leqslant 0. \tag{4}$$

It is to be emphasized that the limit is to be taken along null lines coming into the space-time from past null infinity.

In the next section, condition (4) will be evaluated for zrm scalar fields, and electrodynamics.

3. ZERO-REST-MASS FIELDS IN FLAT SPACETIME

A spin zero, zrm field φ satisfies the scalar wave equation

$$\Box \varphi = \rho \tag{5}$$

The support of the source function ρ is to be uniformly spatially bounded for all time earlier than some finite time. In vacuum, the energy-momentum tensor of this is fiels given by

$$T_{ab} = (\nabla_a\varphi)(\nabla_b\varphi) - \frac{1}{2}\eta_{ab}(\nabla_c\varphi)(\nabla^c\varphi). \quad (6)$$

One expects the field of a spatially bounded source to vanish at infinity in some sense. Unfortunately, there are several inequivalent and at least one insufficient way of making this sense precise. It is, for example, not sufficient to require that $\varphi = O(1/r)$ along all generators of all past null cones with vertices on a single timelike straight line, since it can then happen that φ fails to be $O(1/r)$ along some other null straight line. On the other hand, the requirement that φ be $O(1/r)$ along all pastdirected null straight lines (the « peeling » condition [5]) will often be difficult to verify in practice, A condition which can be formulated for a single family of past null cones, and is sufficient, is that as r goes to infinity, $r\varphi$ be not only bounded, but that the limit actually exist and be continuous. This, however, is stronger than the peeling condition. It is difficult to find a necessary condition for vanishing at infinity which can be formulated along a single family of past null cones. It will be shown in section 4 that retarded fields produced by arbitrarily prescribed sources will in general not satisfy the peeling condition at past null infinity (compare also [*13*], [*15*]). In view of the lack of compelling physical arguments for a « weakest possible » condition, a « most convenient » one will be imposed. This is that φ be weakly asymptotically smooth of order $1/r$ along all generators of all past null cones with vertices on a single timelike straight line, and that the limits as r goes to infinity of $r\varphi$ exist and be continuous. It follows from the continuity that the condition is in fact independent of the choice of timelike straight line. Weak asymptotic smoothness is required since derivatives enter into the expression for the stress-energy tensor.

It is now possible to evaluate the energy condition (4). The vectors u^a and q^a can be written as linear combinations of l^a and n^a with coefficients of order unity, and the covariant derivatives occurring in the energy-momentum tensor can be reexpressed in terms of the directional derivatives (1). The resulting condition is

$$D\varphi(v, r, \zeta) = O(1/r) \quad (7)$$

The classical zrm field of spin one is the electromagnetic field. It is most convenient, for present purposes, to work with certain complex components of the Maxwell field tensor F_{ab}, defined as follows. F_{ab} is equivalent to a symmetric, valence two spinor field φ^{AB}, with

$$F_{ab} = \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\overline{\varphi}_{A'B'}. \quad (8)$$

The stress-energy tensor of the electromagnetic field in vacuum is then given by

$$T_{ab} = 2\varphi_{AB}\overline{\varphi}_{A'B'}. \quad (9)$$

With respect to the spin basis introduced earlier, define three complex scalar fields by

$$\phi_0 = \varphi_{AB}o^A o^B \quad , \quad \phi_1 = \varphi_{AB}o^A \iota^B \quad , \quad \phi_2 = \varphi_{AB}\iota^A \iota^B \tag{10}$$

In an appropriate frame, ϕ_0, ϕ_1, and ϕ_2 are the components of the complex three-vector $\vec{E} + i\vec{B}$, where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field vectors respectively.

As in the scalar case, the vanishing of the electromagnetic field at infinity is expressed by requiring that each of ϕ_0, ϕ_1, and ϕ_2 be weakly asymptotically smooth of order $1/r$ along the generators of the past null cones $v = \text{const}$, and that the limits as $r \to \infty$ of $r\phi_0$, $r\phi_1$, and $r\phi_2$ exist and are continuous. No condition on derivatives of the field is required in the electromagnetic case. Substituting into (4) using (10) then results in the condition

$$\phi_0(v, r, \zeta) = {}^*o(1/r). \tag{11}$$

Somewhat more insight into the radiation conditions (7) and (11) can be gained by reformulating them at past null infinity itself [5]. Recall that if Ω is a conformal factor which makes $\mathscr{I}^-$ finite, and $\varphi_{A_1 \dots A_{2s}}$ is a solution of the zrm field equation of spin s, then $\Omega^{-1}\varphi_{A_1 \dots A_{2s}} = : \widehat{\varphi}_{A_1 \dots A_{2s}}$ is a solution of this equation in the space-time whose metric is conformally rescaled by Ω^2. $\Omega^{-1}\varphi_{A_1 \dots A_{2s}}$ is called the conformally rescaled field. Having chosen a time-like straight line in flat space-time, $\Omega = 1/r$ is a convenient conformal factor. The condition for vanishing of the field at infinity in flat space-time adopted above then becomes simply the condition that the conformally rescaled field be continuous on $\mathscr{I}^-$. In the case of spin zero, in addition, differentiability of the conformally rescaled field on $\mathscr{I}^-$ was required.

With the above remarks, a sufficient condition for absence of incoming radiation in a solution of the scalar wave equation can be formulated as follows:

> The conformally rescaled field $\widehat{\varphi}$ is smooth of class C^1, and the derivative of $\widehat{\varphi}$ in the direction of the generators of $\mathscr{I}^-$ vanishes: $\partial\widehat{\varphi}/\partial v = 0$ on $\mathscr{I}^-$. (12)

In electrodynamics, a sufficient condition for absence of incoming radiation in a solution of Maxwell's equations is:

> The conformally rescaled field $\widehat{\varphi}_{AB}$ is continuous on $\mathscr{I}^-$, and the radiation field $\widehat{\phi}_0 = \Omega^{-1}\phi_0$ vanishes on $\mathscr{I}^-$. (13)

As was remarked earlier, these are not the weakest possible conditions guaranteeing absence of incoming radiation. Weaker conditions would, however, require a certain asymptotic behaviour along all past directed null straight lines in flat spacetime, and would consequently be clumsier to state and be more difficult to verify than the conditions (12) and (13).

Also, there would appear to be no physical reason for wishing to have a weakest possible condition.

It can be shown with the help of a Kirchoff-type integral representation [*8*] of the fields discussed here that a globally regular solution of the homogeneous, source-free, field equation, which satisfies the given radiation condition (12) or (13), vanishes identically. It follows that there exists at most one solution of the inhomogeneous field equation for a given source which satisfies the radiation condition.

The manner in which the field enters the radiation condition depends on the spin of the field. For spin zero, a derivative of the field vanishes on $\mathscr{I}^-$, while for spin one, the radiation field itself vanishes on $\mathscr{I}^-$. It will be shown later that for spin two (General Relativity), it is an integral of the radiation field on $\mathscr{I}^-$ which plays the corresponding role.

4. RETARDED FIELDS IN FLAT SPACETIME

It has sometimes been assumed [*3*], [*4*], [*11*] that retarded solutions of zrm field equations automatically satisfy a radiation condition and consequently admit no incoming radiation. That the validity of such an assumption cannot be assured without further restriction may be inferred from the following argument. The peeling theorem [*5*], [*12*] for retarded fields produced by spatially bounded sources is valid along generators of future null cones. Consider, however, a sequence of space-time points diverging to spacelike or past null infinity. Past-directed null straight lines, along which retarded fields propagate (in the reverse sense), joining points of this sequence to points of a source distribution, will asymptotically tend to some generator of past null infinity as the sequence is followed out to infinity. Consequently, the asymptotic behaviour of retarded fields, either on spacelike hypersurfaces or along generators of past null cones, will be determined by the asymptotic behaviour in time of the sources in the infinite past.

This point can be illustrated by a simple example. For behaviour of the field in the past, the behaviour of the sources to the future of any finite time is irrelevant. For some finite time $t_0 < 0$, therefore, consider the retarded solution

$$\varphi = \frac{1}{4\pi}(r - t)/t \tag{14}$$

for $t < t_0$ of the scalar wave equation, corresponding to the source function

$$\rho = -\, t\delta(r) \quad , \quad t < t_0 < 0. \tag{15}$$

The retarded field given by (14) satisfies neither the radiation condition formulated here, nor that formulated at spacelike infinity by Sommerfeld [1]. What is more, the stress-energy tensor of this retarded field has the behaviour

$$T_{ab} = (1/8\pi^2)(r^{-2} + O(r^{-3}))n_a n_b \tag{16}$$

along generators of past null cones. This shows that every idealized ergometer will measure incoming energy, despite the purely retarded nature of the field.

The feature of the above retarded solution which results in incoming radiation is not so much that the source function blows up in the infinite past, but that the source still sends out waves in the infinite past. These waves become infinitely compacted by the conformal rescaling, so that the conformally rescaled field ceases to be defined (it can happen, for example with $\rho = \sin t\delta(r)$, that the amplitude of the rescaled field remains bounded, but the frequency diverges) on $\mathscr{I}^-$. Thus one may think of the source as being spread out over the whole of $\mathscr{I}^-$. Viewed in this way, it is not surprising that even the retarded solution should have incoming radiation.

A theorem asserting the absence of incoming radiation of retarded zrm fields must therefore contain a restriction on the asymptotic time dependence of the sources on approaching past timelike infinity. Among the strongest such conditions are those assuring regularity (C^1) of the conformally rescaled field on $\mathscr{I}^-$. The following two sets of conditions are sufficient for this regularity in the cases of scalar waves and electrodynamics respectively: for spin zero:

a) the source function ρ is of differentiability class C^3 everywhere,
b) there is a positive real number R and a time t_0 such that the source function vanishes outside a world tube of radius R for times earlier then t_0, (17)
c) the source function and its first time derivative are bounded at all times earlier than t_0,
d) $|\partial\rho/\partial t| = O(t^{-1})$ for $t \to -\infty$ along timelike lines within the support of ρ,

and for spin one:

a) the charge-current vector j^a is of differentiability class C^3 everywhere,
b) there is a positive real number R and a time t_0 such that the charge-current vector vanishes outside a world tube of radius R for times earlier than t_0, (18)
c) the charge-current vector and its first and second time derivatives are bounded at all times earlier than t_0,
d) there are static vectorfields $j^a_{(0)}$ and $j^a_{(1)}$ such that

$$j^a = j^a_{(0)} + t^{-1}j^a_{(1)} + O(t^{-2})$$

for $t \to -\infty$ along timelike lines within the support of j^a.

From the appropriate integral representation of the retarded fields in flat spacetime one can then deduce the following theorem:

If the source satisfies condition (17) (resp. (18)), then the retarded field due to this source is the unique solution of the vacuum field equation which satisfies the radiation condition (12) (resp. 13)).

5. THE CLASSICAL ZRM FIELD OF SPIN TWO: GENERAL RELATIVITY

The case of gravitation differs from scalar waves or electrodynamics in flat space-time in a number of important respects. Space-time is no longer flat, there is no global integral representation of vacuum gravitational fields, and the vacuum gravitational field has no energy-momentum tensor. All three (related) phenomena complicate an analysis of the above kind. Nevertheless one might expect the space-time corresponding to a spatially bounded source distribution to be asymptotically flat. If this were true in the sense that the null hypersurface $\mathscr{I}^-$ at past null infinity were to exist for such a space-time, one could argue as follows.

A measure of the total energy of a gravitational system is its mass, computed on a section of $\mathscr{I}^-$. Choosing the conformal factor Ω in the definition of $\mathscr{I}^-$ so that the (degenerate) metric on $\mathscr{I}^-$ is that of a unit sphere, and picking a single section of $\mathscr{I}^-$, one can construct a standard Bondi-type coordinate system $\{ v, \zeta \}$ and frame-field l^a, n^a, m^a, on $\mathscr{I}^-$. ζ is a complex stereographic coordinate on the sphere, and v labels a family of spherical sections of $\mathscr{I}^-$ with $v = 0$ being the originally chosen section. The vector l^a is covariantly constant on $\mathscr{I}^-$ and tangential to its generators, and n^a and m^a are parallel along the generators of $\mathscr{I}^-$. A spin frame o^A, ι^A related to the null vectors as before, can be introduced. Such a coordinate system and frame-field has some of the features of advanced null coordinates and the corresponding frame in flat space-time. It also has important differences, however. In particular, the (complex) asymptotic shear σ^0 of the null hypersurfaces $v =$ const will not in general vanish, as it does for null cones in flat space-time. Defining the Bondi-Sachs news function N (at past null infinity !) to be minus the v-derivative of the complex conjugate of σ^0, and taking Bondi *et al*'s definition [*9*] of mass (cf. also [*10*], [*14*]), the rate of gain of mass by the system due to incoming gravitational radiation across $\mathscr{I}^-$ is given by

$$\frac{dm}{dv} = \int_S \mathrm{N}\bar{\mathrm{N}} dS, \tag{19}$$

S being one of the spherical sections $v =$ const. of $\mathscr{I}^-$. Hence the boundary condition arising from an energy argument in the case of General Relativity is that the news function vanish identically on past null infinity. This vanishing of the news function is independent both of the choice of conformal factor and of initial section of $\mathscr{I}^-$.

Now the incoming gravitational radiation field on $\mathscr{I}^-$, Ψ_0^0 (the coefficient of that part of the Weyl tensor which falls off along past-directed null geodesics as the inverse of an affine or luminosity-distance parameter) is given by the time derivative $\partial N/\partial v$ of the news function. Hence the above boundary condition can be written

$$\forall v, \int_{-\infty}^{v} \Psi_0^0 dv = 0. \tag{20}$$

One sees that the condition is on an integral of the radiation field.

It was assumed in the above that $\mathscr{I}^-$ existed as a regular null hypersurface in the conformally rescaled space-time. Yet we have seen that, in special-relativistic theories, the conformally rescaled retarded fields of bounded sourced can be singular on $\mathscr{I}^-$. In General Relativity, this behaviour would presumably imply that $\mathscr{I}^-$ either did not exist, or existed only as a singular null hypersurface in some sense. In such a case it is not clear how or whether one could impose a radiation condition of the type discussed here. Intuitively, one would expect that a condition of the sort given in (17) or (18), that the sources of the gravitational field become stationary sufficiently rapidly on approaching past timelike infinity, would ensure the existence of $\mathscr{I}^-$ (« $\mathscr{I}^-$ should be alright if the system did not radiate in the infinite past »), but such an analysis is enormously complicated due to the lack of an integral representation of space-time. Indeed, it is not *a priori* clear whether non-stationary solutions with a regular $\mathscr{I}^-$ exist at all.

If asymptotically flat space-times are intended to model space-time around an isolated gravitational system, then some condition ensuring absence of incoming radiation is essential. Further, the notions of energy and radiation-field in General Relativity have only been defined unambiguously at infinity. It is therefore of great importance to determine whether the hypersurface at past null infinity exists for physically realistic isolated gravitational systems.

ACKNOWLEDGMENTS

Bernd Schmidt suggested that the problem might be interesting. Jürgen Ehlers shared his knowledge with us. Michael Streubel constructed examples of functions which are $O(1/r)$ along generators of one family of past null cones but which fail to be $O(1/r)$ on athers. Abbay Ashtekhar provided helpful criticism. Rosita Strasser typed the manuscript. To all of them we owe a debt of gratitude.