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On the lattice structure of quantum logics

by

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ABSTRACT. — The problem of the lattice structure of a quantum logic is solved by embedding the logic into the so-called phase geometry of the physical system, being an atomistic complete lattice. The latter is constructed using pure states of the physical system.

1. INTRODUCTION

The important problem of the lattice structure of a quantum logic was considered in many papers, however, in the author's opinion, only the paper by Bugajska and Bugajski [*2*] may be considered as giving the satisfactory solution to this question. In our paper, following to the Bugajska and Bugajski's idea, we propose to solve this problem by embedding the logic into an atomistic complete lattice, however the latter is now the so-called phase geometry of the physical system under study (see [*4*]) and is constructed using pure states of the system.

There should be emphasized the following advantages of the approach presented here. Firstly, the « projection postulate » of Bugajska and Bugajski [*2*] becomes now superfluous and can therefore be omitted. As this postulate is neither obvious nor generally unquestionable, our axiom system seems to be more plausible. Secondly, we do not need here the structure of an orthomodular σ-orthoposet for the logic L. Our assumptions are much more modest: L is assumed to be a poset with an involution and with the least and greatest elements in it.

2. DEFINITIONS AND NOTATION

With each physical system two important sets are always associated: the set 0, whose elements will be called observables, and the set S of states of the system, whose members are mappings from 0 to the set of all probability measures on B(R), B(R) being the σ-algebra of all Borel subsets of the real line R. Every state $m \in S$ assigns to each observable $A \in 0$ its « probability distribution in the state m » denoted by $m(A)$. According to the usual interpretation (see, e. g. [*5*]), the number $[m(A)](E)$, where $m \in S$, $A \in 0$ and $E \in B(R)$, is meant as the probability that a measurement of an observable A for the system in a state m gives a value lying in a Borel set E.

Within every physical theory we are interested in the possibility of the verification of propositions of the following form: « a measurement of an observable A gives a value in a Borel set E ». Denote such propositions by $\langle A, E \rangle$. They may be identified, according to the usual convention (see [*5*], [*7*]), with the equivalence classes of the equivalence relation $\sim$ defined in the set $0 \times B(R)$ as follows:

$$(A, E) \sim (B, F) \quad \text{iff} \quad [m(A)](E) = [m(B)](F) \qquad \text{for all } m \in S.$$

Namely, we identify the proposition $\langle A, E \rangle$ with the equivalence class $(A, E)/_\sim$.

The set $L = (0 \times B(R))/_\sim$ of all equivalence classes of the relation $\sim$ admits a natural partial ordering defined by [*7*]:

$$(A, E)/_\sim \leqslant (B, F)/_\sim \quad \text{iff} \quad [m(A)](E) \leqslant [m(B)](F) \qquad \text{for all } m \in S;$$

moreover, $(A, \emptyset)/_\sim$ and $(A, R)/_\sim$ are the least and the greatest elements in L, respectively, and the mapping $(A, E)/_\sim \to (A, R - E)/_\sim$ is a well-defined involution in L. (By an *involution* of a partially ordered set L we mean a map $' : L \to L$ with the following properties: for any pair $a, b \in L$ with $a \leqslant b$ one has $b' \leqslant a'$, and $a'' = a$ for every $a \in L$.)

DEFINITION. — The partially ordered set $(L, \leqslant)$ endowed with the involution $' : (A, E)/_\sim \to (A, R - E)/_\sim$ is called the *logic of a physical system* (briefly, a *logic*, see [*7*], compare also [*5*]) and its elements will be denoted by small letters $a, b, c, \ldots$, etc.

DEFINITION. — We say that two propositions $a, b \in L$ are *orthogonal* and write $a \perp b$, if $a \leqslant b'$.

Observe ([*7*], [*5*]) that each state m may be identified with the mapping $(A, E)/_\sim \to [m(A)](E)$, which maps the logic L into the closed interval [0, 1]. This mapping will be denoted by the same letter m, that is, we put by the definition $m(a) = [m(A)](E)$, whenever $a = (A, E)/_\sim$.

DEFINITION. — We shall say that two states m_1 and m_2 are *mutually* *exclusive* or *orthogonal* [3], and write $m_1 \perp m_2$, if for some proposition $a \in L$ one has $m_1(a) = 1$ and $m_2(a) = 0$.

This orthogonality relation is, of course, symmetric, i. e. $m_1 \perp m_2$ implies $m_2 \perp m_1$.

DEFINITION. — A state m is said to be *pure*, if it cannot be written as a non-trivial convex combination of other states.

Pure states will be denoted by small letters $p, q, r, \ldots$, etc., and the whole set of all pure states we shall denote by P. It could happen, of course, that P is empty. Suppose at the moment that pure states exist, i. e. that $P \neq \emptyset$, and let $S \subseteq P$. Define $S^\perp$ to be the set of all pure states p such that $p \perp S$ (read : $p \perp q$ for all $q \in S$) and write S^- instead of $S^{\perp\perp}$. Obviously, $S \subseteq S^-$. If $S = S^-$, we call the set S *closed.*

DEFINITION. — The set P of pure states endowed with the orthogonality relation $\perp$ will be called the *phase space* of the physical system. The family C(P) of all closed subsets of P we shall call the *phase geometry* associated with the system (see [4]).

It is not difficult to check (see [4], also [1]) that, under set inclusion, C(P) becomes a complete lattice whose joins and meets are given by

$$\bigvee_j M_j = \Big(\bigcup_j M_j\Big)^- \quad \text{and} \quad \bigwedge_j M_j = \bigcap_j M_j$$

($\{ M_j \}$ being an arbitrary family of closed subsets of P).

Moreover, it can also easily be shown that the correspondence $S \to S^\perp$ defines an orthocomplementation in C(P). (For the empty set $\emptyset$ we put, by the definition, $\emptyset^\perp = P$. This leads immediately to $\emptyset, P \in C(P)$.)

Let now $K \subseteq L$, $T \subseteq S$ and $j = 0$ or 1. The following abbreviations will be used throughout this paper:

$$K^j = \{ m \in S : m(a) = j \quad \text{for all} \quad a \in K \},$$

$$T^j = \{ a \in L : m(a) = j \quad \text{for all } m \in T \}.$$

If the set K consists of one point only, say, $K = \{ a \}$, then we write a^j instead of $\{ a \}^j$. Analogously, we write m^j instead of $\{ m \}^j$.

3. AXIOMS

AXIOM 1. — (*i*) For every non-zero proposition $a \in L$ there exists a pure state p such that $p(a) = 1$; moreover:

(*ii*) If $b \not\perp a$, then the state p can be chosen in such a way that $p(b) > 0$.

Formally, the Axiom 1 can be written as follows:

$$\forall_{a \in L, a \neq 0} \quad \forall_{b \in L, b \not\perp a} \quad \exists_{p \in P} \quad p(a) = 1 \quad \& \quad p(b) > 0 .$$

The first part of the Axiom 1 assumes that P, the set of pure states, is not only non-empty, but sufficiently large. Such a postulate was assumed, for instance, by Mac Laren [6]. The second part of this axiom we easily find to be equivalent to the following statement (taken as a postulate, for example, by Gudder [3]):

(∗) If for each pure state $p \in P$ with $p(a) = 1$ we have also $p(b) = 1$ ($a, b \in L$), then $a \leqslant b$.

In fact, suppose that $a \nleqslant b$. Then $a \not\perp b'$, and by Axiom 1 there exists a pure state $p \in P$ with $p(a) = 1$ and $p(b') > 0$, hence $p(b) < 1$, which proves the implication: Axiom 1 $\Rightarrow$ (∗). Conversely, assume the validity of the first part of the Axiom 1. Then (∗) implies (*ii*). Indeed, let $b \not\perp a$, $a \neq 0$, then $a \nleqslant b'$, and therefore, by (∗), there exists $p \in P$ such that $p(a) = 1$ and $p(b') < 1$, the latter being equivalent to $p(b) > 0$. (The existence of at least one pure state p with $p(a) = 1$ is guaranteed by (*i*).)

Therefore our Axiom 1 may be formulated in the following equivalent form:

AXIOM 1′. — (*i*) For every non-zero proposition $a \in L$ there exists a pure state $p \in P$ with $p(a) = 1$; moreover

(*ii*) If for each pure state $p \in P$ for which $p(a) = 1$ we have also $p(b) = 1$ for some $b \in L$, then $a \leqslant b$.

Our second (and last) axiom is:

AXIOM 2. — For every pure state $p \in P$ there exists a proposition $a \in L$ such that $p(a) = 1$ and $q(a) < 1$ for all pure states q distinct from p.

The Axiom 2 (which was assumed as a postulate e. g. by Mac Laren [6]) asserts that pure states may be realized in the laboratory: there exists a measuring device answering the experimental question « Is the physical system in the pure state p? ».

4. THE EMBEDDING THEOREM

DEFINITION. — The proposition $a \in L$ is said to be a *carrier* of a state $m \in S$ (see [9], also [8]), if

(*i*) $m(a) = 1$,

(*ii*) $b \not\perp a$ implies $m(b) > 0$.

Notice that the carrier of a state m, whenever it exists, is uniquely determined by m, since it is the smallest element of the set m^1. The carrier of m, if exists, will be denoted by carr m.

LEMMA 1. — Each pure state p has the carrier, and $q(\text{carr } p) < 1$ for every pure state $q \neq p$.

Proof. — Let $p \in P$. By the Axiom 2 there exists $a \in L$ $(0 < a < 1)$ such that $p(a) = 1$ and $q(a) < 1$ for all pure states $q \neq p$. Let $b \in L$, $b \not\perp a$. By the Axiom 1 there is $r \in P$ such that $r(a) = 1$ and $r(b) > 0$, but, owing to Axiom 2, $r = p$. Therefore we have proved the following:

$$\forall_{p \in P} \quad \exists_{a \in L} \quad p(a) = 1 \quad \& \quad \forall_{b \in L, b \not\perp a} \quad p(b) > 0\,,$$

that is, $a = \operatorname{carr} p$.

At the same time we proved (see above) that $q(\operatorname{carr} p) < 1$ for every pure state $q \neq p$.

LEMMA 2. — The logic L is atomic and the correspondence $\operatorname{carr} : p \to \operatorname{carr} p$, $p \in P$, is a one-to-one mapping of the set P of pure states onto the set of all atoms of the logic L.

Proof. — Let p be an arbitrary pure state, and let $0 < b \leqslant \operatorname{carr} p$. By the Axiom 1 there exists a pure state q with $q(b) = 1$, hence $q(\operatorname{carr} p) = 1$, hence $q = p$ by Lemma 1, and therefore $p(b) = 1$. Hence $\operatorname{carr} p \leqslant b$, which shows that $\operatorname{carr} p$ is an atom indeed.

Let now $a \in L$, $a \neq 0$, and let $p(a) = 1$, $p \in P$ (such a pure state p there exists owing to the Axiom 1). Then $a \geqslant \operatorname{carr} p$, hence, as $\operatorname{carr} p$ is an atom, we find the logic L to be atomic. If, in particular, a is an atom itself, then $a = \operatorname{carr} p$ which shows that the mapping carr is a surjection. Finally, if $\operatorname{carr} p = \operatorname{carr} q$ $(p, q \in P)$, then $q(\operatorname{carr} p) = q(\operatorname{carr} q) = 1$, hence $q = p$ by Lemma 1, which proves that carr is one-one.

Furthermore, the logic L is atomistic, that is every non-zero proposition $a \in L$ is the least upper bound of atoms contained in it. This is a consequence of the following statement:

LEMMA 3. — For every non-zero proposition $a \in L$ one has

$$a = \vee \{ \operatorname{carr} p : p \in a^1 \cap P \}\,.$$

Proof. — As $p \in a^1 \cap P$ implies $\operatorname{carr} p \leqslant a$, we find a to be an upper bound for the set $\{ \operatorname{carr} p : p \in a^1 \cap P \}$. It remains to be shown that a is the least upper bound for this set. Suppose $b \geqslant \operatorname{carr} p$ for all $p \in a^1 \cap P$, then obviously $p(b) = 1$ for every $p \in a^1 \cap P$. Thus we have proved that $a^1 \cap P \subseteq b^1 \cap P$, hence $a \leqslant b$ by $(*)$. This completes the proof of the lemma.

LEMMA 4. — Let T be an arbitrary non-empty subset of P. Then $T^\perp = \emptyset$ if and only if $T^1 = \{ 1 \}$ (1 denotes the greatest element of L).

Proof. — Assume $T^\perp = \emptyset$ and suppose that there exists $a \in T^1$, $a \neq 1$. Since $a' \neq 0$, there exists, by the Axiom 1, a pure state p with $p(a') = 1$, hence $p(a) = 0$. Thus $p \in T^\perp$, which contradicts the assumption.

Conversely, suppose that $T^1 = \{ 1 \}$ and that $T^\perp \neq \emptyset$. Let $p \in T^\perp$; then, of course, $\operatorname{carr} p \perp \operatorname{carr} q$ for all $q \in T$, hence $q((\operatorname{carr} p)') = 1$ for all $q \in T$, that is $(\text{carr } p)' = T^1$. Hence, by the assumption, $(\text{carr } p)' = 1$, i. e. $\text{carr } p = 0$, which is impossible, as carr p is an atom.

LEMMA 5. — For every non-empty subset T of the set P of pure states one has.

$$T^- = \{ p \in P : T^1 \subseteq p^1 \} = T^{11} \cap P .$$

Proof. — CASE I : $T^\perp = \emptyset$.
Then $T^1 = \{ 1 \}$ by Lemma 4, and therefore $\{ p \in P : T^1 \subseteq p^1 \} = P = \emptyset^\perp = T^-$.

CASE II : $T^\perp \neq \emptyset$.

Suppose that $T^1 \subseteq p^1$ for some $p \in P$. We shall show that $p \in T^-$. Indeed, let $q \in T^\perp$, then $\text{carr } q \perp \text{carr } r$ for all $r \in T$, hence $(\text{carr } q)' \in T^1$, hence $(\text{carr } q)' \in p^1$ by the assumption, hence $\text{carr } p \perp \text{carr } q$, which implies $p \perp q$. This shows that $p \perp T^\perp$, or that $p \in T^{\perp\perp} = T^-$, and therefore we have shown the inclusion $\{ p \in P : T^1 \subseteq p^1 \} \subseteq T^-$.

To prove the inverse inclusion suppose $p \in T^-$, and let $a \in T^1$, $a \in L$. We shall show that $a \in p^1$. One can assume without any loss of generality that $a \neq 1$. (Since $T^\perp \neq \emptyset$, such a proposition there exists by Lemma 4.) Then, as it easily follows from Lemma 3, $a' = \vee \{ \text{carr } q : q \in a^0 \cap P \}$, hence $a \geqslant \text{carr } p$, which implies $a \in p^1$, as claimed. In fact, $q \in a^0 \cap P$ implies $q \in T^\perp$, hence $q \perp p$, as $p \in (T^\perp)^\perp$. This implies $\text{carr } p \perp \text{carr } q$ for all $q \in a^0 \cap P$, hence also $\text{carr } p \perp a'$ or, equivalently, $\text{carr } p \leqslant a$.

Let now q be an arbitrary pure state. Applying the Lemma 5 we find that $\{ q \}^- = \{ p \in P : q^1 \subseteq p^1 \}$, hence $\{ q \}^- = \{ q \}$ by the Axiom 2. Hence, as a direct corollary we obtain:

LEMMA 6. — The phase geometry C(P) is atomistic.

THEOREM. — For every $a \in L$ the set $a^1 \cap P$ belongs to C(P), and the mapping $j : a \to a^1 \cap P$ is an orthoinjection of the logic L into the phase geometry C(P).

Proof. — Let $a \in L$, $a \neq 0$. If $p \in (a^1 \cap P)^-$, then $(a^1 \cap P)^1 \subseteq p^1$ by the Lemma 5, hence $p(a) = 1$, i. e. $p \in a^1 \cap P$. This shows that $(a^1 \cap P)^- = a^1 \cap P$, i. e. $a^1 \cap P \in C(P)$. Since also $0^1 \cap P = \emptyset \in C(P)$, we have $a^1 \cap P \in C(P)$ for each $a \in L$.

The implication $a \leqslant b \Rightarrow j(a) \subseteq j(b)$ is obvious, and the converse one follows directly from the Axiom 1 (see (*)). To prove that j preserves the orthocomplementation, let $p \in j(a')$, $0 < a < 1$. Then $p(a) = 0$, which implies $p \perp q$ for all $q \in a^1 \cap P$, that is $p \in j(a)^\perp$. Conversely, since (see Lemma 3) $a = \vee \{ \text{carr } q : q \in a^1 \cap P \}$, $p \in j(a)^\perp = (a^1 \cap P)^\perp$ implies $\text{carr } p \perp a$, hence $p \in a'^1 \cap P = j(a')$. Since also $j(0') = j(1) = j(0)^\perp$ and $j(1') = j(0) = j(1)^\perp$, we have $j(a') = j(a)^\perp$ for all $a \in L$. This completes the proof of the theorem.

REFERENCES

[*1*] G. BIRKHOFF, Lattice theory, *Amer. Math. Soc. Coll. Publ.*, Vol. **25**, New York, 1948.
[*2*] K. BUGAJSKA and S. BUGAJSKI, The lattice structure of quantum logics. *Ann. Inst. Henri Poincaré*, vol. **19**, 1973, p. 333-340.
[*3*] S. P. GUDDER, A superposition principle in physics. *J. Math. Phys.*, vol. **11**, 1970, p. 1037-1040.
[*4*] W. GUZ, A modification of the axiom system of quantum mechanics. *Reports Math. Phys.*, vol. **7**, 1975, p. 313-320.
[*5*] G. W. MACKEY, *The mathematical foundations of quantum mechanics*, W. A. Benjamin, Inc., New York, 1963.
[*6*] M. D. MAC LAREN, Notes on axioms for quantum mechanics, Argonne Nat. Lab. *Report ANL-7065*, 1965.
[*7*] M. J. MACZYNSKI, Boolean properties of observables in axiomatic quantum mechanics. *Reports Math. Phys.*, vol. **2**, 1971, p. 135-150.
[*8*] J. C. T. POOL, Semimodularity and the logic of quantum mechanics. *Commun. math. Phys.*, vol. **9**, 1968, p. 212-228.
[*9*] N. ZIERLER, Axioms for non-relativistic quantum mechanics. *Pacific J. Math.*, vol. **11**, 1961, p. 1151-1169.

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