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Section A :
Physique théorique.

Equilibrium configurations and small oscillations of some dynamical systems

by

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ABSTRACT. — It is shown that the equilibrium configurations for the certain sequence of dynamical systems coincide. The relation between the frequencies of small oscillations for these systems is also established. For some systems the explicit expression of the equilibrium positions and of the frequencies of small oscillations is exhibited.

1. In a recent paper [*1*] Calogero has shown that the equilibrium configurations for the two dynamical systems described by the Hamiltonians

$$H_1 = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + U_1(q) \tag{1}$$

and

$$H_2 = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + U_2(q) \tag{2}$$

with potential energies

$$U_1 = \frac{1}{2}\sum_{j=1}^{n} q_j^2 - \sum_{j>k} \ln (q_j - q_k) \tag{1'}$$

and

$$U_2 = \frac{1}{2}\sum_{j=1}^{n} q_j^2 + \sum_{j>k} (q_j - q_k)^{-2} \tag{2'}$$

coincide.

For the system with Hamiltonian (1) and potential energy (1′) the equilibrium configuration is given by Stieltjes's result [2]: it coincides with the n zeroes of the Hermite polynomial $H_n(x)$ (1). These zeroes also provide the equilibrium configuration for the second problem.

In the present paper it is shown that the equilibrium configurations for the systems (1) and (2) coincide for a more general class of potential energies than those given above in (1′) and (2′). In particular, for the systems of type VB_n in terminology of reference [4] using Stieltjes results [2] and the results of reference [5] it is proved that the equilibrium configuration coincides with square roots of the zeroes of the Laguerre polynomial $L_n^\mu(x)$ of degree n (1).

The relation between the frequencies of small oscillations of these dynamical systems is also established, and for some systems the explicit expression for the frequencies is also given. For the systems of type VA_{n-1} (with potential energy of type (2′)) and for those of type VB_n the explicit expression for the frequencies can be also obtained using the results of reference [6].

2. Consider the sequence of dynamical systems with Hamiltonians of type (1) and (2) and potential energy of the type

$$U_s = \frac{1}{2}(\partial_i U_1)(\partial_i \partial_j U_1) \ldots (\partial_k \partial_l U_1)(\partial_l U_1), \qquad s = 2, 3, \ldots \tag{3}$$

where

$$\partial_i U_1 = \partial U_1/\partial q_i, \qquad s\text{-degree } U_s \text{ in } U_1 .$$

Let us suppose that the system (1) has a stable isolated equilibrium position $q^0 = (q_1^0, \ldots, q_n^0)$.

THEOREM 1. — The system with potential energies (3) for any s has a stable isolate equilibrium position which coincide with the equilibrium position of the system (1). The eigenfrequencies of small oscillations around the equilibrium position for the system with potential energy U_s (3) are equal to the sth degree of the corresponding eigenfrequencies of small oscillations for the system (1). The normal modes of small oscillations for all the considered systems coincide.

Proof. — The potential energy of the system (1) near the equilibrium position has the form

$$U_1(q) = U_1(q^0) + \frac{1}{2}a_{ij}(q_i - q_i^0)(q_j - q_j^0) + \ldots \tag{4}$$

where

$$a_{ij} = (\partial_i \partial_j U_1(q))_{q=q^0}, \tag{5}$$

(1) For the classical polynomials we use the notation of book [3].

det $a \neq 0$ and the matrix a is positive definite. Now from (3) it follows that at $q \sim q^0$

$$U_s(q) = \frac{1}{2} a_{ij}^{(s)}(q_i - q_i^0)(q_j - q_j^0) + \ldots, \qquad s = 2, 3, \ldots \tag{6}$$

where

$$a_{ij}^{(s)} = [a^s]_{ij} \tag{7}$$

The statement of the theorem is now evident.

3. Let us now apply the theorem to the system (1) with

$$U_1(q) = \frac{1}{2}\sum_{j=1}^{n} q_j^2 - \sum_{\alpha \in R_+} g_\alpha \ln q_\alpha \tag{8}$$

Here $R = \{ \alpha \}$ is one of the so called root systems connected with a finite group W generated by reflections (Coxeter group), i. e. certain system of vectors in n-dimensional space, R_+ is the subsystem of positive roots, $q_\alpha = (q, \alpha)$, g_α are positive constants which are equal to one another for equivalent roots, i. e. for roots which are related one another by transformations of the group W (²).

All these systems are completely classified and we list below their definitions.

A_{n-1} : $U_1(q)$ has the form (1′)

$B_n = C_n = BC_n$:

$$U_1(q) = \frac{1}{2}q^2 - g\sum_{j>k}[\ln (q_j - q_k) + \ln (q_j + q_k)] - f\sum_{j=1}^{n} \ln q_j \tag{9}$$

D_n : $U_1(q)$ is given by the formula (9) with $f = 0$

$$I_2(2n) : U_1(q) = \frac{1}{2}q^2 - g \ln (r^n \cos n\varphi) - f \ln (r^n \sin n\varphi) \tag{10}$$

$$I_2(2n + 1) : U_1(q) = \frac{1}{2}q^2 - g \ln [r^{(2n+1)} \cos (2n + 1)\varphi] \tag{11}$$

$$q = (q_1, q_2) = (r \cos \varphi, r \sin \varphi)$$

There exist moreover six exceptional systems, H_3, H_4, F_4, E_6, E_7 and E_8 which we will not describe here (see [7]). Let us note only the isomorphism between these systems : $A_2 \sim I_2(3)$; $B_2 \sim C_2 \sim I_2(4)$ and $I_2(6)$ is isomorphic to the exceptional system G_2.

(²) A detailed description of the groups generated by reflections and the root systems can be found in [7].

Now from formula (8) using the relation (3) we immediately obtain

$$U_2(q) = \frac{1}{2}\Big(q_j - \sum_{\alpha\in R_+} g_\alpha \alpha_j q_\alpha^{-1}\Big)\Big(q_j - \sum_{\beta\in R_+} g_\beta \beta_j q_\beta^{-1}\Big)$$

$$= \frac{1}{2}\sum_{j=1}^{n} q_j^2 + \sum_{\alpha\in R_+} g_\alpha^2 \,|\,\alpha\,|^2 q_\alpha^{-2} - C + F \quad (12)$$

where

$$C = \sum_{\alpha\in R_+} g_\alpha \quad (13)$$

$$F = \frac{1}{2} \sum_{\substack{\alpha\neq\beta \\ \alpha,\beta\in R_+}} g_\alpha g_\beta (\alpha, \beta) q_\alpha^{-1} q_\beta^{-1} \quad (14)$$

Now we use .the identity

$$F \equiv 0 \,. \quad (15)$$

The proof of this identity is given in paper [5] although under some restrictions. Note however that this proof is valid also in the general case.

COROLLARY 1. — The absolute minimum of the potential energy

$$U = \frac{1}{2}\sum_{j=1}^{n} q_j^2 + \frac{1}{2}\sum_{\alpha\in R_+} g_\alpha^2 \,|\,\alpha\,|^2 q_\alpha^{-2} \quad (16)$$

is equal to

$$U(q^0) = \sum_{\alpha\in R_+} g_\alpha \quad (17)$$

and the equilibrium configuration q^0 of the system (3) with $U_s(q)$ of type (16) is determined by the system of equations

$$q_j^0 = \sum_{\alpha\in R_+} g_\alpha \alpha_j (q_\alpha^0)^{-1} \qquad \text{or} \qquad q_j^0 = \sum_{\alpha\in R_+} g_\alpha^2 q_\alpha^{-3} \,|\,\alpha\,|^2 \alpha_j \quad (18)$$

Using the relation (8) it is not difficult to obtain the explicit expression for the matrix a in the formula (4):

$$a_{ij} = \delta_{ij} + \sum_{\alpha\in R_+} g_\alpha \alpha_i \alpha_j (q_\alpha^0)^{-2} \quad (19)$$

Here the vector $q^{(0)}$ is determined by the system of equations (18).

THEOREM 2. — The eigenvalues of the matrix a for all systems (1) with $U_1(q)$ of type (8) possibly with the exception of systems of type H_3 and H_4 are equal to the orders of the basic invariants $\nu_1, \ldots, \nu_n$ of the group W. (Note that in the case of the system with $U_1(q)$ of type (1'), $\nu_j = j$).

Proof. — From the results of References [8] and [5] there follows that in all cases except possibly for the systems of type $I_2(n)$, H_3 and H_4, for certain values of the constants g_α, there exist n functions $B_{\nu_1}(p, q), \ldots, B_{\nu_n}(p, q)$ that depend on $\bar{p}_j$ polynomially and on t exponentially. Namely,

$$B_{\nu_j}(t) = B_{\nu_j}(0) \exp(-i\nu_j t).$$

From this it follows that the systems under consideration are periodic in time and have frequencies $\omega_j = \nu_j$. For the system of type $I_2(n)$ the frequencies ω_1 and ω_2 can be obtained by direct calculations and one finds $\omega_1 = \nu_1 = 2$ and $\omega_2 = \nu_2 = n$. Due to theorem 1 these frequencies coincide with the eigenvalues of matrix a; hence for certain values of the constants g_α theorem 2 is proved. Rescaling the space and time coordinates these results are immediately extended to the case with arbitrary coupling constants.

COROLLARY. — Using theorem 1 one concludes that the frequencies of small oscillations of the system (1) are $\sqrt{\nu_1}, \ldots, \sqrt{\nu_n}$ while those for the system (3) are $\nu_1^{s/2}, \ldots, \nu_n^{s/2}$.

Remark 1. — The theorem 2 is presumably valid also for the exceptional systems of type H_3 and H_4.

Remark 2. — For the systems of type A_{n-1} and B_n this theorem may be verified by direct calculation of the matrix a and by comparison with the matrices A and B of Reference [6].

Remark 3. — From relation (19) there follows that

$$a_{ij}q_j^0 = 2q_i^0 \tag{20}$$

Hence the vector $q^0 = (q_1^0, \ldots, q_n^0)$ provides the normal mode of the small oscillations of the system with U_2 with frequency $\omega = \nu_1 = 2$.

Remark 4. — From formula (16) we get the explicit expression for the matrix $a^{(2)}$

$$a_{ij}^{(2)} = a_{ik}a_{kj} = \delta_{ij} + 3\sum_{\alpha\in R_+} g_\alpha^2 \,|\, \alpha \,|^2 q_\alpha^{-4}\alpha_i\alpha_j \tag{21}$$

Since $a^{(2)} = a^2$ there follows that the eigenvalues of the matrix $a^{(2)}$ are equal to the squares $\nu_1^2, \ldots, \nu_n^2$ of the basic invariants of the group W.

We conclude this section reporting the values of the invariants $\nu_1, \nu_2, \dots \nu_n$ for the finite groups generated by reflections:

$$\begin{aligned}
A_{n-1} &: 2,\ 3,\ \dots,\ n \\
B_n = C_n &: 2,\ 4,\ \dots,\ 2n \\
D_n &: 2,\ 4,\ \dots\ 2(n-1),\ n \\
I_2(n) &: 2,\ n \\
H_3 &: 2,\ 6,\ 10 \\
H_4 &: 2,\ 12,\ 18,\ 30 \\
F_4 &: 2,\ 6,\ 8,\ 12 \\
E_6 &: 2,\ 5,\ 6,\ 8,\ 9,\ 12 \\
E_7 &: 2,\ 6,\ 8,\ 10,\ 12,\ 14,\ 18 \\
E_8 &: 2,\ 8,\ 12,\ 14,\ 18,\ 20,\ 24,\ 30\,.
\end{aligned} \tag{22}$$

4. Let us now proceed to consider specific examples.

The system of type A_{n-1} was treated by Calogero [*1*] [*6*], who, using Stieltjes results, proved that the equilibrium positions coincide with the zeroes of the Hermite polynomial $H_n(x)$.

Let us consider the system of type VB_n. For this system the equations (18) take the form

$$q_j = g \sum_k{}' [(q_j - q_k)^{-1} + (q_j + q_k)^{-1}] + f q_j^{-1} \tag{23}$$

Going from the variables q_j to the new variables

$$r_j = g^{-1} q_j^2 \tag{24}$$

we obtain instead of (23) the new system

$$\sum_k{}' (r_j - r_k)^{-1} = \frac{1}{2}(1 - g^{-1} f r_j^{-1}) \tag{25}$$

However from Stieltjes' results it follows that the quantities r_j are the zeroes of the Laguerre polynomial $L_n^{(\mu)}(x)$ where

$$\mu = g^{-1} f - 1 \tag{26}$$

Hence we prove

THEOREM 3. — The equilibrium position $q^{(0)}$ of the system (2) with potential energy

$$U(q) = \frac{1}{2} \sum_{j=1}^{n} q_j^2 + g^2 \sum_{j>k} [(q_j - q_k)^{-2} + (q_j + q_k)^{-2}] + \frac{f^2}{2} \sum_{j=1}^{n} q_j^{-2} \tag{27}$$

is given by the formula

$$q_j^0 = \sqrt{g r_j} \tag{28}$$

where r_j are the zeroes of the Laguerre polynomial

$$L_n^{(\mu)}(r_j) = 0, \qquad j = 1, \ldots, n, \qquad \mu = g^{-1}f - 1 \tag{29}$$

Explicit evaluation of the matrix a for this system and comparison of a with the matrix B of Reference [6] implies that the eigenvalues of a are 2, 4, ... $2n$, in agreement with theorem 2.

5. In those cases when for the system with $\omega = 0$ L − M pair is known (these are the cases A_{n-1} and B_n) for the system with $\omega \neq 0$ we can use the results of paper [8].

A. Let us consider first the case A_{n-1}. If we introduce the matrices

$$\begin{aligned} X_{jk} &= (1 - \delta_{jk})(q_j - q_k)^{-1} \\ M_{jk} &= \delta_{jk}\left(\sum_l (q_k - q_l)^{-2}\right) - (1 - \delta_{jk})(q_j - q_k)^{-2} \\ Q_{jk} &= \delta_{jk} q_k, \quad Q^{\pm} = Q \pm X \end{aligned} \tag{30}$$

Then the result of Ref. [8] imply for the equilibrium configuration

$$[M, Q^{\pm}] = \pm Q^{\pm}, \qquad (\text{but } [Q^-, Q^+] \neq \lambda M) \tag{31}$$

Hence if u_k are the eigenvectors of the matrix M with eigenvalues μ_k, then from (31) we get

$$Q^{\pm} u_k = \lambda_k^{\pm} u_{k\pm1}, \qquad Q^- u_1 = 0, \qquad Q^+ u_n = 0 \tag{32}$$

and $\mu_{k\pm1} = \mu_k \pm 1$.

It is not difficult to show further that

$$\begin{aligned} u_1 &= (1, 1, \ldots, 1) \quad \text{and} \quad \mu_1 = 0 \\ u_2 &= (q_1^0, \ldots, q_n^0) \quad \text{and} \quad \mu_2 = 1 \end{aligned} \tag{33}$$

From this we obtain

$$\mu_k = k - 1, \qquad k = 1, 2, \ldots, n \tag{34}$$

Comparing the matrices a and M we find

$$a = 1 + M \tag{35}$$

Hence the eigenvalues of the matrix a are $1, \ldots, n$. Note that the matrices $Q^{\pm}$ are nilpotent $(Q^{\pm})^n = 0$ and the quantities $B_k = \mathrm{Sp}\,(Q^-)^k$ vanish at equilibrium. In particular from this there follows that $\mathrm{Sp}\, Q^2 = - \mathrm{Sp}\, X^2$.

From this there also follows that all the eigenvectors of the matrix M, that identify normal modes of the small oscillations, are easily obtained from the vector u_1 through the formula $u_k = c_k(Q^+)^{k-1}u_1$ and have the form

$$u_{k+1} = \left(P_k(q_1^0), \ldots, P_k(q_n^0)\right) \tag{36}$$

where $P_k(q)$ is a polynomial of degree k having definite parity.

B. In the case C_n the matrices X, M, Q, $Q^{\pm}$ are of order $2n$ (or $2n + 1$)

$$\begin{aligned} X_{jk} &= (1 - \delta_{jk})(1 - \delta_{j-k})(q_j - q_k)^{-1} + \delta_{j,-k} f q_j^{-1} ; \\ &\qquad\qquad\qquad\qquad j, k = -n, \ldots, -1, 1, \ldots, n \\ Q_{jk} &= \delta_{jk} q_k , \\ M_{jk} &= \delta_{jk}\left(\sum_{l \neq \pm j} (q_j - q_l)^{-2} + f q_j^{-2}\right) \\ &\qquad - (1 - \delta_{jk})(1 - \delta_{j,-k})(q_j - q_k)^{-2} - f \delta_{j,-k} q_j^{-2} \end{aligned} \tag{37}$$

The relations (31) for these matrices are also valid. Hence just as in the previous case the eigenvalues of the matrix M are $0, 1, \ldots, (2n - 1)$. (Analogous results are also obtained for the matrix M of order $(2n + 1)$).

Note that in the case under consideration the eigenvectors have the form

$$u_k = (v_k, (-1)^{k+1} v_k) , \qquad u_k = \left(v_1^{(k)}, \ldots, v_n^{(k)}\right) \tag{38}$$

and the matrix M has the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{11} \end{pmatrix} \tag{39}$$

From this there follows that

$$a = M_{11} + M_{12} + 2 \tag{40}$$

and

$$a v_{2j-1} = 2j v_{2j-1} ; \qquad j = 1, 2, \ldots, n$$

Moreover

$$(M_{11} - M_{12} + 1) v_{2j} = 2j v_{2j} ; \qquad j = 1, 2, \ldots, n \tag{41}$$

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