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Section A :
Physique théorique.

Definite material scheme in relativistic magnetohydrodynamics

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ABSTRACT. — The stress-energy tensor for Definite Material Scheme in relativistic magnetohydrodynamics is defined by considering the partial pressures along the space-like congruences. It is shown that (*i*) the conformally flat class one space-time is incompatible with the relativistic magnetofluid. (*ii*) the magnetofluid admits Einstein collineation with respect to flow vector if the flow is geodesic and expansion-free.

1. INTRODUCTION

Action principle in general theory of relativity was used by Maugin [*1*] to obtain the field equations for thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability. A stress-energy tensor found by Maugin differs from that of Lichnerowicz's [*2*] by additional magnetic field term. For unit value of magnetic permeability, both field equations agree in form. Lichnerowicz's [*2*] field equations are studied extensively (see Date [*3*] and the references sited in it). Recently, Patil and Date [*4*] have investigated some properties of Maugin's [*1*] magnetofluid. Explicit occurance of magnetic field term in equation of continuity for Maugin's magnetofluid gives rise to interesting results.

It is proposed to study the Definite Magnetofluid Scheme (DMS) by modifying Maugin's formulation on the lines of Definite Material Schemes in relativistic hydrodynamics (Lichnerowicz [5]) and in relativistic magnetohydrodynamics (Shaha [6]). For DMS, we specify partial pressures p_α ($\alpha = 1, 2, 3$) along the space-like congruences. On formulating field equations in Section 2, differential identities are obtained in Section 3. A necessary and sufficient condition for Riemannian manifold to be of embedding class one (Barnes [7]) is used in Section 4 to prove that DMS is incompatible with conformally flat class one space-time. In Section 5, Einstein Collineation (see Khade and Radhakrishna [8], Asgekar and Date [9] [10]) in DMS is investigated.

Throughout the investigations, arbitrary coordinates x^a are used to describe the 4-dimensional Riemannian space-time with metric g_{ab} of signature + 2. Latin indices *a*, *b*, *c*... range and sum over 1 to 4. Covariant differentiation is denoted by a semi-colon and partial differentiation by a comma. The round brackets and square brackets around the suffixes denote symmetrization and anti-symmetrization respectively.

2. FIELD EQUATIONS

We consider the stress-energy tensor for DMS as

$$T_{ab} = (r + m)u_a u_b + (p_1 + m)v_a v_b + (p_2 + m)w_a w_b + (p_3 + m)n_a n_b - \mu h_a h_b + \mu/2(1 - \mu)\boldsymbol{h}^2 g_{ab} \quad (2.1)$$

where,

$$3p - p_1 - p_2 - p_3 = 0, \quad (2.2)$$

$$-u^a u_a = v^a v_a = w^a w_a = n^a n_a = 1,$$

$$u^a h_a = u^a v_a = u^a w_a = u^a n_a = v^a w_a = v^a n_a = w^a n_a = 0,$$

and

$$h^a h_a = \boldsymbol{h}^2, \qquad m = \frac{\mu}{2}\boldsymbol{h}^2 \quad (2.3)$$

Here, r is the energy density, p_α ($\alpha = 1, 2, 3$) are partial pressures, h^a is magnetic field vector, u^a is the 4-velocity vector and v^a, w^a, n^a are space-like vectors.

The Einstein equations are

$$R_{ab} - \frac{1}{2}R g_{ab} = KT_{ab}, \quad (2.4)$$

where K is a constant of gravitation.

The Maxwell equations are

$$(u^a h^b - u^b h^a)_{;b} = 0 . \tag{2.5}$$

Equations connecting thermodynamical variables are:

$$\mathrm{T}d\mathrm{S} - di + \frac{dp}{\rho} = 0 , \tag{2.6}$$

$$r = \rho(1 + \varepsilon) , \tag{2.7}$$

$$i = \varepsilon + \frac{p}{\rho} , \tag{2.8}$$

where T is the proper temperature, S is the specific entropy, ρ is the matter density, ε is the internal energy density and i is the enthalpy.

On using

$$g_{ab} = - u_a u_b + v_a v_b + w_a w_b + n_a n_b . \tag{2.9}$$

the equation (2.1) can be written as

$$\mathrm{T}_{ab} = (r + m)u_a u_b + [p_1 + m(2 - \mu)]v_a v_b + [p_2 + m(2 - \mu)]w_a w_b + [p_3 + m(2 - \mu)]n_a n_b - \mu h_a h_b . \tag{2.10}$$

Contracting (2.10) in succession with u^a, v^a, w^a, and n^a we get

$$\mathrm{T}_{ab} u^a = - [r + m\mu]u_b , \tag{2.11}$$

$$\mathrm{T}_{ab} v^a = [p_1 + m(2 - \mu)]v_b - \mu h_a h_b v^a , \tag{2.12}$$

when h^a is along v^a we have

$$\mathrm{T}_{ab} v^a = [p_1 - m\mu]v_b , \tag{2.13}$$

$$\mathrm{T}_{ab} w^a = [p_2 + m(2 - \mu)]w_b - \mu h_a h_b w^a , \tag{2.14}$$

$$\mathrm{T}_{ab} n^a = [p_3 + m(2 - \mu)]n_b - \mu h_a h_b n^a . \tag{2.15}$$

It is clear that v^a, w^a, n^a will correspond to the eigen-vectors of T_{ab} when magnetic field vector is along v^a or w^a or n^a. If $p_1 = p_2 = p_3 = p$ we get the Maugin's [*1*] stress-energy tensor for the perfect general relativistic magnetofluid, viz.

$$\underset{(\mathrm{M})}{\mathrm{T}_{ab}} = (r + p + 2m)u_a u_b + [p + m(2 - \mu)]g_{ab} - \mu h_a h_b . \tag{2.16}$$

The Hawking-Ellis [*11*] energy condition for all known forms of matter and all predicted equations of states is

$$\mathrm{T}_{ab} u^a u^b \geqslant (- 1/2)\mathrm{T} \tag{2.17}$$

This condition is satisfied for the magnetofluid if

$$r + 3p + 2m(2 - \mu) \geqslant 0 . \tag{2.18}$$

Thus, if $\mu \geqslant 2$, then the energy condition is automatically satisfied.

3. DIFFERENTIAL IDENTITIES

If the space-like vector v^a is chosen along the magnetic field vector h^a, the stress-energy tensor for DMS becomes

$$T_{ab} = (r + m\mu)u_a u_b + (p_1 - m\mu)v_a v_b + [p_2 + m(2 - \mu)]w_a w_b + [p_3 + m(2 - \mu)]n_a n_b . \quad (3.1)$$

Consequently

$$\begin{aligned} R_{ab} &= K\left[T_{ab} - \frac{1}{2}Tg_{ab}\right] \\ &= K[(r + m\mu)u_a u_b + (p_1 - m\mu)v_a v_b \\ &\quad + \{ p_2 + m(2 - \mu) \}w_a w_b + \{ p_3 + m(2 - \mu) \}n_a n_b \\ &\quad - \frac{1}{2} g_{ab}\{ - r + 3p + 4m(1 - \mu) \}] . \end{aligned} \quad (3.2)$$

The equation of continuity is

$$ru^a{}_{;a} + \{ r + m(\mu - 1) \}_{;a}u^a + u_{(a;b)}(p_1 v^a v^b + p_2 w^a w^b + p_3 n^a n^b) = 0 . \quad (3.3)$$

and the differential system of the steam lines is

$$\begin{aligned} &(r + m\mu)u_{a;b}u^b + (ru^b)_{;b}u_a + (p_1 v_a v^b + p_2 w_a w^b + p_3 n_a n^b)_{;b} \\ &+ \{ 2m(1 - \mu/2) \}_{;b}p_a{}^b + \frac{\mu^2}{2}\boldsymbol{h}^2{}_{;b}u_a u^b + 2mu_a u^b{}_{;b} - (2mv_a v^b)_{;b} = 0 . \end{aligned} \quad (3.4)$$

Here p_{ab} is the projection operator defined by

$$p_{ab} = g_{ab} + u_a u_b . \quad (3.5)$$

It satisfies the relations

$$p_a{}^a = 3, \qquad p_{ab}p^{bc} = p_a{}^c , \qquad p_{ab}u^b = 0 . \quad (3.6)$$

4. SPACE-TIMES OF EMBEDDING CLASS ONE IN DMS

A necessary and sufficient condition for n-dimensional Riemannian manifold to be of embedding class one is the existence of a symmetric tensor satisfying the Gauss and Codazzi equations [7]

$$R^{ab}_{cd} = 2ef^a{}_{[c}f_{d]}{}^b , \quad (4.1)$$

$$f^a{}_{[b;c]} = 0 . \quad (4.2)$$

where $e = +1$ or -1 when the normal to the manifold is space-like or time-like respectively.

In the four-dimensional case, let

$$f_{ab} = -\lambda_0 u_a u_b + \lambda_1 v_a v_b + \lambda_2 w_a w_b + \lambda_3 n_a n_b. \tag{4.3}$$

On contracting equation (4.1), we get

$$R^a_b = e(f^a_c f^c_b - f f^a_b), \tag{4.4}$$

where

$$f = f^c_c.$$

From equation (4.3) and (4.4) we have

$$\begin{aligned} R^a_b = e[&(\lambda_0 u^a u_c + \lambda_1 v^a v_c + \lambda_2 w^a w_c + \lambda_3 n^a n_c) \\ &\times (-\lambda_0 u^c u_b + \lambda_1 v^c v_b + \lambda_2 w^c w_b + \lambda_3 n^c n_b) \\ &- (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)(-\lambda_0 u^a u_b + \lambda_1 v^a v_b + \lambda_2 w^a w_b + \lambda_3 n^a n_b)]. \end{aligned} \tag{4.5}$$

Substituting (4.5) in Einstein equations (2.4), we have

$$\begin{aligned} KT_{ab} = e[&-(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3)u_a u_b + (\lambda_0\lambda_2 + \lambda_0\lambda_3 + \lambda_2\lambda_3)v_a b_b \\ &+ (\lambda_0\lambda_1 + \lambda_0\lambda_3 + \lambda_1\lambda_3)w_a w_b + (\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2)n_a n_b]. \end{aligned} \tag{4.6}$$

Comparison of the equations (3.1) and (4.6) yields

$$K(r + m\mu) = e(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3), \tag{4.7}$$

$$K(p_1 - m\mu) = e(\lambda_0\lambda_2 + \lambda_0\lambda_3 + \lambda_2\lambda_3), \tag{4.8}$$

$$K\{ p_2 + m(2 - \mu) \} = e(\lambda_0\lambda_1 + \lambda_0\lambda_3 + \lambda_1\lambda_3), \tag{4.9}$$

$$K\{ p_3 + m(2 - \mu) \} = e(\lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2). \tag{4.10}$$

Pandey and Gupta [*12*] have shown that if the space-time is conformally flat and class one then

$$\lambda_1 = \lambda_2 = \lambda_3. \tag{4.11}$$

Therefore, equations (4.7, 4.10) produce

$$p_1 - m\mu = p_2 + m(2 - \mu) = p_3 + m(2 - \mu). \tag{4.12}$$

Thus for the perfect general relativistic magnetofluid (where $p_1 = p_2 = p_3 = p$), $2m = \mu \boldsymbol{h}^2 = 0$. Therefore, we conclude that the perfect general relativistic magnetofluid scheme is incompatible with conformally flat class one space-time.

5. EINSTEIN COLLINEATION

THEOREM. — *In DMS Einstein Collineation with respect to flow vector implies that the flow is geodesic and expansion-free.*

Proof. — The Einstein Collineation [*9*] with respect to flow vector for DMS leads to

$$\begin{aligned} \underset{u}{£}\{ &(r + m\mu)u_a u_b + (p_1 - m\mu)v_a v_b + [p_2 + m(2 - \mu)]w_a w_b \\ &+ [p_3 + m(2 - \mu)]n_a n_b \} = 0. \end{aligned} \tag{5.1}$$

Contracting (5.1) successively we get

$$(r + m\mu)_{;c}u^c = 0\,, \tag{5.2}$$

$$(\gamma_1 + m\mu)\dot{u}_a v^a = 0\,, \tag{5.3}$$

$$(r + \mu m)\dot{u}_a w^a = 0\,, \tag{5.4}$$

and

$$(r + \mu m)\dot{u}_a n^a = 0\,. \tag{5.5}$$

Now the conservation law $(T^a_b u^b)_{;a} = 0$ corresponding to the Einstein Collineation [10] for DMS becomes

$$(r + m\mu)_{;a}u^a + (r + m\mu)u^a{}_{;a} = 0\,. \tag{5.6}$$

By (5.2) and (5.6) we get

$$u^a{}_{;a} = 0 \qquad \text{as} \qquad r + m\mu \neq 0 \tag{5.7}$$

From the equations (5.3), (5.4) and (5.5) we have

$$\dot{u}_a v^a = \dot{u}_a w^a = \dot{u}_a n^a = 0 \Rightarrow \dot{u}_a = 0\,. \tag{5.8}$$

Thus we have

$$\underset{u}{\pounds} T_{ab} = 0 \Rightarrow \dot{u}_a = u^a{}_{;a} = 0\,. \tag{5.9}$$

Proof of the theorem is complete.

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