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The Yukawa quantum field theory: the Matthews-Salam formulas

by

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ABSTRACT. — We prove the Matthews-Salam integral representation for the quantum field theory with the Hamiltonian

$$d\Gamma_b(\mu_1) \otimes I_f + I_b \otimes d\Gamma_f(\omega_1) + H_I,$$

where μ_1 and ω_1 are (rather arbitrary) boson and fermion one-particle operators and H_I is the interaction Hamiltonian of the (cut-off) Yukawa theory.

RÉSUMÉ. — On démontre la représentation intégrale de Matthews et Salam pour la théorie quantique à l'hamiltonien

$$d\Gamma_b(\mu_1) \otimes I_f + I_b \otimes d\Gamma_f(\omega_1) + H_I,$$

où μ_1, ω_1 sont des opérateurs (suffisamment arbitraires) définis sur les sous-espaces à une particule Bose ou Fermi de l'espace Fock et H_I est l'hamiltonien de l'interaction de la théorie de Yukawa à cut-offs.

1. INTRODUCTION

In the present paper we prove the Matthews-Salam formulas [*1*, *2*, *3*, *4*] for the Yukawa interaction in the two-dimensional space-time ($= Y_2$) with a space-time and ultraviolet cut-off. Since we have cut-offs the two-dimensional restriction is not essential. For the main results and references on the Y_2 theory, see, for instance [*5*-*18*].

The proof of the Matthews-Salam formulas has been considered in Refs. [*7*, *9*, *12*, *16*]. The exposition of Ref. [*12*] is rather recapitulative. Gross's arguments have used the fact that the Euclidean fermion function is the kernel of the operator the inverse of which is local. In addition, the Matthews-Salam formulas by itself do not need the existence of the Euclidean fermion fields.

Here we give the proof of the Matthews-Salam representation, which does not depend on the locality of the inverse of the two-point Euclidean fermion function and on the existence of the Euclidean fermion fields. Our proof is close in the spirit to that of Gross [*16*], but instead of locality we use the commutation relations to deduce the Matthews-Salam formulas.

In Ref. [*17*] we use the Matthews-Salam formulas to prove a linear N_τ bound and in Ref. [*18*]—to prove the Lorentz invariance of the Y_2 quantum field theory.

In the following $f^\sim, f^\wedge$ denote the direct and inverse Fourier transform of the function f. We define $\det_n$ as

$$\det_n (1 + A) : = \det \left[(1 + A) \exp \left[\sum_{k=1}^{n-1} (- A)^k/k \right] \right].$$

By $c_1, c_2, \ldots$ we denote strictly positive constants possibly depending on unessential variables.

2. INTEGRAL REPRESENTATION OF MATTHEWS-SALAM

We want to obtain the integral representation of Matthews-Salam type for the Hamiltonian expressions of the form

$$(\Omega_0, \exp(- t_1 H) F_1 \exp((t_1 - t_2)H) F_2 \ldots \exp((t_{n-1} - t_n)H)\Omega_0),$$

where Ω_0 is the free vacuum vector in the Fock space $\mathcal{F}$, F is either a fermion field ψ or its Dirac conjugate $\overline{\psi} : = \psi^+ \gamma_0$, or a function of the boson field $\phi(0, x)$ at time zero, $H : = H_0 + H_I$, where H_I is the interaction Hamiltonian of the Y_2 theory and

$$\begin{aligned} H_0 &= H_{0,b} + H_{0,f}, \\ H_{0,b} &= d\Gamma_b(\mu_1) \otimes I_f, \\ H_{0,f} &= I_b \otimes d\Gamma_f(\omega_1). \end{aligned}$$

We suppose that ω_1 is the positive self-adjoint operator in the one-particle fermion complex Hilbert space $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ and that μ_1 is the positive self-adjoint operator in the one-particle boson real Hilbert space $L_2(\mathbb{R})$ and that

$$0 < c_1 \leq \mu_1 \leq c_2 \mu_0^n,$$

where μ_0 is the operator of multiplication in the momentum space by the function $\mu_0(k) = (k^2 + m_0^2)^{1/2}$.

To deduce the Matthews-Salam formulas it is technically convenient to consider the boson measure as a measure on continuous sample paths which consists of a Banach space Q of continuous functions from the real line to some Hilbert space. The corresponding construction of the space Q is analogous to that given by Gross [*16*, p. 190-192] and may be described as follows.

Let $L_2^r(\mathbb{R})$ be the completion of the real Schwartz space $\mathcal{S}_{Re}(\mathbb{R})$ in the norm

$$\| f \|_{L_2^r(\mathbb{R})} = (\mu_0^{2r} f, f)^{1/2}_{L_2(\mathbb{R})}.$$

The dual space of $L_2^r(\mathbb{R})$ may be identified with $L^{-r}_2(\mathbb{R})$ by the pairing

$$\langle f, g \rangle = \int dx f(x) g(x).$$

The two-point boson function G is given by ($f, g \in \mathcal{S}_{Re}(\mathbb{R}^2)$)

$$\langle G, fg \rangle = \int dt ds (\Omega_0, \phi(0, f(t,.)) \exp(-\mid t-s \mid H_0) \phi(0, g(s,.)) \Omega_0)$$
$$= (2\pi)^{-1} \int dt ds (\mu_0^{-1/2} f(t,.), \exp(-\mid t-s \mid \mu_1) \mu_0^{-1/2} g(s,.))_{L_2(\mathbb{R})}.$$

This expression may be rewritten in the following form

$$(2\pi)^{-1} \int dt ds (f(t,.), \exp(-\mid t-s \mid \mu_2) g(s,.))_{L_2^{-1/2}(\mathbb{R})},$$

where μ_2 is the generator of the semigroup

$$\mu_{1/2} \exp(-t\mu_1) \mu_{-1/2}$$

and

$$\mu_{1/2} : L_2(\mathbb{R}) \to L_2^{-1/2}(\mathbb{R}), \qquad \mu_{-1/2} : L_2^{-1/2}(\mathbb{R}) \to L_2(\mathbb{R})$$

are the continuous linear operators generated by the operators $\mu_0^{1/2}$ and $\mu_0^{-1/2}$, respectively.

For sufficiently large β the operator

$$\alpha = \mu_0^{-\beta}(1 + x^2)^{-\beta}$$

is a Hilbert-Schmidt operator on $(L_2^{-1/2}(\mathbb{R}))' = L_2^{1/2}(\mathbb{R})$. Fix such a β. Let $\mathcal{H}$ be the real Hilbert space, which is the completion of $L_2^{1/2}(\mathbb{R})$ in the norm

$$\| \alpha(\mu_2^{-1/2})' f \|_{L_2^{1/2}(\mathbb{R})},$$

where a prime denotes the adjoint operator on $(L_2^{-1/2}(\mathbb{R}))' = L_2^{1/2}(\mathbb{R})$.

By Proposition 5.1 [*16*] $\mathscr{H}$ may be identified with the state space for a Gaussian process $\phi(t)$ with continuous sample paths and covariance

$$\int_{\text{path space}} d\mu(\phi(.)) \langle \phi(t,.), f \rangle \langle \phi(s,.), g \rangle$$
$$= (\exp(-\mid t-s \mid \mu_2) f, g)_{L_2^{-1/2}(\mathbb{R})}, \qquad f, g \in \mathscr{S}_{\mathrm{Re}}(\mathbb{R}).$$

We may regard the path space measure μ as a measure on the space of continuous functions from $\mathbb{R}$ into $\mathscr{H}$. The seminorms

$$\| \phi \|_n = \sup \{ \| \phi(t,.) \|_{\mathscr{H}} : n \leq t \leq n+1 \}$$

on this space are measurable and by Fernique's theorem [*19*, *20*] are integrable. Since the process is stationary, they all have the same distribution. Hence $\sum_{n=-\infty}^{\infty} a_n \| \phi \|_n$ is integrable whenever $\sum_{n=-\infty}^{\infty} | a_n | < \infty$. In particular, it follows that the norm

$$\| \phi \| = \sup_{-\infty < t < \infty} \{ \| \phi(t,.) \|_{\mathscr{H}} (1+t^2)^{-1} \} \tag{2.1}$$

is finite almost every where and so the Banach space Q (= the completion of $\mathscr{S}_{\mathrm{Re}}(\mathbb{R}^2)$ in the norm (2.1)) is a set of path space measure one. We henceforth take μ as a countably additive Borel measure on the Banach space of continuous path Q.

We remark that the Gaussian measure μ is hypercontractive and has, at least, the primitive Markov property in the temporal direction [*21*], but, generally speaking, it has no Markov property in the spatial direction [*22*].

We want to obtain the Matthews-Salam formulas for the interactions of the form

$$H_I = \int dx [: \bar{\psi}_\sigma(x) \Gamma \psi_\sigma(x) : W(\phi_k(0,x)) g(x) + W_1(\phi_k(0,x)) g_1(x)],$$

$\Gamma = \alpha + i\beta\gamma_5$ with real α, β and $\gamma_5 = \gamma_5^+$ and where $\psi_\sigma(x) = \int dy \sigma(x-y)\psi(y)$ and $\sigma(x) = \sigma(-x)$ is a function from $\mathscr{S}_{\mathrm{Re}}(\mathbb{R})$. Let, for simplicity, W be a bounded analytic real-valued function on $\mathbb{R}$, $W_1 \in \mathscr{S}_{\mathrm{Re}}(\mathbb{R})$ and an ultraviolet cut-off k be made with the help of a function from $\mathscr{S}_{\mathrm{Re}}(\mathbb{R})$.

Let us define the unnormalized Schwinger functions. Let $\chi(t)$ be piecewise constant function with a bounded support taking the values 0 or 1. Let $H(t) = H_0 + \chi(t) H_I$.

The Euclidean propagator for $H(t)$ is the strongly continuous two parameter family of bounded operator $U(t, s)$ in the Fock space, defined for $t \leq s$ and for points where $\chi(t)$ is continuous by the equations

$$\begin{aligned} \partial U(t,s)/\partial s &= -U(t,s)H(s), \qquad t < s, \\ U(t,t) &= 1. \end{aligned} \tag{2.2}$$

The existence of a unique solution of the equations (2.2) follows from the self-adjointness and boundedness below of $H_0 + \chi(t)H_I$ and from the fact that $\chi(t)$ is a piecewise constant function. The resulting family $U(t, s)$ is strongly continuous and satisfies (2.2) on $\mathscr{D}(H_0)$ for all but finitely many s.

Since $\chi(t) = 0$ for sufficiently large $|t|$, then $(\Omega_0, U(t, s)F)$ is independent of t for large negative t and $(F, U(t, s)\Omega_0)$ is independent of s for large s. We write $(\Omega_0, U(-\infty, t)F)$ and $(F, U(t, \infty)\Omega_0)$ for the corresponding limits as $t \to -\infty$ or $s \to \infty$.

Let $\mathbb{R}_0^n = \{ x \in \mathbb{R}_0^n \mid x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ x_i \neq x_j \text{ for } i \neq j \}$.

We define the unnormalized Schwinger functions for the Hamiltonian $H(t)$.

We put

$$S_0 = (\Omega_0, U(-\infty, \infty)\Omega_0).$$

If $t_1 < t_2 \ldots < t_n$, then we put

$$S_n(t_1, F_1, \ldots, t_n, F_n) = (\Omega_0, U(-\infty, t_1)F_1U(t_1, t_2)F_2 \ldots U(t_n, \infty)\Omega_0),$$

where F are either bounded functions of the time zero beson field, or the time zero fermion fields

$$\psi(f) = \sum_{\alpha=1}^{2} \int dx \psi_\alpha(x) f_\alpha(x) \qquad \text{or} \qquad \bar{\psi}(f) = \sum_{\alpha=1}^{2} \int dx \bar{\psi}_\alpha(x) f_\alpha(x).$$

If $(t_1, t_2, \ldots, t_n) \in \mathbb{R}_0^n$ we put

$$S_n(t_1, F_1, \ldots, t_n, F_n) = (-1)^{p(\pi)} S_n(t_{\pi(1)}, F_{\pi(1)}, \ldots, t_{\pi(n)}, F_{\pi(n)}),$$

where π is the permutation that puts $t_1, \ldots, t_n$ in increasing order. That is, $t_{\pi(1)} \ldots t_{\pi(n)}$. And $p(\pi)$ is the number of transpositions of fermion fields in the permutation π. The time ordering operation T may be used to express S_n as

$$S_n(t_1, F_1, \ldots, t_n, F_n) = (\Omega_0, TU(-\infty, t_1)F_1U(t_1, t_2) \ldots \Omega_0).$$

T reorders the factors following it in accordance with increasing time and introduces the appropriate sign change.

We note that by charge symmetry all S_n with unequal number of ψ and $\bar{\psi}$ are zero. Moreover, the functions S_n are continuous and uniformly bounded in $\mathbb{R}_0^n$ and are locally integrable in $\mathbb{R}^n$.

Let

$$\begin{aligned} &\mathfrak{S}_{k+2m}(t_1, F_1, \ldots, t_k, F_k; f_1, \ldots, f_m; f_{m+1}, \ldots, f_{2m}) \\ &= \int ds_1 \ldots ds_{2m} S_{k+2m}(t_1, F_1, \ldots, t_k, F_k, s_1, F_{k+1}(s_1), \ldots, s_{2m}, F_{2m}(s_{2m})), \end{aligned}$$

where $F_1, \ldots, F_k$ are bounded functions of the time zero boson field and

$$F_{k+j}(s_{k+j}) = \psi(f_j(s_j)),$$
$$F_{k+m+j}(s_{k+m+j}) = \bar{\psi}(f_{k+m+j}(s_{k+m+j}))$$

for $j = 1, \ldots, m, f_1, \ldots, f_{2m} \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and having bounded supports.

Now we define the operator $V_\phi : L_2(\mathbb{R}^2) \otimes \mathbb{C}^2 \to L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$.
For each point $\phi \in Q$ we put

$$v_{\phi(t)}(x) = W\left(\int dy \phi(t, y) k(x - y) \right) g(x).$$

Then $v_{\phi(t)}(.)$ is a continuous function on $\mathbb{R}$ with compact support for each t and we define the operator $V_\phi(t) : L_2(\mathbb{R}) \otimes \mathbb{C}^2 \to L_2(\mathbb{R}) \otimes \mathbb{C}^2$ by

$$V_\phi(t)u = \sigma * \{ \Gamma v_{\phi(t)}(\sigma * u) \}, \tag{2.3}$$

that is, $V_\phi(t)$ is a multiplication by $\Gamma v_{\phi(t)}$ surrounded by convolution by σ. We define V_ϕ as the operator on $L_2(\mathbb{R}; L_2(\mathbb{R}) \otimes \mathbb{C}^2) = L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ given by

$$(V_\phi f)(t) = V_\phi(t) f(t). \tag{2.4}$$

We also introduce the Euclidean fermion two-point function

$$S(t_1 - t_2, x_1 - x_2)_{\alpha\beta} = \begin{cases} (\Omega_0, \psi_\alpha(x_1) \exp((t_1 - t_2)H_0)\bar{\psi}_\beta(x_2)\Omega_0) & \text{for } t_1 \leq t_2 \\ -(\Omega_0, \bar{\psi}_\beta(x_2) \exp((t_2 - t_1)H_0)\psi_\alpha(x_1)\Omega_0) & \text{for } t_1 > t_2 \end{cases} \tag{2.5}$$

The following theorem is valid.

THEOREM 2.1. — (Matthews-Salam formulas). *Let* $f_1, \ldots, f_m$,

$$f_{m+1}, \ldots, f_{2m} \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$$

and have bounded supports. Let $F_1, \ldots, F_k$ *be bounded function of the time zero boson field. Let* $(t_1, \ldots, t_k) \in \mathbb{R}^k$. *Then*

$$\mathfrak{S}_{k+2m}(t_1, F_1, \ldots, t_k, F; f_1, \ldots, f_m ; f_{m+1}, \ldots, f_{2m})$$
$$= \int d\mu(\phi) \det_2 (1 + SV_\phi\chi)(-1)^{m(m-1)/2} \left\langle \bigwedge_{j=1}^{m} f_j, \bigwedge_{j=1}^{m} \{ (1 + SV_\phi\chi)^{-1} S f_{m+j} \} \right\rangle$$
$$\prod_{i=1}^{k} F_k(\phi(t_k)) \exp\left[- \int dt dx W_1(\phi(t, x))\chi(t) g_1(x) \right], \tag{2.6}$$

where S *is the integral operator in* $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ *with the integral kernel* $S(s - t, x - y)_{\alpha\beta}$, χ *is the multiplication operator in* $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ *by the function* $\chi(t)$, *the product* $\Lambda_{j=1}^{m}$ *are ordered with larger j to the right, and denotes the duality on* $\Lambda^m[L_2(\mathbb{R}^2) \otimes \mathbb{C}^2]$, *i. e., the bilinear (rather than sesquilinear) form.*

3. THE MATTHEWS-SALAM FORMULAS FOR AN EXTERNAL, TIME-DEPENDENT FIELD

We prove the Matthews-Salam formulas for the interaction of fermions with an external field.

Let

$$H_f(t) = H_{0,f} + \lambda H_I(t) + v_1(t),$$

where

$$H_I(t) = \int dx[:\bar{\psi}_\sigma(x)\Gamma\psi_\sigma(x):v(t, x)]$$

and $v(t, x)$, $v_1(t)$ are piecewise constant in t and smooth in x functions with bounded supports.

Let $V(t)$ and V be the operators defined by (2.3) and (2.4) where $v(t, x)$ stands instead of $v_{\phi(t)}$.

The Euclidean propagator for $H_f(t)$ is the strongly continuous in $\mathscr{F}_f$ (= the fermion Fock space) two parameter family of bounded operators $U_f(t, s)$ defined for $t \leq s$ and for points where $H_I(t)$ is continuous by the equations

$$\begin{aligned} \partial U_f(t, s)/\partial s &= - U_f(t, s)H_f(s), \qquad t < s, \\ U_f(t, t) &= 1. \end{aligned} \tag{3.1}$$

The existence of a unique solution of the equations (3.1) follows from the self-adjointness and positiveness of $H_{0,f}$ and from the fact that $H_I(t)$ is a piecewise constant function taking the values in the set of bounded operators. The resulting family $U_f(t, s)$ is strongly continuous in $\mathscr{F}_f$ and satisfies (3.1) on $\mathscr{D}(H_{0,f})$ for all but finitely many s.

Similarly define the unnormalized Schwinger functions for the theory with an external field.

Let

$$S_0^f = (\Omega_{0,f}, U_f(-\infty, \infty)\Omega_{0,f}).$$

If $t_1 < t_2 < \ldots < t_n$ and $f_1, \ldots, f_n$ are two-component test functions we put

$$S_n^f(t_1, f_1, \ldots, t_n, f_n) = (\Omega_{0,f}, U_f(-\infty, t_1)\psi^{\#}(f_1)U_f(t_1, t_2) \ldots \Omega_{0,f}),$$

where $\psi^{\#}$ is either ψ or $\bar{\psi}$ at time zero. If $(t_1, t_2, \ldots, t_n) \in \mathbb{R}_0^n$, then we put

$$S_n^f(t_1, f_1, \ldots, t_n, f_n) = \operatorname{sgn} \pi S_n^f(t_{\pi(1)}, f_{\pi(1)}, \ldots, t_{\pi(n)}, f_{\pi(n)}), \tag{3.2}$$

where π is the permutation that puts $t_1, \ldots, t_n$ in increasing order.

Let $f_1, \ldots, f_{2m} \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and have bounded supports. We put

$$\begin{aligned} &\mathfrak{S}_{2m}^f(f_1, \ldots, f_m; f_{m+1}, \ldots, f_{2m}) \\ &\qquad = \int dt_1 \ldots dt_{2m} S_m^f(t_1, f_1(t_1), \ldots, t_{2m}, f_{2m}(t_{2m})), \end{aligned} \tag{3.3}$$

where the test functions $f_1, \ldots, f_m$ correspond to the fields ψ and the test functions $f_{m+1}, \ldots, f_{2m}$—to the fields $\bar{\psi}$.

To calculate the Schwinger functions we prove some lemmata.

Let $\psi(f) = \psi^{(+)}(f) + \psi^{(-)}(f)$ be the decomposition of ψ in the creation-annihilation operators and let $(.)_t$ be the operator on $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ such that

$$f_t = \exp(-t\omega_1)f.$$

LEMMA 3.1. — *Let $t < s$, then*

$$U_f(t, s)\psi^{(+)}(f) = \psi^{(+)}(f_{s-t})U_f(t, s) - \lambda \int_t^s dr\, U_f(t, r)\psi((S(s-r)V(r))^{tr} f)U_f(r,s),$$

$$\psi^{(-)}(f)U_f(t, s) = U_f(t, s)\psi^{(-)}(f_{s-t}) - \lambda \int_t^s dr\, U_f(t, s)\psi((S(t-r)V(r))^{tr} f)U_f(r,s),$$

where $S(t)$ is the integral operator in $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ with the kernel $S(t, x-y)_{\alpha\beta}$ and $(\ )^{tr}$ denotes the transpose (i. e., the adjoint) of an operator in $L_2(\mathbb{R}) \otimes \mathbb{C}^2$.

Proof of Lemma 3.1. — Let us consider the case of an annihilation operator. Let us write the commutation relations

$$\psi^{(-)}(f) \exp(-tH_{0,f}) = \exp(-tH_{0,f})\psi^{(-)}(f_t), \tag{3.4}$$

$$\left[\psi^{(-)}(f_t), \int dx : \bar{\psi}_\sigma(x)\Gamma\psi_\sigma(x) : v(r, x)\right] = \psi((S(-t)V(r))^{tr} f). \tag{3.5}$$

Using the fact that $H_I(t)$ is a piecewise constant function, we apply the Trotter formula writing it in the following form

$$U_f(t, s) = s\text{-}\lim_{n\to\infty} \prod_{r\in A_n(t,s)} U_n(r), \tag{3.6}$$

where the factors in the product are ordered from left to right in correspondence with the increase of r and where

$$U_n(r) = \exp\left(-\frac{1}{n}H_{0,f}\right)\left(1 - \frac{1}{n}H_I(r)\right)$$

and

$$A_n(t, s) = \{ x \in \mathbb{R} \mid t < x < s,\ x = i/n,\ i \in \mathbb{Z}^1 \}.$$

Commuting $\psi^{(-)}(f)$ to the right and using the commutation relation (3.4) and (3.5) we obtain

$$\psi^{(-)}(f)U_f(t, s) = U_f(t, s)\psi^{(-)}(f_{s-t}) - \lambda\ s\text{-}\lim_{n\to\infty} \sum_{r\in A_n(t,s)} n^{-1} \prod_{u\in A_n(t,r)} U_n(u) \exp\left(-\frac{1}{n}H_{0,f}\right) \psi((S(t-r)V(r))^{tr} f) \prod_{u\in A_n(r,s)} U_n(u). \tag{3.7}$$

Since the convergence in (3.6) is uniform in s, t for bounded s, t, $U_f(t, s)$ is strongly continuous in s, t and the operator $\psi((S(t-r)V(r))^{tr} f)$ is strongly piecewise continuous in r, so taking the limit in the right side of (3.7) we obtain the statement of the lemma.

In the same way we consider the case of a creation operator. Lemma 3.1 is proved.

Lemma 3.1 implies the following assertion

LEMMA 3.2. — *Let $f_1, \ldots, f_{2m} \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and have bounded supports. Then*

$$\mathfrak{S}_{2m}^f(f_1, \ldots, f_m ; f_{m+1}, \ldots, f_{2m})$$
$$= \sum_{i=1}^{m} (-1)^{m+i} S(f_1, f_{m+i}) \mathfrak{S}_{2m-2}^f(f_2, \ldots, f_m ; f_{m+1}, \ldots, f/_{m+i}, \ldots, f_{2m})$$
$$- \lambda \mathfrak{S}_{2m}^f((SV)^{tr} f_1, f_2, \ldots, f_m ; f_{m+1}, \ldots, f_{2m}). \quad (3.8)$$

Here $f/_{m+i}$ denotes that the corresponding fermion field is missed, S *is the integral operator in $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ with the kernel $S(s-t, x-y)_{\alpha\beta}$ and $(\ )^{tr}$ denotes the transpose (i. e., the adjoint) of an operator on $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$.*

Proof of Lemma 3.2. — We write $\psi(f_1(t_1)) = \psi^{(+)}(f_1(t_1)) + \psi^{(-)}(f_1(t_1))$ and, using eqs. (3.2), (3.3), the commutation relation of Lemma 3.1, we commute $\psi^{(+)}$ to the left and $\psi^{(-)}$ to the right. It is easy to see that as a result we obtain eq. (3.8). Lemma 3.2 is proved.

LEMMA 3.3. — *Let the operator $1 + \lambda SV$ be invertible in $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and let $\mathfrak{S}_0^f := (\Omega_{0,f}, U(-\infty, \infty)\Omega_{0,f}) \neq 0$. Then the operator*

$$D = (1 + \lambda SV)^{-1} S$$

is the integral operator in $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ with the kernel

$$\langle u, D(s, t)v \rangle_{L_2(\mathbb{R}) \otimes \mathbb{C}^2} = \mathfrak{S}_2^f(s, u ; t, v)/\mathfrak{S}_0^f, \ u, v \in L_2(\mathbb{R}) \otimes \mathbb{C}^2,$$

where $\langle\ ,\ \rangle$ is the duality on $L_2(\mathbb{R}) \otimes \mathbb{C}^2$, i. e., bilinear (rather than sesquilinear) form.

The kernel $D(s, t)$ is strongly continuous in (s, t) for $s \neq t$. The jump at $s = t$ is

$$D(t_+, t) - D(t_-, t) = -\gamma_0 I.$$

Proof of Lemma 3.3. — Lemma 3.2 implies that

$$\mathfrak{S}_2^f((1 + \lambda(SV)^{tr})f_1 ; f_2) = S(f_1, f_2)\mathfrak{S}_0^f.$$

If $1 + \lambda SV$ is invertible, then $1 + \lambda(SV)^{tr}$ is also invertible. The above equality implies that

$$\langle f_1, Df_2 \rangle_{L_2(\mathbb{R}^2) \otimes \mathbb{C}^2} = \mathfrak{S}_2^f(f_1 ; f_2)/\mathfrak{S}_0^f.$$

This representation and the strong continuity of the Euclidean propagator $U_f(t, s)$ for $t < s$ imply the statements of Lemma 3.3. Lemma 3.3 is proved.

LEMMA 3.4. — *Let* A *be the Hilbert-Schmidt operator. Suppose that* g *is an entire function such that* $g(0) = a$ *and*

$$dg(\lambda)/d\lambda = \operatorname{Tr}\,[(1 + \lambda A)^{-1}A - A]g(\lambda)$$

for those complex λ *in some neighbourhood of zero for which* $1 + \lambda A$ *has a bounded inverse. Then*

$$g(\lambda) = a \det{}_2 (1 + \lambda A).$$

Proof of Lemma 3.4. — The proof of the lemma is analogous to the proof of Lemma 4.1 of Gross [*16*]. Lemma 3.4 is proved.

LEMMA 3.5. — SV *is a Hilbert-Schmidt operator in* $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ *and*

$$\mathfrak{S}_0^f = (\Omega_{0,f}, U_f(-\infty, \infty)\Omega_{0,f}) = \mathfrak{S}_0^f\,|_{\lambda=0} \det{}_2 (1 + \lambda SV).$$

Proof of Lemma 3.5. — It is evident that the kernel of the operator SV is square integrable and, hence, SV is a Hilbert-Schmidt operator in $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$.

To prove Lemma 3.5 it is sufficient to prove that $\mathfrak{S}_0^f$ is an entire function of the coupling constant λ and that for sufficiently small in absolute value complex λ

$$d\mathfrak{S}_0^f/d\lambda = \mathfrak{S}_0^f \operatorname{Tr}\,[(1 + \lambda SV)^{-1}SV - SV].$$

Then the assertion follows from Lemma 3.4.

For this purpose let us consider the operator $A = ((1 + \lambda SV)^{-1}S - S)V$ supposing that the operator $(1 + \lambda SV)^{-1}$ exists as a bounded operator. Then we show that the operator A satisfies the conditions of Lemma 4.6 [*16*].

Since

$$A = -(1 + \lambda SV)^{-1}(SV)^2,$$

SV is Hilbert-Schmidt, $(1 + \lambda SV)^{-1}$ is a bounded operator, so A is a trace class operator.

By Lemma 3.3 both $D(s, t) = [(1 + \lambda SV)^{-1}S](s, t)$ and $S(s - t)$ have a (strong) jump of $-\gamma_0 I$ at $t = s$. Thus, the operator

$$R(s, t) = D(s, t) - S(s - t)$$

is strongly continuous in (s, t). Thus, for any compact operator C, $R(s, t)C$ is norm continuous in (s, t). Writing v as a product of two functions we see that the operator $V(t)$ is a product of two Hilbert-Schmidt operators, and, hence, is a trace class operator. For each given t we may find a compact strictly positive operator $C(t)$ such that $V(t) = C(t)W(t)$, where $W(t)$ is trace class. Let $V(t_j)$, $j = 1, \ldots, n$, be the distinct nonzero values of $V(.)$

and $C = \left(\sum_{j=1}^{n} C(t_j)^2\right)^{1/2}$. Then, $C^{-1}C_j$ is bounded and $W'(t) = C^{-1}V(t)$ is piecewise constant and is trace class for all t. But then

$$A(s, t) = (R(s, t)C)W'(t)$$

is piecewise continuous in (s, t) from $\mathbb{R}^2$ into the space of trace class operators and is continuous in s into this space for each t. This verifies hypothesis (b) of Lemma 4.6 [*16*]. The hypotheses (a) and (c) are also fulfilled since $\| R(s, t) \|$ is bounded in s and t.

Thus, applying Lemma 4.6 [*16*], we have

$$\mathrm{Tr}\,[(1 + \lambda SV)^{-1}SV - SV] = \int dt\, \mathrm{tr}_{L_2(\mathbb{R})\otimes\mathbb{C}^2}[(((1 + \lambda SV)^{-1}S)(t_-, t) - S(t_-, t))V(t)]. \quad (3.9)$$

Then, Lemma 4.4 [*16*] implies that

$$d\mathfrak{S}_0^f/d\lambda = -(\Omega_{0,f}, U_f(-\infty, t)H_I(t)U_f(t, \infty)\Omega_{0,f}). \quad (3.10)$$

Let $u_1, u_2, \ldots$ be an orthonormal basis in $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ and

$$a_{ij}(t) = (u_i, V(t)u_j)_{L_2(\mathbb{R})\otimes\mathbb{C}^2}.$$

Since $V(t)$ is trace class for each t, the series

$$-\sum_{i,j} a_{ij}(t)(\psi(u_i^*)\bar{\psi}(u_j) - (\Omega_{0,f}, \psi(u_i^*)\bar{\psi}(u_j)\Omega_{0,f}))$$

converges in operator norm and is equal to the operator $H_I(t)$. The equality follows from the fact that these operators satisfy the same commutation relations with $\bar{\psi}(f)$, $\psi(f)$ and because of irreducibility of the set of the operators

$$\bigcup_{f\in L_2(\mathbb{R})\otimes\mathbb{C}^2} \{\bar{\psi}(f), \psi(f)\}.$$

Now

$$(\Omega_{0,f}, U_f(-\infty, t)H_I(t)U_f(t, \infty)\Omega_{0,f}) = -\sum_{i,j} a_{ij}(t)[(\Omega_{0,f}, U_f(-\infty, t)\psi(u_i^*)\bar{\psi}(u_j)U_f(t, \infty)\Omega_{0,f}) - (\Omega_{0,f}, \psi(u_i^*)\bar{\psi}(u_j)\Omega_{0,f})(\Omega_{0,f}, U_f(-\infty, \infty)\Omega_{0,f})].$$

Lemma 3.3 implies that the last expression may be written in the form

$$-\mathfrak{S}_0^f \sum_{i,j} a_{ij}(t)((u_i, D(t_-, t)u_j) - (u_i, S(-0)u_j)) = -\mathfrak{S}_0^f \sum_i ((u_i, D(t_-, t) - S(-0))V(t)u_i) = -\mathfrak{S}_0^f\, \mathrm{tr}_{L_2(\mathbb{R})\otimes\mathbb{C}^2}[(D(t_-, t) - S(-0))V(t)].$$

Integration over t and eqs. (3.9), (3.10) imply the equality

$$d\mathfrak{S}_0^f/d\lambda = \mathfrak{S}_0^f \operatorname{Tr}\,[(1+\lambda SV)^{-1}SV - SV].$$

$(1 + \lambda SV)$ is invertible for small $|\lambda|$, thus, this equation holds for all λ in some neighbourhood of zero. We apply Lemma 3.4 to conclude the proof of Lemma 3.5. Lemma 3.5 is proved.

LEMMA 3.6. — *If* $(1 + \lambda SV)$ *has a bounded inverse in* $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$, *then*

$$\mathfrak{S}_0^f(f_1, \ldots, f_m; f_{m+1}, \ldots, f_{2m}) = (-1)^{m(m-1)/2} \left\langle \bigwedge_{j=1}^{m} f_j, \bigwedge_{j=1}^{m} (1+\lambda SV)^{-1} S f_{m+j} \right\rangle \det{}_2 (1+\lambda SV)(\mathfrak{S}_0^f|_{\lambda=0}).$$

Proof of Lemma 3.6. — Lemma 3.6 follows from the statements of Lemmas 3.2 and 3.5. Lemma 3.6 is proved.

4. THE PROOF OF THE MATTHEWS-SALAM FORMULAS

LEMMA 4.1. — *The Gaussian measure* μ *is nondegenerate, i. e., the only closed subspace of* Q *of measure* 1 *is* Q.

Proof of Lemma 4.1. — For the covariance G of the measure may be written the following expression ($f, g \in \mathcal{S}_{Re}(\mathbb{R}^2)$)

$$\begin{aligned}\langle G, fg \rangle &= \int dt ds \int d\mu \langle \phi(t,.), f(t,.) \rangle \langle \phi(s,.), g(s,.) \rangle \\ &= \int dp (\mu_0^{-1/2} f^{\sim}(-p,.),\ \mu_1(p^2+\mu_1^2)^{-1}\mu_0^{-1/2} g^{\sim}(p,.))_{L_2(\mathbb{R})} \\ &= (\mu_0^{-1/2} f^{\sim}(-.,.),\ \mu_1(p_1^2+\mu_1^2)^{-1}\mu_0^{-1/2} g^{\sim}(.,.))_{L_2(\mathbb{R};L_2(\mathbb{R}))}. \end{aligned} \quad (4.1)$$

Let $\mathcal{H}(G)$ be the completion of $\mathcal{S}_{Re}(\mathbb{R}^2)$ in the scalar product (4.1) (it is easy to see that $\mu_1(p_1^2 + \mu_1^2)^{-1}$ is a positive operator in $L_2(\mathbb{R}; L_2(\mathbb{R}))$.

The inequality $0 < c_1 \leq \mu_1 \leq c_2 \mu_0^n$ and Theorem VI.2.21 [23] imply that in $L_2(\mathbb{R}; L_2(\mathbb{R}))$

$$\mu_1(p_1^2 + \mu_1^2)^{-1} \geq (p_1^2/c_1 + c_2\mu_0^n)^{-1}$$

and

$$\langle G, ff \rangle \geq (f^{\sim}(-.,-.),\ \mu_0(p_2)^{-1}(p_1^2/c_1 + c_2\mu_0(p_2)^n)^{-1} g^{\sim}(.,.))_{L_2(\mathbb{R}^2)}. \quad (4.2)$$

Let $\mathcal{H}_1$ be the completion of $\mathcal{S}_{Re}(\mathbb{R}^2)$ in the scalar product

$$(f^{\sim}(-.,-.),(p_1^2/c_1 + c_2\mu_0(p_2)^n)^{-1}\mu_0(p_2)^{-1} g^{\sim}(.,.))_{L_2(\mathbb{R}^2)}.$$

The inequality (4.2) implies that

$$\mathscr{H}(G) \subset \mathscr{H}_1.$$

With respect to the pairing

$$\langle f, g \rangle = \int d^2x f(x) g(x) \tag{4.3}$$

$\mathscr{H}'_1$ may be identified with the completion of $\mathscr{S}_{\mathrm{Re}}(\mathbb{R}^2)$ in the scalar product

$$(f^{\sim}(-\,.\,,\,-\,.), (p_1^2/c_1 + c_2\mu_0(p_2)^n)\mu_0(p_2)g^{\sim}(.\,,.))_{L_2(\mathbb{R}^2)}.$$

Thus, with respect to the pairing (4.3)

$$\mathscr{H}'_1 \subset \mathscr{H}(G)'$$

and so $\mathscr{H}(G)' \supset \mathscr{S}_{\mathrm{Re}}(\mathbb{R}^2)$.

If, now, a linear subspace A has a nonzero measure, $A \subset Q$, then, since μ is the normal distribution over $\mathscr{H}(G)$, $A \supset \mathscr{H}(G)'$ and, thus, $A \supset \mathscr{S}_{\mathrm{Re}}(\mathbb{R}^2)$ and so is dense in Q in Q norm and if A is closed it coincides with Q. Lemma 4.1 is proved.

LEMMA 4.2. — *The operator* $1 + \lambda SV_\phi\chi$ *has a bounded inverse in*

$$L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$$

for μ *almost every* $\phi \in Q$. *Equivalently,* $\det_2(1 + \lambda SV_\phi\chi) \neq 0$ μ *almost everywhere on* Q.

Proof of Lemma 4.2. — μ is nondegenerate mean zero Gaussian measure on a separable real Banach space and the proof of the lemma follows from Lemma 5.4 [*16*] and is analogous to the proof of Theorem 5.2 [*16*]. The equivalence of the invertibility and of the nonvanishing of the determinant follows from Corollary 6.3 [*24*]. Lemma 4.2 is proved.

Proof of Theorem 2.1. — The proof may be given in the same way as that of Theorem 5.5 [*16*] with Gross's $H_{0,b}$, $H_{0,f}$ being replaced by our $H_{0,b}$, $H_{0,f}$, etc.

Theorem 2.1 is proved.

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