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WAWRZYNIEC GUZ

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An improved formulation of axioms for quantum mechanics

by

Wawrzyniec GUZ
Institute of Physics, Gdansk University, 80-952 Gdansk, Poland

ABSTRACT. — We modify the quantum axiomatics presented recently in this journal by excluding from it some nonphysical assumptions. This is done in such a way that all important consequences deduced previously still remain valid.

1. INTRODUCTION

The purpose of the present paper is to modify the quantum axiomatics presented recently in [*2*] by excluding from it the nonphysical assumptions in such a way that all important consequences deduced previously will remain valid.

The main objection may be connected with the Axiom 7 in [*2*], as this assumption seems to be unrealistic from a physical point of view. In the axiom system presented here we replace this postulate by a weaker one, expressed in terms of intensities of beams only, and similarly, the partial ordering and the orthogonality are here defined by using the concept of the intensity of a beam, and not by pure algebraic terms, as it was done in [*2*].

The axioms that we assume here split naturally into four groups (similarly as those in [*2*]), so that the consequences of these are also divided into four parts (see Section 3). All the consequences of our previous axiom system still remain true in the present axiomatic framework, so we find our new axiom system to be a good alternative for the quantum logic axiomatic scheme.

2. AXIOMS

With every physical system we associate a triple (B, F, d) consisting of two nonvoid sets, the set B (*beams*) and the set F of some mappings from B to B (*filters*), and the function d: $B \to R_+$ (*intensity functional*) from B to the nonnegative real numbers, such that the following requirements are satisfied (compare [2], where also the interpretation of the axioms that we assume can be found):

AXIOM 1

$$\forall_{a \in F} a^2 = a$$

AXIOM 2

$$\forall_{a \in F} \forall_{m \in B} d(am) \leqslant d(m)$$

AXIOM 3 *a*

$$[\forall_{a \in F} d(am_1) = d(am_2)] \Rightarrow m_1 = m_2$$

AXIOM 3 *b*

$$[\forall_{m \in B} d(a_1 m) = d(a_2 m)] \Rightarrow a_1 = a_2$$

AXIOM 4

$$\exists_{m_0 \in B} d(m_0) = 0$$

It follows from Axioms 2 and 3 *a* that the beam m_0 must necessarily be unique; we denote it by 0. Evidently, $a0 = 0$ for all $a \in F$.

AXIOM 5. — O, I $\in$ F, where O and I are the zero and the identity transformations from B to B, respectively (defined respectively by $Om = 0$ and $Im = m$ for all $m \in B$).

AXIOM 6

$$\forall_{m_1, m_2 \in B} \forall_{t_1, t_2 \geqslant 0} \exists_{m \in B} \forall_{a \in F} d(am) = t_1 d(am_1) + t_2 d(am_2)$$

Axiom 6 imposes on the set B of beams the structure of a cone [2], since owing to Axiom 3a the beam m appearing in Axiom 6 must necessarily be unique. It is denoted by $t_1 m_1 + t_2 m_2$ and interpreted as the *mixture of* m_1 *and* m_2 *in the proportion* $t_1 : t_2$ *and with the intensity*

$$t_1 d(am_1) + t_2 d(am_2).$$

Also we have for all $a \in F$, $m_1, m_2 \in B$, $t_1, t_2 \in R_+$ (see [2]):

$$d(a(t_1 m_1 + t_2 m_2)) = t_1 d(am_1) + t_2 d(am_2) = d(t_1(am_1) + t_2(am_2)).$$

In particular,

$$d(t_1 m_1 + t_2 m_2) = t_1 d(m_1) + t_2 d(m_2) \quad , \quad \text{all } m_1, m_2 \in B, \ t_1, t_2 \in R_+.$$

Now, for $a, b \in \mathrm{F}$ we define:

$$a \leqslant b \quad \text{iff} \quad d(bam) = d(am) \qquad \text{for all } m \in \mathrm{B},$$
$$a \perp b \quad \text{iff} \quad d(bam) = d(abm) = 0 \qquad \text{for all } m \in \mathrm{B}.$$

Observe that $a \perp b$ if and only if $ab = ba = 0$.

AXIOM 7

$$a \leqslant b (a, b \in \mathrm{F}) \Rightarrow \forall_{m \in \mathrm{B}} d(am) = d(abm)$$

AXIOM 8

$$a \perp b (a, b \in \mathrm{F}) \Rightarrow \exists_{c \in \mathrm{F}} \forall_{m \in \mathrm{B}} d(am) + d(bm) = d(cm)$$

Note that if $a \perp b$, then for all $m \in \mathrm{B}$ we have $d(cm) = d((a + b)m)$, where $a + b$ denotes the ordinary sum of a and b. Note also that by Axiom 3 *b* the filter c is necessarily unique; we denote it by $a \dotplus b$. We must remember, however, that in general $a \dotplus b$ may differ from $a + b$.

Axioms 7 and 8 have two important consequences; they are:

$$a \leqslant b \Rightarrow \forall_{m \in \mathrm{B}} d(am) \leqslant d(bm),$$
$$a \perp b \Rightarrow \forall_{m \in \mathrm{B}} d(am) + d(bm) \leqslant d(m).$$

AXIOM 9

$$\forall_{a \in \mathrm{F}} \exists_{b \in \mathrm{F}, b \perp a} \forall_{m \in \mathrm{B}} d(am) + d(bm) = d(m)$$

The filter b defined in Axiom 9, being necessarily unique by Axiom 3 *a*, is denoted as a'.

AXIOM 10. — *i*) For every non-zero filter $a \in \mathrm{F}$ there exists a homogeneous beam [1] p such that $d(ap) = d(p)$.

Moreover:

ii) If, at the same time, $b \not\perp a$, then the beam p can be chosen in such a way that $d(bp) > 0$.

AXIOM 11. — For every homogeneous beam p there exists a filter $a \in \mathrm{F}$ such that $d(ap) = d(p)$ and $d(aq) < d(q)$ for all homogeneous beams q that are not proportional to p.

AXIOM 12. — *i*) Every filter $a \in \mathrm{F}$ transforms homogeneous beams into homogeneous ones.

Moreover:

ii) $a \in \mathrm{F}$, when restricted to the set B_h of homogeneous beams, is a positively homogeneous mapping, that is, for $p \in \mathrm{B}_h$ and $t \in \mathrm{R}_+$

$$a(tp) = t.ap.$$

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[1] A non-zero beam m is said to be *homogeneous* if it cannot be written in the form $m = t_1 m_1 + t_2 m_2$, where t_1, t_2 are positive real numbers and m_1, m_2 are two other non-zero beams, being not proportional to one another.

iii) $a \leqslant b$ $(a, b \in \mathrm{F})$ implies $a = ba = ab$ on B_h.

Remark. — The assumption *iii*) can easily be obtained as a consequence of a stronger axiom, which however possesses more physical justification; this is the following one:

iii') $a \leqslant b$ $(a, b \in \mathrm{F}) \Rightarrow ba, ab \in \mathrm{F}$.

AXIOM 13. — For any sequence $\{ e_i \}_{i=1}^{\infty}$ of pairwise orthogonal atomic filters [2] there exists a filter $a \in \mathrm{F}$ such that for every homogeneous beam $p \in \mathrm{B}_h$

$$d(ap) = \sum_{i=1}^{\infty} d(e_i p).$$

Remark. — In the sequel we will often write $m(a)$ instead of $d(am)$.

3. CONSEQUENCES

(A) CONSEQUENCES OF AXIOMS 1-9:

(1) For every $a \in \mathrm{F}$ we have $a \dotplus a' = \mathrm{I}$ and $a'' = a$.
(2) $a \leqslant b \Rightarrow a \perp b'$.
(3) $ba = 0 \Rightarrow a \leqslant b'$; in particular, $a \perp b \Rightarrow a \leqslant b'$.
(4) $a \leqslant b \Rightarrow b' \leqslant a'$.
(5) $(\mathrm{F}, \leqslant, ')$ is a partially ordered orthocomplemented set.
(6) If a, b, c are pairwise orthogonal, then $a \perp b \dotplus c$.
(7) $a \perp b \Rightarrow a \dotplus b = a \vee b$ [3].
(8) F is orthomodular, *i. e.*, $a \leqslant b$ implies $b = a \vee c$ for some $c \perp a$.

Proofs. — (1) Obvious.

(2) For each $m \in \mathrm{B}$ we have $d(ab'm) = d(abb'm) \leqslant d(bb'm) = 0$, hence $d(ab'm) = 0$, and by Axiom 9,

$$d(b'am) = d(am) - d(bam) = d(am) - d(am) = 0.$$

Therefore $a \perp b$.

(3) Assume $d(bam) = 0$ for all $m \in \mathrm{B}$. Then

$$d(am) = d((b \dotplus b')am) = d(bam) + d(b'am) = d(b'am),$$

all $m \in \mathrm{B}$, which means that $a \leqslant b'$.

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(2) A filter $e \in \mathrm{F}$ is said to be *atomic* if $e \neq 0$ and if $a \leqslant e$ $(a \in \mathrm{F})$ implies either $a = e$ or $a = 0$.

(3) The symbols $\vee$ and $\wedge$ are used to denote the least upper bound and the greatest lower bound in F, respectively.

(4) Follows from (2) and (3).

(5) Among the requirements for a partial ordering only the transitivity is here not quite evident. Assume $a \leqslant b$ and $b \leqslant c$. Then, for all $m \in \mathrm{B}$, $d(cam) = d(am) - d(c'am) = d(am) - d(c'b'am)$, since $c' \leqslant b'$ by (4), but $d(c'b'am) = 0$, the latter being a consequence of the inequality

$$d(c'b'am) \leqslant d(b'am) = 0.$$

In view of (1) and (4), to prove that $'$ is an orthocomplementation on F it suffices to show that $a \leqslant b$ and $a \leqslant b'$ will imply $a = 0$. But this is indeed the case, as we then have for each $m \in \mathrm{B}$

$$d(am) = d(bam) = d(b'am);$$

hence

$$2d(am) = d((b \dotplus b')am) = d(am),$$

so that $d(am) = 0$, all $m \in \mathrm{B}$, and therefore $a = 0$.

(6) First, let us observe that $a, b \leqslant a \dotplus b$, since for every $m \in \mathrm{B}$ we have $d((a \dotplus b)am) = d(am) + d(bam) = d(am)$, and similarly

$$d((a \dotplus b)bm) = d(bm).$$

Let now $c \geqslant a, b$. It remains to be shown that $c \geqslant a \dotplus b$, but this follows from (6). Indeed, a, b, c' are pairwise orthogonal, so $a \dotplus b \perp c'$ by (6), that is (see (3)), $a \dotplus b \leqslant c$.

(8) We will follow the well known path, and prove that c may be chosen as $(a \dotplus b')'$. In fact, for every $m \in \mathrm{B}$ one has

$$\begin{aligned} d((a \dotplus b')'m) &= d(m) - d((a \dotplus b')m) \\ &= d(m) - d(am) - d(b'm) = d(bm) - d(am), \end{aligned}$$

so that

$$d((a \dotplus (a \dotplus b')')m) = d(am) + d((a \dotplus b')'m) = d(bm), \quad , \quad \text{all } m \in \mathrm{B},$$

and thus (see Axiom 3 b)

$$a \dotplus (a \dotplus b')' = b.$$

Remark. — Observe that for all $m \in \mathrm{B}$ and all pairs $a, b \in \mathrm{F}$ with $a \leqslant b$ we have $d((b \wedge a')m) = d((a \dotplus b')') = d((b - a)m)$, where $b - a$ is the ordinary difference of b and a. For this reason we will write $b \dot{-} a$ in place of $b \wedge a'$, similarly as we write $b \dotplus a$ instead of $b \vee a$, when $b \perp a$.

(B) CONSEQUENCES OF AXIOMS 1-11:

Having established the statements (1)-(8), we are in a position to prove the following facts (see [*1*]):

(9) F is atomistic, and there is a one—to—one mapping carr: $p \to \operatorname{carr} p$ of the set P of pure states ([4]) onto the set of all atomic filters, such that $\operatorname{carr} p \leqslant a$ if and only if $p(a) = 1$.

([4]) By a *state* we mean any normalized beam $m \in \mathrm{B}$, i. e., satisfying $d(m) = 1$. Any homogeneous state is also called a *pure state*.

(10) The phase geometry C(P, $\perp$) associated with a physical system ([5]) is atomistic.

(11) For every $a \in F$ the set $a^1 := \{ p \in P : p(a) = 1 \}$ belongs to C(P, $\perp$), and the mapping $a \to a^1$ is an orthoinjection of F into C(P, $\perp$).

(C) CONSEQUENCES OF AXIOMS 1-12:

(12) If filters a and b are compatible ([6]), $a \leftrightarrow b$, then $ab = ba = a \wedge b$ on B_h.

Proof. — We begin with a trivial but very useful remark that a beam $am(a \in F, m \in B)$ can be written as $am = m(a)p$, where p is a state satisfying $p(a) = 1$.

Let $a \leftrightarrow b$, *i. e.*, $a = a_1 \dotplus c$ and $b = b_1 \dotplus c$ for some pairwise orthogonal $a_1, b_1, c \in F$. It is sufficient to show that $ab = ba = c$ on B_h, since $c = a \wedge b$ (see, *e. g.* [3]). Since, by (6), $a_1 \perp b_1 \dotplus c = b$, we have for every $m \in B_h$

$$0 = a_1 bm = a_1 abm = m(b)p(a)a_1 q,$$

where p, q are pure states satisfying $p(b) = q(a) = 1$ such that $bm = m(b)p$, $ap = p(a)q$.

Hence either $a_1 q = 0$, whenever $m(b)p(a) \neq 0$, which leads to $q(a_1) = 0$, or $m(b)p(a) = 0$, which is equivalent to $abm = 0$, since $abm = m(b)p(a)q$. For the case when $a_1 q = 0$ we have

$$q(c) = q(a) - q(a_1) = q(a) = 1;$$

hence

$$cq = q(c)r = r,$$

r being a pure state with $r(c) = 1$.

Since carr $r \leqslant c$, we get

$$(\text{carr } r)q = (\text{carr } r)cq = (\text{carr } r)r = r;$$

([5]) The *phase geometry* C(P, $\perp$) is defined as a family of all subsets $S \subseteq P$ satisfying $S^{\perp\perp} = S$, where $S^{\perp}$ is defined as the set of all pure states $p \in P$ such that $p \perp q$ for all $q \in S$, and the orthogonality relation $\perp$ in P is defined by

$$p \perp q \text{ iff } p(a) = 1 \ \& \ q(a) = 0 \text{ for some } a \in F.$$

It is easily seen that under set inclusion C(P, $\perp$) becomes a complete lattice with joins and meets given by

$$\bigvee_j S_j = \Big(\bigcup_j S_j\Big)^{\perp\perp} \quad \text{and} \quad \bigwedge_j S_j = \bigcap_j S_j$$

and with the orthocomplementation defined as $S \to S^{\perp}(S_j, S \in C(P, \perp))$. For the empty set $\varnothing$ we put, by definition, $\varnothing^{\perp} = P$, which leads immediately to $\varnothing, P \in C(P, \perp)$.

([6]) The compatibility relation $\leftrightarrow$ in F is defined as follows:

$$a \leftrightarrow b \text{ iff } a = a_1 \vee c \ \& \ b = b_1 \vee c \text{ for some mutually orthogonal } a_1, b_1, c \in F.$$

hence

$$q(\text{carr } r) = d(r) = 1,$$

which leads to

$$\text{carr } r \geqslant \text{carr } q.$$

Since carr r, carr q are atoms, we get carr r = carr q, and therefore $r = q$, since the mapping carr is a bijection. Thus, finally, $cq = q$.

By using the inequalities $c \leqslant a$ and $c \leqslant b$ we now find for each $m \in \mathrm{B}_h$

$$cm = cabm = c(m(b)p(a)q) = m(b)p(a)cq = m(b)p(a)q = abm;$$

hence $c = ab$ on B_h, as claimed.

Let us now consider the second case, when $m(b)p(a) = 0$. We then have

$$0 = abm = m(b)p(a)q = m(b_1 \dotplus c)p(a_1 \dotplus c)q;$$

hence either $m(b_1) + m(c) = 0$, which implies $m(b_1) = m(c) = 0$, or $p(a_1) + p(c) = 0$, hence $p(a_1) = p(c) = 0$, the latter implying carr $p \leqslant c'$. But $p(b) = 1$ implies carr $p \leqslant b$, and therefore

$$\text{carr} \leqslant b_1 = b \dot{-} c = b \wedge c';$$

hence

$$p(b_1) = 1.$$

Since $b_1 \leqslant b$, we have

$$b_1 m = b_1 bm = b_1(m(b)p) = m(b)b_1 p;$$

hence

$$m(b_1) = m(b)p(b_1) = m(b),$$

and therefore

$$m(c) = m(b) - m(b_1) = 0.$$

Thus in the second case we have always $m(c) = 0$, which leads to $cm = 0 = abm$. Therefore, we have shown that in both cases $ab = c$ on B_h. By symmetry (since $a \leftrightarrow b$ implies $b \leftrightarrow a$) we also have $ba = c$ on B_h, and the proof is complete.

Having proved (12), we are able to show that (see [*2*]):

(13) If $a \neq 0$ ($a \in \mathrm{F}$), and if $e \in \mathrm{F}$ is an arbitrary atom not contained in a' (i. e., $e \nleqslant a'$), then there exists $e \vee a'$ in F, $e \vee a' \dot{-} a'$ is an atom, and

$$ap_e = (e \vee a' \dot{-} a')p_e = p_e(a)p_{e \vee a' \dot{-} a'},$$

where p_e denotes the unique pure state such that carr $p_e = e$.

(D) CONSEQUENCES OF AXIOMS 1-13:

(14) The completion by cuts of F (being orthoisomorphic to C(P, $\perp$)) is orthomodular and the covering law holds in it.

REFERENCES

[*1*] W. GUZ, *Ann. Inst. H. Poincaré*, t. **28**, 1978, p. 1.
[*2*] W. GUZ, *Ann. Inst. H. Poincaré*, t. **29**, 1978, p. 357.
[*3*] V. S. VARADARAJAN, *Commun. Pure Appl. Math.*, t. **15**, 1962, p. 189.

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