ANNALES DE L'I. H. P., SECTION A

JEAN-Y. VINET

Elasto-optical detection of gravitational waves

Annales de l'I. H. P., section A, tome 30, nº 3 (1979), p. 251-262

<http://www.numdam.org/item?id=AIHPA_1979__30_3_251_0>

Ann. Inst. Henri Poincaré,
Vol. XXX, n° 3, 1979, p. 251-262.

Section A :
Physique théorique.

Elasto-optical detection of gravitational waves

by

Jean-Y. VINET

Laboratoire de Physique Théorique, Institut Henri Poincaré
11, rue Pierre-et-Marie-Curie, 75231 Paris Cedex 05

ABSTRACT. — The use of the photoelastic effect in a transparent solid in order to detect stresses induced by a gravitational wave is proposed. A deductive way leading from a covariant theory of elasticity to equations describing the crystal response under gravitational forces is developed. The birefringence induced by the gravitational stresses is then examined, and orders of magnitude estimated.

To day's improvements in ellipsometric measures with coherent light allow detection and measure of the birefringence induced in certain transparent solids by even very weak stresses. Therefore, it seems interesting to estimate the sensitivity of a gravitational antenna based on this principle. Most of working gravitational antennæ are essentially constituted by large metallic solids, the vibration state of which is analyzed. In certain cases, the small displacements of the bar's end are recorded; in other cases the observed variable is the weak strain at the middle of the bar. The present survey relates to a transparent crystalline bar, the strains of which are analyzed through the polarization changes of a travelling light ray.

The theory of elastic deformations of a continuous medium induced by a passing gravitational wave has been considered by several authors, whose conclusions are not obviously equivalent. For instance, J. Weber introduces gravitational forces as volume forces, being exerted inside any type of a solid, while Dyson and Gambini argue that gravitational forces are developed only upon the limiting surface in the case of an isotropic body.

Therefore, we pay as much attention to the theoretical framework for the study of the effect of a gravitational wave upon a solid of cubic class, as to orders of magnitude of the induced birefringence.

I. ACTION OF A GRAVITATIONAL WAVE UPON A CRYSTAL

I.1 Covariant theory of elasticity

The first estimations of the cross-section of an elastic gravitational antenna are due to Weber [*1*], [*2*], who added to the dynamical equation of displacement waves, a driving term containing the Riemann tensor, following the general form of geodesic deviation equation, plus a phenomenologic damping term, in order to introduce the quality factor of the resonator. Further works were done in a deductive way, starting on a covariant theory of elasticity, and using then the principles of general relativity. Use will be made here of Papapetrou's [*3*] theory and of discussions by Gambini [*4*] and Tourrenc [*5*] of visco-elasticity and boundary conditions. The aim of the present section is to present a synthesis of these two approaches, and to construct a system of equations both consistent with general principles, and useful for any experimental calculation.

Considered as a continuous medium, the set of all particles of matter that constitute a body defines a congruence of trajectoires, and a 4-velocity field $u^\mu(x^\lambda)$. The spatial projector is defined as

$$\overset{*}{g}_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor, of signature $(+ - - -)$. The rate-of-strain tensor $E_{\alpha\beta}$ is given by

$$E_{\alpha\beta} = \frac{1}{2} \mathscr{L}_u \overset{*}{g}_{\alpha\beta} \tag{2}$$

where $\mathscr{L}_u$ refers to the Lie derivative with respect to u^α. $E_{\alpha\beta}$ is essentially tridimensional, for the obvious consequence of (1) and (2) is

$$E_{\alpha\beta} u^\beta = 0 \tag{3}$$

The stress tensor is generalized by $\Theta^{\alpha\beta}$ with the orthogonality condition

$$\Theta^{\alpha\beta} u_\beta = 0 \tag{4}$$

Hooke's law relying strain to stress, is rewrited as

$$\mathscr{L}_u \Theta_{\alpha\beta} = - C_{\alpha\beta}{}^{\mu\nu} E_{\mu\nu} \tag{5}$$

Papapetrou has pointed out that the tensor of elastic stiffness $C_{\alpha\beta}{}^{\mu\nu}$ has only 21 independent components, as in classical theory. The energy-momentum tensor $T^{\mu\nu}$ has the usual form

$$T^{\mu\nu} = \rho c^2 u^\mu u^\nu - \Theta^{\mu\nu}$$

where ρ is the invariant density of matter. General relativity provides us the dynamical equation

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{6}$$

which with (5) constitutes the basis of a covariant theory of elasticity. Following Gambini, we take into account stresses that are related to internal damping by a supplementary term in $T^{\mu\nu}$:

(7) $$T^{\mu\nu} = \rho c^2 u^\mu u^\nu - \Theta^{\mu\nu} - \Theta'^{\mu\nu}$$

$\Theta'^{\mu\nu}$ is related to the rate of strain tensor $E_{\alpha\beta}$ by

(8) $$\Theta'^{\mu\nu} = - C'^{\mu\nu\alpha\beta} E_{\alpha\beta}$$

where $C'^{\mu\nu\alpha\beta}$ generalizes the visco-elastic tensor of the matter. The following orthogonality condition holds:

$$\Theta'^{\mu\nu} u_\nu = 0$$

so that $C'^{\mu\nu\alpha\beta}$ has 21 independent components, as the elastic stiffness tensor does.

I.2 Weak-field and weak-strain approximations

We consider a gravitational field that differs slightly from zero, so that the metric tensor differs in turn slightly from its Minkowskian form:

$$g_{\mu\nu} = n_{\mu\nu} + h_{\mu\nu}$$

with $n_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$. $h_{\mu\nu}$ is a plane, transverse, traceless wave, of weak amplitude h in the laboratory frame :

(9) $$h^{\mu\nu}{}_{,\nu} = 0 \qquad h_{0\nu} = 0 \qquad h_\mu{}^\mu = 0$$

In the sequel, we consider only the first order terms with respect to h (linearized theory). The amplitudes of strains are characterized by a small dimensionless parameter ε, and we neglect nonlinear terms in ε. We also neglect terms containing εh. As a last hypothesis, we suppose the laboratory to be located in a region of space of small extension with respect to the gravitational wave-length for the significant part of the emission spectrum.

I.3 Tensorial and vectorial equations of elasticity

A) The dynamical equation (6) determines the motion of any point particle of matter. We turn rather our attention towards a small domain in the neighbourhood of this particle: let us consider two neighbouring points of world-lines X^α, Y^α. It can be shown that, at first order with respect to ε, both trajectories may be parametrized by the same proper time s which is nothing but the laboratory time. Thus, the spatial vector

$$N^\alpha = Y^\alpha - X^\alpha$$

has a vanishing Lie-derivative:

(10) $$\mathscr{L}_u N^\alpha = 0$$

We shall consider only linear terms in N^α. By applying the operator $N^\lambda\nabla_\lambda$ to (6), we obtain:

$$\rho c^2(N^\lambda\nabla_\lambda)(u^\sigma\nabla_\sigma)u^\alpha + (u^\sigma\nabla_\sigma u^\alpha)(N^\lambda\nabla_\lambda\rho c^2) = N^\lambda\nabla_\lambda\nabla_\sigma(\Theta^{\alpha\sigma} + \Theta'^{\alpha\sigma})$$

Now, we use the identity

$$[N^\lambda\nabla_\lambda, u^\sigma\nabla_\sigma]\mathscr{V}^\alpha \equiv R^\alpha{}_{\sigma\lambda\tau}\mathscr{V}^\sigma N^\lambda u^\tau - \mathscr{L}_u N^\sigma.\nabla_\sigma\mathscr{V}^\alpha$$

and get

$$(11)\quad \rho c^2\{(u^\lambda\nabla_\lambda)(u^\sigma\nabla_\sigma)N^\alpha + R^\alpha{}_{\beta\gamma\delta}u^\beta N^\gamma u^\delta + \mathscr{L}_u\mathscr{L}_u N^\alpha\} + (u^\sigma\nabla_\sigma u^\alpha)(N^\lambda\nabla_\lambda\rho c^2) = N^\lambda\nabla_\lambda\nabla_\sigma(\Theta^{\alpha\sigma} + \Theta'^{\alpha\sigma})$$

B) We attach to each point of the initial 3-space (in the absence of gravitational wave), a free observer at rest in the laboratory frame. Then we attach to each observer a tetrad, *i. e.* a set of four independent 4-vectors

$$\{{}_0Z_{\bar\beta}{}^\alpha : \bar\beta = 0, 1, 2, 3\}$$

${}_0Z_{\bar 0}{}^\alpha$ being a time-like vector, while $\{{}_0Z_{\bar k}{}^\alpha : \bar k = 1, 2, 3\}$ is a triad of space-like vectors. We choose

$${}_0Z_{\bar 0}{}^\alpha \equiv {}_0\mathscr{U}^\alpha$$

where ${}_0\mathscr{U}^\alpha$ is the 4-velocity field of the observers. We require the following orthogonality relations:

$$(12)\qquad \eta_{\mu\nu}\,{}_0Z_{\bar\alpha}{}^\mu\,{}_0Z_{\bar\beta}{}^\nu = \eta_{\bar\alpha\bar\beta} \quad ; \quad \eta^{\bar\alpha\bar\beta}\,{}_0Z_{\bar\alpha}{}^\mu\,{}_0Z_{\bar\beta}{}^\nu = \eta^{\mu\nu}$$

where the scalar arrays $n_{\bar\alpha\bar\beta} = n^{\bar\alpha\bar\beta} = \text{diag}\,(1, -1, -1, -1)$ enable us to raise or lower tetradic indices. We require further the tetrad to be parallely propagated along the observer's world line, so that

$${}_0Z_{\bar 0}{}^\alpha\,{}_0Z_{\bar\beta,\alpha} = 0 \qquad (\bar\beta = 0, 1, 2, 3)$$

The simplest choice of such a tetrad is obviously

$${}_0Z_{\bar\beta}{}^\alpha = \delta_{\bar\beta}{}^\alpha$$

Let us introduce now the gravitational TT plane wave. The 4-velocity field $\mathscr{U}^\alpha$ of observers is not to be modified, but the orthogonality conditions (12) become

$$(13)\qquad g_{\mu\nu}Z_{\bar\alpha}{}^\mu Z_{\bar\beta}{}^\nu = \eta_{\bar\alpha\bar\beta} \quad , \quad \eta^{\bar\alpha\bar\beta}Z_{\bar\alpha}{}^\mu Z_{\bar\beta}{}^\nu = g^{\mu\nu}$$

and the condition for parallel propagation is now

$$Z_{\bar 0}{}^\alpha Z_{\bar\beta;\alpha} = 0$$

The initial tetrad $\{{}_0Z_{\bar\beta}{}^\alpha\}$ must therefore be slightly modified so as to fulfil the new conditions (13). The resulting tetrad $\{Z_{\bar\beta}{}^\alpha\}$ we get is

$$(14)\qquad Z_{\bar\beta}{}^\alpha = {}_0Z_{\bar\beta}{}^\alpha - \frac{1}{2}h^\alpha{}_\sigma\,{}_0Z_{\bar\beta}{}^\sigma$$

C) It is easy to verify that

$$u^\lambda Z^{\bar{k}}{}_{\alpha;\lambda} = 0(\varepsilon^2)$$

where u^λ is the 4-velocity field of matter. By contracting $Z^{\bar{k}}{}_\alpha$ with (11), we get at first order:

$$\rho c^2 (N^{\bar{k}}{}_{,\bar{0}\bar{0}} + R^{\bar{k}}{}_{\bar{0}\bar{l}\bar{0}} N^{\bar{l}}) = N^{\bar{l}}(\Theta^{\bar{k}\bar{m}} + \Theta'^{\bar{k}\bar{m}})_{;\bar{l}\bar{m}}$$

Now, $\partial_{\bar{0}}$ is nothing but $c^{-1}\partial_t$, consequently,

$$\rho\left(\ddot{N}^{\bar{k}} - \frac{1}{2}\ddot{h}^{\bar{k}}{}_{\bar{l}} N^{\bar{l}}\right) = N^{\bar{l}}(\Theta^{\bar{k}\bar{m}} + \Theta'^{\bar{k}\bar{m}})_{;\bar{l}\bar{m}} \tag{15}$$

where the dot refers to partial derivative with respect to time. Let $\vec{n}$ be the spatial vector joining the two particles of trajectories X^α, Y^α in the unstrained crystal in flat space-time (reference state), and ${}_0n^{\bar{k}}$ its components in the initial tetrad $\{ {}_0Z_{\bar{\beta}}{}^\alpha \}$. Let $\varepsilon^{*\bar{l}}$ be the displacement field of the solid actually recorded by the local observer: we can define it by setting

$$N^{\bar{k}} = {}_0n^{\bar{k}} + {}_0n^{\bar{l}} \varepsilon^{*\bar{k}}{}_{,\bar{l}} \tag{16}$$

The ${}_0n^{\bar{k}}$ being constants, (15) becomes:

$$\rho\left[{}_0n^{\bar{l}} \ddot{\varepsilon}^{*\bar{k}}{}_{,\bar{l}} - \frac{1}{2}\ddot{h}^{\bar{k}}{}_{\bar{l}}\, {}_0n^{\bar{l}}\right] = {}_0n^{\bar{l}}(\Theta^{\bar{k}\bar{m}} + \Theta'^{\bar{k}\bar{m}})_{,\bar{l}\bar{m}}$$

But $\vec{n}$ was arbitrarily chosen, whence

$$\rho\left(\ddot{\varepsilon}^{*\bar{k}}{}_{,\bar{l}} - \frac{1}{2}\ddot{h}^{\bar{k}}{}_{\bar{l}}\right) = (\Theta^{\bar{k}\bar{m}} + \Theta'^{\bar{k}\bar{m}})_{,\bar{l}\bar{m}} \tag{17}$$

All terms in eq. (17) being of order ε or h, tetradic indices arise from the initial tetrad ${}_0Z_{\bar{\beta}}{}^\alpha = \delta_{\bar{\beta}}{}^\alpha$ so that tetradic components may be identified with tensorial components in the initial flat space-time. We can thus rewrite (17) under the familiar symmetrized form for ordinary 3-space:

$$\rho\left(\ddot{\varepsilon}^*_{ij} + \frac{1}{2}\ddot{h}_{ij}\right) = \frac{1}{2}[(\Theta_j{}^l + \Theta'^l_j)_{,li} + (\Theta_i{}^l + \Theta'^l_i)_{,lj}] \tag{18}$$

This equation, in absence of gravitational wave, reduces to the well-known tensorial equation for damped elastic strain waves. We are going to consider (18) as the generalized equation for strain waves, and call

$$\varepsilon^*_{ij} = \frac{1}{2}(\varepsilon^*_{i,j} + \varepsilon^*_{j,i})$$

the « physical strain » tensor. Thus (18) appears as a wave equation plus a gravitational driving term.

The trajectories of our two neighbouring particles of matter may be parametrized as follows:

$$X^\alpha = \begin{cases} ct \\ x^i + \varepsilon^i(x^k, t) \end{cases} , \quad Y^\alpha = \begin{cases} c\,t \\ x^i + n^i + \varepsilon^i(x^k + n^k, t) \end{cases}$$

where (ct, x^i) is the coordinate system of the laboratory, or the reference frame of the unstrained solid. We thus have:

$$N^\alpha = Y^\alpha - X^\alpha = \begin{cases} 0 \\ n^i + n^j \varepsilon^i{}_{,j} \end{cases}$$

The tetradic components of N^α can be obtained from (14):

$$N^{\bar{k}} = n^k + n^l\left(\varepsilon^k{}_{,l} + \frac{1}{2} h^k{}_l\right)$$

which by comparison with (16) gives the relation

$$\varepsilon^*_{ij} = \varepsilon_{ij} - \frac{1}{2} h_{ij} \tag{19}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(\varepsilon_{i,j} + \varepsilon_{j,i})$$

Besides (18) we need an approximation of the generalized Hooke's law. At the same order, we have from (5):

$$\dot{\Theta}^{ij} = c^{ijkl} \dot{\varepsilon}^*_{kl}$$

which can be integrated, leading to

$$\Theta^{ij} = c^{ijkl} \varepsilon^*_{kl} \tag{20}$$

(see [3])

Similarly, from (8) we get

$$\Theta'^{ij} = c'^{ijkl} \dot{\varepsilon}^*_{kl} \tag{21}$$

With (20) and (21), equation (18) gives rise to a wave equation in terms of the strain ε^*_{ij}.

The dimensions of the body being small compared to the gravitational wave-length, spatial derivatives of $h_{\mu\nu}$ will be neglected. By setting the origin of the laboratory frame at the center of mass of the unstrained body, eq. (18) may be thought of as deriving from the vectorial equation

$$\rho \ddot{\varepsilon}^{*i} - (\Theta^{ij} + \Theta'^{ij})_{,j} = -\frac{1}{2} \rho \ddot{h}^{ij} x^j \tag{22}$$

where the force density that appears in the right-hand side has a vanishing integral over the solid.

D) Following Gambini and Tourrenc, we require the following conditions to hold at the boundary 2-surface of the solid:

$$(\Theta^{ij} + \Theta'^{ij})l_j = 0 \tag{23}$$

where $\vec{l}$ is the external vector normal to the surface.

Finally, the action of a gravitational wave upon an elastic body can be deduced from the system

$$(24)\quad \begin{cases} \rho\ddot{\varepsilon}^{*i} - (\Theta^{ij} + \Theta'^{ij})_{,j} = -\frac{1}{2}\rho\ddot{h}^{ij}x^j & (i) \\ \Theta^{ij} = c^{ijkl}\varepsilon^*_{kl} & (ii) \\ \Theta'^{ij} = c'^{ijkl}\dot{\varepsilon}^*_{kl} & (iii) \\ (\Theta^{ij} + \Theta'^{ij})l_j = 0 & (iv) \end{cases}$$

These formulas are the classical ones, except for the driving term arising from Riemann tensor, which can be regarded as a volume force density, in agreement with Weber. No surface force appear. Let us note that use of the « unphysical strain » tensor ε_{ij} leads for an isotropic solid to surface instead of volume forces, as in Dyson [*6*], and Gambini's papers. The correct damping term is however presumably (24-*iii*), and not Weber's one (see for instance W. G. Cady [*7*]). Let us emphasize that these differences are negligible in practice only in the limit of weak internal damping at resonance (high-Q devices).

II. BIREFRINGENCE INDUCED BY A GRAVITATIONAL WAVE

II.1 Photoelastic effect

It is well-known [*8*] that applied stresses can modify the dielectric tensor ν_{ij} of a crystalline solid, and therefore affect the propagation of light. Let us consider a cubic crystal, the relative dielectric tensor of which is isotropic:

$$\nu_{ij} = \nu\delta_{ij} \qquad (\nu = n_0^2)$$

(n_0 being the refractive index). In the strained crystal, the new dielectric tensor is now

$$\nu'_{ij} = \nu\delta_{ij} + \delta\nu_{ij}$$

and is no more isotropic. Diagonalization of ν'_{ij} makes the three principal axes to appear, with the three corresponding principal dielectric constants

$$\nu + \delta\nu_{11} \quad , \quad \nu + \delta\nu_{22} \quad , \quad \nu + \delta\nu_{33}$$

which differ one from another, so that an incident light ray will result in general in two different refracted rays (double refraction). In the special case of a light ray propagating along a principal direction assumed to be normal to the boundary plane, there is no change in direction, but a change in velocity depending on the polarization state. The refractive index for general orientation can be determined by means of the indicatrix, *i. e.* the quadric of equation

$$B_1x_1^2 + B_2x_2^2 + B_3x_3^2 = 1$$

with respect to the principal axes. B_i stands for $1/n_i^2$, n_i being the principal refractive index associated with the principal direction x_i. For a crystal of cubic class, we have the initial indicatrix

$$B_{ij}x^ix^j \equiv n_0^{-2}(x_1^2 + x_2^2 + x_3^2) = 1$$

In the strained crystal, the perturbed indicatrix is now

$$B'_{ij}x_ix_j = 1$$

with respect to the same axes. The difference $B'_{ij} - B_{ij} \equiv \Delta B_{ij}$ is related to the stress tensor Θ^{kl} by the so-called elasto-optic tensor π_{ijkl}:

$$\Delta B_{ij} = \pi_{ijkl}\Theta^{kl}$$

For a uniaxial stress directed along one of the cubic axes (*e. g.* Ox_1), the principal axes remain the cubic axes (Ox_1, Ox_2, Ox_3), but there is a loss of degeneracy: The changes of the refractive indices with respect to their common value n_0 are:

$$\Delta n_1 = -\frac{1}{2}n_0^3\pi_{11}\Theta_1$$

for an Ox_1-polarized wave, and

$$\Delta n_2 = -\frac{1}{2}n_0^3\pi_{12}\Theta_1$$

for an Ox_2-polarized wave respectively. π_{ij} is the elasto-optic tensor written in the standard 6-dimensional notation for symetric tensors, where pairs of tensorial indices are replaced by single vectorial ones, according to the rule

$$(1,1) \to 1 \quad (2,2) \to 2 \quad (3,3) \to 3 \quad (2,3) \to 4 \quad (3,1) \to 5 \quad (1,2) \to 6$$

Thus Θ_1 refers to the (1,1) component of Θ^{ij}. This notation will be used throughout the sequel for any symmetrical tensor of the 3-space.

For a differential experiment involving for instance two light rays travelling along Ox_3 and polarized along Ox_1 and Ox_2 respectively, the optical path change results from

$$\Delta n \doteq -\frac{1}{2}n_0^3(\pi_{11} - \pi_{12})\Theta_1 \qquad (25)$$

II.2 Stresses developed in a cubic crystal by a gravitational wave

We consider again a crystal of cubic class, cut as a lengthened bar of length L, directed along the Ox_1 cubic axis:

$$- L/2 \leqslant x \leqslant + L/2$$

of width l, directed along the Oy axis:

$$- l/2 \leqslant y \leqslant + l/2$$

and thickness e, along the Oz axis:

$$- e/2 \leqslant z \leqslant + e/2$$

The assumption that the bar is lengthened is precisely

$$L \gg l \quad , \quad L \gg e$$

We are only interested in longitudinal modes with uniaxial stresses and strains along Ox. We assume that the frequencies of other modes do not match with the frequency spectrum of the gravitational wave, and cannot resonate (the frequencies are well-separated because of the preceding condition).

In a cubic crystal, stresses Θ_i are related to strains ε_i by only three independent elastic coefficients:

$$\begin{aligned} \Theta_1 &= c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{12}\varepsilon_3 & \Theta_4 &= c_{44}\varepsilon_4 \\ \Theta_2 &= c_{12}\varepsilon_1 + c_{11}\varepsilon_2 + c_{12}\varepsilon_3 & \Theta_5 &= c_{44}\varepsilon_5 \\ \Theta_3 &= c_{12}\varepsilon_1 + c_{12}\varepsilon_2 + c_{11}\varepsilon_3 & \Theta_6 &= c_{44}\varepsilon_6 \end{aligned} \tag{26}$$

Now, we require $\Theta_2 = \Theta_3 = \Theta_4 = \Theta_5 = \Theta_6 = 0$. Thus, the system (26) can easily be inverted, which leads to the relations

$$\varepsilon_1 = (1/Y)\Theta_1 \quad , \quad \varepsilon_2 = \varepsilon_3 = - \sigma\varepsilon_1$$

where we have set

$$\begin{aligned} Y &= \left(c_{11}^2 + c_{11}c_{12} - 2c_{12}^2\right)/(c_{11} + c_{12}) \qquad &\text{(Young's modulus)} \\ \sigma &= c_{12}/(c_{11}+c_{12}) \qquad &\text{(Poisson's ratio)} \end{aligned}$$

The wave equation for elastic longitudinal displacement field derived in I can be written down simply as

$$\rho\ddot{u} - Y\frac{\partial^2 u}{\partial x^2} - F\frac{\partial^3 u}{\partial t \partial x^2} = - \frac{1}{2}\rho\ddot{h}_1 x$$

with the boundary condition

$$\left[Y\frac{\partial u}{\partial x} + F\frac{\partial^2 u}{\partial t \partial x}\right]_{x=\pm L/2} = 0$$

(we write u instead of ε_1 so as to avoid confusion with the strain tensor in the 6×1 notation). F is the damping factor, deduced from c'_{ij} as Y from c_{ij}. For a general polarization of the gravitational wave, there would be an additional driving term

$$- \frac{1}{2} \rho \ddot{h}_6 y - \frac{1}{2} \rho \ddot{h}_5 z$$

which we shall not take into account here, the conditions for resonance being quite different.

Let us consider a monochromatic incident gravitational wave polarized so that

$$h_1 = h \cos \omega t$$

The spatial dependence of h_1 is to be neglected, as seen above. When the length of the bar is chosen as

$$L = \frac{\pi}{\omega} \sqrt{\frac{Y}{\rho}}$$

the resonant part of the strain is

$$\varepsilon_1 = (2hQ/\pi) \cos kx \sin \omega t$$

where $k = \pi/L = \omega/v$ ($v \equiv \sqrt{Y/\rho}$ is the velocity of compressional acoustic waves). Q is the quality factor of the bar, related to Y and F by

$$Q = Y/\omega F$$

and assumed to be great compared to unity. The uniaxial stress is then

$$\Theta_1 = (2hQY/\pi) \cos kx \sin \omega t$$

The resulting change in the birefringence along the Oz axis ($x = 0$) is in turn, with (25):

$$\Delta n = - n_0^3(\pi_{11} - \pi_{12}) hQY\pi^{-1} \sin \omega t \tag{26}$$

II.3 Numerical estimations

The following parameters can be found for a cubic crystal such that $Ba(NO_3)_2$ (Barium Nitrate):

$\rho = 3{,}240$ kg m^{-3} (CRC *Handbook of Chemistry*).
$c_{11} = 6.02\ 10^{-10}$ N m^{-2}.
$c_{12} = 1.86\ 10^{-10}$ N m^{-2}.
$\pi_{11} - \pi_{12} = -\ 23\ 10^{-12}$ m^2 N^{-1} ([9]).
$n_0 = 1.570$ (Landolt, Bornstein, 1933).

(for optical wave length 5,893 Å).

The dimensionless photoelastic coupling factor

$$- n_0^3\pi^{-1}(\pi_{11} - \pi_{12})Y$$

is about 1.46 so that

$$\Delta n \simeq 1.46\ \mathrm{Q}h \sin \omega t$$

which shows that we have no coupling losses by using optical instead of mechanical techniques.

Now, we take the following figures as dimensions of the bar:

L = 0.2 m, $l = e = 0.02$ m. The mass is thus about 0.26 kg, and the fundamental frequency about $\nu_0 = 10^4$ Hz. The relative phase difference between two light rays of same frequency travelling along Oz and orthogonally polarized will be

$$\Delta\varphi = \frac{2e\pi}{\lambda_{\text{opt}}} \Delta n$$

In terms of the gravitational perturbation, this becomes

$$\Delta\varphi \sim 3.10\ 10^5\ \mathrm{Q}h \sin \omega t$$

With a proper device, one can realize a synchronous detection of the beat of the two light rays. The minimum detectable phase difference $(\Delta\varphi)_{\min}$ is determined essentially by the laser photon noise, in relation with integration time. Detailed calculation [*10*] allows us to hope, for an optical signal-to-noise ratio of order 1, a minimum detectable phase difference better than

$$(\Delta\varphi)_{\min} \sim 10^{-11}$$

with an integration time of 0.1 s and 60 mW of detectable light power [*11*]. Two orders of magnitude might be won by including the crystal in a Fabry-Perot cavity; consequently, we get, for the lowest detectable h, the limit

$$h_{\min} \sim 10^{-24}$$

with a mechanical quality factor $\mathrm{Q} \sim 10^6$.

The most important limiting factor remains, as customary, the thermal noise in the solid. Experimenters will remark, however, that it is easier to achieve very low temperatures with less than 1 kg masses than with several tons cylinders.

Work supported by the Centre d'Études Théoriques de la Détection et des Communications (convention 77/1282 DRET).

REFERENCES

[*1*] J. Weber, *General relativity and gravitational waves*, Interscience, 1961.

[*2*] J. Weber, In *Experimental gravitation* (Proceedings of the international school of physics E. Fermi), Academic Press, 1974.

[*3*] A. Papapetrou, *Ann. Inst. H. Poincaré*, A **16**, 1972, p. 63.

[*4*] R. Gambini, *Ann. Inst. H. Poincaré*, A **23**, 1975, p. 389.

[5] P. TOURRENC, *Thèse de Doctorat ès Sciences*, Université Paris VI, 1976.
[6] F. J. DYSON, *Astrophysical Journal*, vol. **156**, 1969, p. 529.
[7] W. G. CADY, *Piezoelectricity*, vol. I, Dover Pub. Inc., 1964.
[8] J. F. NYE, *Physical properties of crystals*, Oxford, Clarendon Press, 1957.
[9] S. BAGHAVANTAM, K. V. KRISHNA RAO, *Acta Cryst.*, **6**, 1953, p. 799.
[10] G. BOYER, B. LAMOUROUX, B. PRADE, J.-Y. VINET (to be published).
[11] R. L. FORWARD, Paper presented at the Eighth GRG conference, University of Waterloo, August 1977.

(Manuscrit reçu le 5 mars 1979)