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Section A :
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On the computation of bounds for low energy Compton scattering parameters: proof of a conjecture

by

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ABSTRACT. — A general method proposed by Raszillier [1] to obtain constraints on low energy Compton scattering parameters in terms of upper bounds on the cross sections above photoproduction threshold, is shown to give optimal results.

1. INTRODUCTION

Recently, a method has been proposed by Raszillier [1] to solve an extremum problem which arises in the derivation of dispersion inequalities for Compton scattering [1, 2].

This method should lead to an improvement of previous bounds [2] on low energy scattering parameters as functions of the fixed transfer cross section above photoproduction threshold. The results appear in the form of the so-called « inner » and « outer » approximations to the optimal bounds. The purpose of the present paper is to show that the « outer approximation » gives in fact the optimal bounds.

As the problem is of general interest, we shall not reproduce here its physical background, which can be found in ref. [1, 2]. We merely state

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it in its reduced mathematical form as presented in [1], i. e. after suitable changes of functions (= amplitudes) and conformal mapping (of the energy plane) have been made.

Let S be the class of complex vector-valued functions

$$\vec{w}(z) = (w_1(z), \ldots, w_n(z)),$$

analytic in the unit disc $|z| < 1$, such that:

$$(1.1) \qquad \sum_{i=1}^{n} |w_i(e^{i\theta})|^2 \leq 1$$

The $w_i(z)'s$ are assumed to be « real analytic »: $w_i(z^*) = w_i^*(z)$. Given n real points x_i $(i = 1, \ldots, n)$ inside the unit disc, consider the set of (real) values:

$$(1.2) \quad \bar{W} = \{ W_\alpha \}_{\alpha=1,\ldots,N} = \{ w_1(x_1), w_1'(x_1), \ldots, w_1^{(k_1)}(x_1); \ldots ; w_n(x_n), w_n'(x_n), \ldots, w_n^{(k_n)}(x_n) \}$$

where $w_i^{(k)}(x_i)$ stands for $\left.\dfrac{d^k w_i(z)}{dz^k}\right|_{z=x_i}$ and $N = \sum_{i=1}^{n} (k_i + 1)$.

The problem is to determine the range D of $\bar{W}$ in $\mathbb{R}^N$ when $\vec{w}(z)$ varies over the whole of S.

Notice that in the special case $n = 1$, this is a well known problem which is easily solved by standard interpolation theory. To deal with the general case, Raszillier introduces an « outer approximation » to D in the following way.

For any positive function $\rho(\theta)$ subjected to the normalization condition:

$$(1.3) \qquad \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) = 1,$$

one defines the class S_ρ of $\vec{w}(z)'s$ such that:

$$(1.4) \qquad \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \sum_{i=1}^{n} |w_i(e^{i\theta})|^2 \leq 1$$

and the corresponding region D_ρ in $\mathbb{R}^N$ (= range of $\bar{W}$ when $\vec{w}(z)$ varies over S_ρ). Clearly, the class S is included in S_ρ, so that D is contained in D_ρ. The « outer approximation » $D_0 \supset D$ is obtained by taking the intersection of all D_ρ's:

$$(1.5) \qquad D_0 = \bigcap_\rho D_\rho$$

where ρ ranges over the class of weight functions normalized according to eq. (1.3).

The interest of this construction lies in the fact that the regions D_ρ can be determined explicitely by using interpolation theory in the Hardy space H^2, as explained in [1]. Moreover, in the three examples worked out in that paper, the outer approximations D_0 turn out to coincide with the exact regions D. Whether the equality $D_0 = D$ is a general property was left undecided however (although implicitely conjectured). We show here that this is true indeed, namely that:

$$D = \bigcap_\rho D_\rho \tag{1.6}$$

The key of the proof, which is given in Section 3, is the use of duality formulae in suitably defined Banach spaces of analytic functions. As a preparatory step, these spaces are defined in the next section, where the problem is also slightly generalized and reformulated in a way allowing the application of the duality argument.

2. PRELIMINARIES

For complex vector-valued function defined on the unit circle, we define the Banach spaces (on the real field):

$$\vec{L}^p = \{ \vec{f}(\theta) = (f_1(\theta), \ldots, f_n(\theta)) \mid f_i(\theta) \in L^p,\ f_i^*(\theta) = f_i(-\theta) \} \tag{2.1}$$

equipped with the norms:

$$\| \vec{f} \|_\infty = \operatorname*{Ess.\,sup}_{-\pi<\theta<\pi} | \vec{f}(\theta) | \quad \text{for} \quad p = \infty \tag{2.2}$$

and:

$$\| \vec{f} \|_p = \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{f}(\theta) |^p \right]^{1/p} \quad \text{for} \quad 1 \leq p < \infty \tag{2.3}$$

(only the values $p = \infty$, 1 and 2 will be used).

Here, $| \vec{f}(\theta) |$ stands for $\left[\sum_{i=1}^{n} | f_i(\theta) |^2 \right]^{\frac{1}{2}}$.

Similarly, starting from the usual Hardy spaces H^p [3], we define the « vectorial » Hardy spaces:

$$\vec{H}^p = \{ \vec{w}(z) = (w_1(z), \ldots, w_n(z)) \mid w_i(z) \in H^p,\ w_i^*(z) = w_i(z^*) \} \quad (p = \infty, 1 \text{ or } 2) \tag{2.4}$$

equipped with the norms $\| \vec{w} \|_p$ above ($\vec{w}(z)$ is identified with its boundary value $\vec{w}(e^{i\theta}) \in \vec{L}^p$, so that the Banach space $\vec{H}^p$ is a (closed) subspace of $\vec{L}^p$).

Given a positive function $\rho(\theta) = \rho(-\theta)$ such that $\mathrm{Log}\,\rho(\theta) \in \mathrm{L}^1$, let us introduce the outer function:

$$G(z) = \exp \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \mathrm{Log}\,\rho(\theta). \tag{2.5}$$

Then:

$$| G(e^{i\theta}) | = \sqrt{\rho(\theta)} \qquad \text{(a. e.)} \tag{2.6}$$

We shall say that $\vec{w}(z) \in \vec{\mathrm{H}}^2_\rho$ if and only if $\vec{h}(z) = G(z)\vec{w}(z) \in \vec{\mathrm{H}}^2$. The correspondence $\vec{w} \leftrightarrow \vec{h}$ then establishes an isometric isomorphism between $\vec{\mathrm{H}}^2_\rho$ and $\vec{\mathrm{H}}^2$ if the norm in $\vec{\mathrm{H}}^2_\rho$ is taken as:

$$\| \vec{w} \|_{2,\rho} = \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \rho(\theta) \, | \vec{w}(e^{i\theta}) |^2 \right]^{\frac{1}{2}} \tag{2.7}$$

The sets S and S_ρ now appear as the unit balls in $\vec{\mathrm{H}}^\infty$ and $\vec{\mathrm{H}}^2_\rho$ respectively:

$$\mathrm{S} = \{ \vec{w} \in \vec{\mathrm{H}}^\infty \mid \| \vec{w} \|_\infty \leq 1 \} \tag{2.8'}$$

$$\mathrm{S}_\rho = \{ \vec{w} \in \vec{\mathrm{H}}^2_\rho \mid \| \vec{w} \|_{2,\rho} \leq 1 \} \tag{2.8''}$$

and the sets D and D_ρ are the images of S and S_ρ in $\mathbb{R}^\mathrm{N}$ through the linear mapping defined by eq. (1.2). Using the appropriate Cauchy representations of $w_i^{(k)}(x_i)$ in H^∞ of H^2_ρ [*3*], it is immediately seen that this mapping is continuous. As a consequence, the sets D and D_ρ are symmetric, convex and bounded. They are also closed. Let us sketch the proof of this last property (for D).

Since the space $\vec{\mathrm{L}}^\infty$ can be identified with the dual of $\vec{\mathrm{L}}^1$, the unit ball in $\vec{\mathrm{L}}^\infty$ is compact in the weak-$*$ topology $\sigma(\vec{\mathrm{L}}^\infty, \vec{\mathrm{L}}^1)$ from the Alaoglu theorem [*4*]. Thus S, which is the intersection of this unit ball with the weak-$*$ closed subspace $\vec{\mathrm{H}}^\infty$ is also weak-$*$ compact. Now the functionals on $\vec{\mathrm{H}}^\infty$ defined by $\vec{w} \to w_i^{(k)}(x_i)$ can be represented by Cauchy kernels, which belong to $\vec{\mathrm{L}}^1$. Therefore, the image of S through the mapping defined by eq. (1.2) is itself compact in $\mathbb{R}^\mathrm{N}$. The proof for D_ρ is even simpler, because $\vec{\mathrm{H}}^2_\rho$ is a reflexive Banach space.

For the application of the duality principle we have in mind, it is necessary to replace the vector-valued functional

$$\vec{w}(z) \to \bar{\mathrm{W}} = \bar{\Delta}(\vec{w}) \tag{2.9}$$

defined by eq. (1.2) by scalar ones. To this end, we shall consider D and D_ρ as intersections of symmetrical strips. To each unit vector $\bar{a} \in \mathbb{R}^\mathrm{N}$, we attach the functional Δ_a defined by the scalar product:

$$\Delta_a = \bar{a}\bar{\Delta} \qquad \left(\text{i. e. } \Delta_a(\vec{w}) = \sum_{\alpha=1}^{\mathrm{N}} a_\alpha \mathrm{W}_\alpha \right) \tag{2.10}$$

and the corresponding strips:

(2.11′) $$B_a = \{ \bar{U} \in \mathbb{R}^N \mid |\bar{a}\bar{U}| \leq \operatorname*{Sup}_{\vec{w} \in S} |\Delta_a(\vec{w})| \}$$

(2.11″) $$B_{a,\rho} = \{ \bar{U} \in \mathbb{R}^N \mid |\bar{a}\bar{U}| \leq \operatorname*{Sup}_{\vec{w} \in S_\rho} |\Delta_a(\vec{w})| \}$$

Notice that the Sup involved in these definitions are in fact attained, for the same reasons which made D and D_ρ closed sets. Now, since D and D_ρ are convex and closed, we can write:

(2.12) $$D = \bigcap_{\bar{a}} B_a, \qquad D_\rho = \bigcap_{\bar{a}} B_{a,\rho}$$

where $\bar{a}$ runs over the whole unit sphere of $\mathbb{R}^N$. Therefore, in order to prove the conjecture (1.6), it is enough to show that, for any $\bar{a}$:

(2.13) $$B_a = \bigcap_{\rho} B_{a,\rho}$$

which in turn is equivalent to:

(2.14) $$\operatorname*{Sup}_{\vec{w} \in S} |\Delta_a(\vec{w})| = \operatorname*{Inf}_{\rho} \operatorname*{Sup}_{\vec{w} \in S_\rho} |\Delta_a(\vec{w})| .$$

As $S \subset S_\rho$, the l. h. s. of this equation is certainly not greater than the r. h. s. Thus eq. (2.14) will be established if we can construct a sequence of functions $\rho(\theta)$ such that $\operatorname*{Sup}_{\vec{w} \in S_\rho} |\Delta_a(\vec{w})|$ converges to $\operatorname*{Sup}_{\vec{w} \in S} |\Delta_a(\vec{w})|$.

3. PROOF

A direct proof of eq. (2.14) does not seem to be easy because one has to deal with spaces of *analytic* functions. We shall circumvent this difficulty by making use of duality formulae, the general form of which is:

(3.1) $$\operatorname*{Sup}_{\substack{w \in T \\ \|w\|_X \leq 1}} |\phi(w)| = \operatorname*{Min}_{\psi \in T^\perp} \| \phi + \psi \|_{X^*}$$

where X is any Banach space, $T \subset X$ a closed subspace, ϕ a continuous functional, and $T^\perp \subset X^*$ the annihilator of T in the dual space X^* ($T^\perp = \{ \psi \in X^* \mid \psi(w) = 0 \ \forall w \in T \}$). The notation Min specifies that the infimum is necessarily attained. To use eq. (3.1) in eq. (2.14), we want to identify the couple (X, T) successively with $(\vec{L}^\infty, \vec{H}^\infty)$ and $(\vec{L}^2, \vec{H}^2)$, and ϕ with the corresponding functionals $\Delta_a(\vec{w})$ ($\vec{w} \in \vec{H}^\infty$) and

$$\Delta_{a,\rho}(\vec{h}) \equiv \Delta_a(\vec{h}/G) \qquad (\vec{h} \in \vec{H}^2) .$$

One has to be careful on two points however:

i) the dual of $\vec{L}^\infty$ does not coincide with $\vec{L}^1((\vec{L}^\infty)^* \supset \vec{L}^1)$,

ii) the kernels which appear in explicit integral representations of the functional Δ_a are not necessarily the same according as Δ_a is supposed to act in $\vec{H}^\infty$ or in $\vec{H}^2_\rho$. When acting in $\vec{H}^\infty$, Δ_a will be for the moment only assumed to admit the representation:

$$(3.2)\qquad \begin{cases} \Delta_a(\vec{w}) = \displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \vec{\Delta}_a^*(\theta)\vec{w}(e^{i\theta}) \qquad \forall \vec{w} \in \vec{H}^\infty, \\ \vec{\Delta}_a(\theta) \in \vec{L}^1 \end{cases}$$

On the other hand, a representation valid in $\vec{H}^2_\rho$ simply results from the well-known representation theorem in H^2 [3] via the isomorphism between $\vec{H}^2$ and $\vec{H}^2_\rho$:

$$(3.3)\qquad \begin{cases} \Delta_a(\vec{w}) = \Delta_{a,\rho}(\vec{h}) = \displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, \vec{\Delta}_{a,\rho}^*(\theta)G(e^{i\theta})\vec{w}(e^{i\theta}) \qquad \forall \vec{w} = \frac{\vec{h}}{G} \in \vec{H}^2_\rho \\ \vec{\Delta}_{a,\rho}(\theta) \in \vec{L}^2 . \end{cases}$$

The representations (3.2) and (3.3) also define natural extensions of the functionals Δ_a and $\Delta_{a,\rho}$ to the spaces $\vec{L}^\infty$ and $\vec{L}^2$ respectively. Let us first apply eq. (3.1) to:

$$(3.4)\qquad \operatorname*{Sup}_{\vec{w}\in S_\rho} |\Delta_a(\vec{w})| = \operatorname*{Sup}_{\substack{\vec{h}\in\vec{H}^2 \\ ||\vec{h}||_2 \leq 1}} |\Delta_{a,\rho}(\vec{h})|$$

We need a characterization of the annihilator $\vec{H}^{2\perp}$ of $\vec{H}^2$ in the dual of $\vec{L}^2$ (which can be identified with $\vec{L}^2$ itself). According to a theorem known for H^2 [3] and readily generalized to $\vec{H}^2$:

$$(3.5)\qquad \psi \in H^{2\perp} \quad \text{iff} \quad \psi(\vec{f}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi}\, e^{i\theta}\, \vec{v}(e^{i\theta})\vec{f}(\theta) \qquad \text{with} \quad \vec{v} \in \vec{H}^2$$

$$\left(\text{clearly } \psi(\vec{w}) = \oint_{|z|=1} dz\, \vec{v}(z)\vec{w}(z) = 0 \text{ for } \vec{w} \in \vec{H}^2\right).$$

Hence, from eqs. (3.3)-(3.5), and noticing that the canonical norm of a functional as element of the dual $(\vec{L}^2)^*$ coincides with the $\vec{L}^2$-norm (2.3) of its kernel, we obtain:

$$(3.6)\qquad \begin{aligned} \operatorname*{Sup}_{\vec{w}\in S_\rho} |\Delta_a(\vec{w})| &= \operatorname*{Min}_{\vec{v}\in\vec{H}^2} ||\, \vec{\Delta}_{a,\rho}^*(\theta) + e^{i\theta}\,\vec{v}(e^{i\theta})\, ||_{\vec{L}^2} \\ &= \operatorname*{Min}_{\vec{v}\in\vec{H}^2} \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} |\, \vec{\Delta}_{a,\rho}^*(\theta) + e^{i\theta}\,\vec{v}(e^{i\theta})\,|^2 \right]^{\frac{1}{2}} \end{aligned}$$

Similarly, would it be true that $(\vec{L}^\infty)^* = \vec{L}^1$, a blind application of eq. (3.1) to $\operatorname{Sup}_{\vec{w}\in\vec{S}} |\Delta_a(\vec{w})|$ would give:

$$(3.7)\qquad \operatorname*{Sup}_{\vec{w}\in\vec{S}} |\Delta_a(\vec{w})| = \operatorname*{Min}_{\vec{u}\in\vec{H}^1} \| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta}) \|_{\vec{L}^1} = \operatorname*{Min}_{\vec{u}\in\vec{H}^1} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta}) |$$

Although this « naive » formula cannot be deduced directly from eq. (3.1), it turns out to be correct *as long as the functional* Δ_a *admits in* $\vec{H}^\infty$ *a representation of the form* (3.2). It is actually the (easy) extension to $\vec{H}^\infty$ of an analogous duality formula known for H^∞. For completeness, we give the full proof in Appendix.

Now, let $\vec{u}_0$ be an element of $\vec{H}^1$ for which the infimum is attained in the r. h. s. of eq. (3.7):

$$(3.8)\qquad \operatorname*{Sup}_{\vec{w}\in\vec{S}} |\Delta_a(\vec{w})| = I \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) | ,$$

and suppose that the kernels of the representations (3.2) and (3.3) can be identified:

$$(3.9)\qquad \vec{\Delta}_{a,\rho}(\theta) G^*(e^{i\theta}) \equiv \vec{\Delta}_a(\theta)$$

By comparing eqs. (3.6) and (3.8), we see that if we were allowed to choose $\rho(\theta)$ and $\vec{v}(z)$ as follows:

$$(3.10)\qquad \begin{cases} \rho(\theta) = | G(e^{i\theta}) |^2 = \dfrac{| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |}{\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |}, \\[2ex] \vec{v}(z) = \dfrac{u_0(z)}{G(z)} \end{cases}$$

then:

$$(3.11)\qquad \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\Delta}_{a,\rho}^*(\theta) + e^{i\theta}\vec{v}(e^{i\theta}) |^2 \right]^{\frac{1}{2}} = \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |^2}{\rho(\theta)} \right]^{\frac{1}{2}} = I$$

and the proof of eq. (2.14) would be completed.

This simple argument has to be refined however because:

i) One has to make sure that the identification (3.9) is compatible with the memberships $\vec{\Delta}_{a,\rho} \in \vec{L}^2$, $\vec{\Delta}_a \in \vec{L}^1$.

ii) The functions $\rho(\theta)$ and $\vec{v}(z)$ given by eq. (3.10) have no reason to belong to their proper class: $\operatorname{Log} \rho(\theta) \in L^1$, $\vec{v} \in \vec{H}^2$.

In order to settle the point *i*), we first remark that in $\vec{H}^\infty$, we can use a

Cauchy-type formula to represent our particular functional Δ_a. Actually, the formulae:

$$(3.12)\quad w_i^{(k)}(x_i) = \int_{-\pi}^{\pi} \frac{d\theta}{d\pi} \Delta_{i,k}^*(e^{i\theta}) w_i(e^{i\theta}), \qquad \Delta_{i,k}(z) = k!\, \frac{z^k}{(1 - x_i z)^{k+1}}$$

are valid when $w_i \in H^\infty$, so that the kernel $\vec{\Delta}_a(\theta)$, which according to eqs. (1.2), (2.10) and (3.2) appears as a linear combination of the $\Delta_{i,k}(e^{i\theta})$'s, obviously belongs to $\vec{L}^1$. We see that $\vec{\Delta}_a(\theta)$ is even continuous. Such a stronger property will turn out to be useful below. In fact, for our proof to be valid, we only need to assume:

$$(3.13)\qquad \vec{\Delta}_a(\theta) \in \vec{L}^2$$

This does not imply yet that $\vec{\Delta}_{a,\rho}(\theta) = \vec{\Delta}_a(\theta)/G(e^{i\theta}) \in \vec{L}^2$, because the function $G(z)$ could have zeros on the unit circle. To prevent this accident, we shall use only strictly positive functions $\rho(\theta)$:

$$(3.14)\qquad m \equiv \underset{-\pi<\theta<\pi}{\text{Ess. inf}}\ \rho(\theta) > 0$$

Under this condition, $|\vec{\Delta}_{a,\rho}(\theta)| \leq \frac{1}{m} |\vec{\Delta}_a(\theta)|$ a. e. and $\vec{\Delta}_a(\theta) \in \vec{L}^2$ entails $\vec{\Delta}_{a,\rho}(\theta) \in \vec{L}^2$.

We now observe that the choice of $\rho(\theta)$ given in eq. (3.10) cannot be made safely. Indeed, it might happen that $|\vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta})|$ vanishes somewhere on $-\pi \leq \theta \leq \pi$, in which case the condition (3.14) would be violated. We shall therefore abandon the choice (3.10), and exhibit instead a family of functions $\rho_\varepsilon(\theta)$ and $\vec{v}_\varepsilon(z)$ meeting all the required conditions and such that:

$$(3.15)\qquad \lim_{\varepsilon \to 0} I_\varepsilon = I$$

where:

$$(3.16)\qquad I_\varepsilon \equiv \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \left| \frac{\vec{\Delta}_a^*(\theta)}{G_\varepsilon(e^{i\theta})} + e^{i\theta} \vec{v}_\varepsilon(e^{i\theta}) \right|^2 \right]^{\frac{1}{2}}$$

and $I\ (> 0)$ is defined in eq. (3.8). Clearly, this is enough to establish the equality (2.14). To construct $\rho_\varepsilon(\theta)$, let us first introduce, for any $\varepsilon > 0$:

$$(3.17)\qquad \delta_\varepsilon = \{\, \theta \mid |\vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta})| \leq \varepsilon I \,\}, \qquad \delta'_\varepsilon = \complement \delta_\varepsilon$$

and:

$$(3.18)\qquad N_\varepsilon = \varepsilon \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} + \frac{1}{I} \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} |\vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta})|$$

From the very definition of I, this can be rewritten as:

$$(3.19)\qquad N_\varepsilon = 1 + \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \left[\varepsilon - \frac{1}{I} |\vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta})| \right]$$

so that:

$$(3.20) \qquad 1 \leq N_\varepsilon \leq 1 + \varepsilon$$

If we now define:

$$(3.21) \qquad \rho_\varepsilon(\theta) = \begin{cases} \dfrac{\varepsilon}{N_\varepsilon}, & \theta \in \delta_\varepsilon \\ \dfrac{| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |}{N_\varepsilon I}, & \theta \in \delta'_\varepsilon \end{cases}$$

We immediately see that $\rho_\varepsilon(\theta) \in L^1$ and $\rho_\varepsilon(\theta) \geq \varepsilon/N_\varepsilon$ a. e.

Thus condition (3.13) is verified and moreover Log $\rho_\varepsilon(\theta) \in L^1$, which allows us to construct the outer function $G_\varepsilon(z)$ according to eq. (2.5). The normalization condition (1.3) is also satisfied.

We choose next:

$$(3.22) \qquad \vec{v}_\varepsilon(z) = \frac{\vec{u}_0(z)}{G_\varepsilon(z)}$$

One has to check that $\vec{v}_\varepsilon \in \vec{H}^2$. First of all $\vec{v}_\varepsilon \in \vec{H}^1$ because $\vec{u}_0 \in \vec{H}^1$ and $1/G_\varepsilon(z)$ is an outer function in $H^\infty(| 1/G_\varepsilon(e^{i\theta}) | \leq \sqrt{N_\varepsilon/\varepsilon})$. Therefore, it is sufficient [3] to show that $\vec{v}_\varepsilon(e^{i\theta}) \in \vec{L}^2$, namely that:

$$(3.23) \qquad || \vec{v}_\varepsilon ||_2^2 = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{| \vec{u}_0(e^{i\theta}) |^2}{\rho_\varepsilon(\theta)} = \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} | \vec{u}_0(e^{i\theta}) |^2 + N_\varepsilon I \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} \frac{| \vec{u}_0(e^{i\theta}) |^2}{| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |}$$

is finite. Using the triangle inequality

$$| \vec{u}_0(e^{i\theta}) | \leq | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) | + | \vec{\Delta}_a^*(\theta) |$$

and the bound

$$| \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) | > \varepsilon I$$

on δ'_ε, we obtain:

$$(3.24) \qquad || \vec{v}_\varepsilon ||_2^2 \leq \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} | \vec{u}_0(e^{i\theta}) |^2 + N_\varepsilon I \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} [\, | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) | + 2 | \vec{\Delta}_a(\theta) | \,] + \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} | \vec{\Delta}_a(\theta) |^2$$

All three integrals in the r. h. s. are finite: the first one because

$$| \vec{u}_0(e^{i\theta}) | \leq \varepsilon I + | \vec{\Delta}_a(\theta) |$$

on δ_ε and $\vec{\Delta}_a(\theta) \in \vec{L}^2$, the second and third ones because $\vec{u}_0 \in \vec{H}^1$ and $\vec{\Delta}_a(\theta) \in \vec{L}^2 \subset \vec{L}^1$. Therefore $|| \vec{v}_\varepsilon ||_2 < \infty$ and $\vec{v}_\varepsilon \in \vec{H}^2$.

Finally, it remains to check eq. (3.15). According to eqs. (3.16), (3.21) and (3.22):

$$(3.25) \qquad I_\varepsilon^2 = \frac{N_\varepsilon}{\varepsilon} \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |^2 + N_\varepsilon I \int_{\delta'_\varepsilon} \frac{d\theta}{2\pi} | \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) |$$

This gives, using eq. (3.18):

$$(3.26)\quad I_\varepsilon^2 - I^2 = \frac{N_\varepsilon}{\varepsilon}\int_{\delta_\varepsilon} \frac{d\theta}{2\pi} \mid \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) \mid^2 - N_\varepsilon I^2 \varepsilon \int_{\delta_\varepsilon} \frac{d\theta}{2\pi} + (N_\varepsilon^2 - 1)I^2$$

and, since $\mid \vec{\Delta}_a^*(\theta) + e^{i\theta}\vec{u}_0(e^{i\theta}) \mid^2 \leq \varepsilon^2 I^2$ on δ_ε:

$$(3.27)\qquad \mid I_\varepsilon^2 - I^2 \mid \leq 2N_\varepsilon I^2 \varepsilon + (N_\varepsilon^2 - 1)I^2$$

Therefore $\lim_{\varepsilon \to 0} I_\varepsilon = I$ on account of eq. (3.20), and the proof is completed.

APPENDIX

We derive here the duality formula:

$$\text{(A.1)} \qquad \underset{\substack{\vec{w}\in\vec{H}^\infty \\ \|\vec{w}\|_\infty \le 1}}{\text{Sup}} |\Delta(\vec{w})| = \underset{\vec{u}\in\vec{H}^1}{\text{Min}} \| \vec{\Delta}^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta}) \|_{\vec{L}^1}$$

where:

$$\text{(A.2)} \qquad \Delta(\vec{w}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \vec{\Delta}^*(\theta)\vec{w}(e^{i\theta}), \qquad \vec{\Delta}(\theta) \in \vec{L}^1$$

Our proof is nothing but a straight forward extension of that given in ref. [3] (chap. 8) for H^∞.

Let $\vec{C}(\subset \vec{L}^\infty)$ be the Banach space consisting of all continuous functions $\vec{f}(\theta)$, equipped with the $\vec{L}^\infty$-norm (2.2), and $\vec{P} \subset \vec{H}^\infty$ the (closed) subspace of $\vec{C}$ generated by $f_i(\theta) = 1$, $e^{i\theta}$, $e^{2i\theta}$, ... Then, according to the general duality formula (3.1):

$$\text{(A.3)} \qquad \underset{\substack{\vec{w}\in\vec{P} \\ \|\vec{w}\|_\infty \le 1}}{\text{Sup}} |\Delta(\vec{w})| = \underset{\psi\in\vec{P}^\perp}{\text{Min}} \| \Delta + \psi \|_{\vec{C}^*}$$

where $\vec{P}^\perp$, the annihilator of $\vec{P}$ in $\vec{C}^*$, has to be characterized. As a consequence of the well-known Riesz representation theorem in C^*, there exists n complex measures $\mu_i(\theta) = \mu_i^*(-\theta)$ such that every functional $\psi \in \vec{C}^*$ has the form:

$$\text{(A.4)} \qquad \psi(\vec{f}) = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{-\pi}^{\pi} d\mu_i(\theta) f_i(e^{i\theta}), \qquad f \in \vec{C}$$

But $\psi \in \vec{P}^\perp$ implies $\psi(\vec{w}) = 0$ for all $\vec{w}$ of the form $\vec{w}(z) = (0, \ldots, 0, z^p, 0, \ldots, 0)$ $(p = 0, 1, 2, \ldots)$. Thus, for $i = 1, \ldots, n$:

$$\text{(A.5)} \qquad \int_{-\pi}^{\pi} d\mu_i(\theta) e^{ip\theta} = 0, \qquad p = 0, 1, 2, \ldots$$

which in turns implies, according to the theorem of F. and M. Riesz [3]:

$$\text{(A.6)} \qquad d\mu_i(\theta) = u_i(e^{i\theta})e^{i\theta}d\theta, \qquad u_i(z) \in H^1$$

Hence $(\Delta + \psi)$ in eq. (A.3) has the representation:

$$\text{(A.7)} \qquad (\Delta + \psi)(\vec{f}) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} [\vec{\Delta}^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta})]\vec{f}(e^{i\theta})$$

with $\vec{u} \in \vec{H}^1$.

Now, for any functional $\phi \in \vec{C}^*$ represented by a kernel $\vec{\phi}(\theta) \in \vec{L}^1$ as in eq. (A.7), one has:

$$\text{(A.8)} \qquad \| \phi \|_{\vec{C}^*} = \| \vec{\phi}(\theta) \|_{\vec{L}^1}$$

as we shall prove in a moment. We conclude that:

$$\text{(A.9)} \qquad \underset{\substack{\vec{w}\in\vec{P} \\ \|\vec{w}\|_\infty \le 1}}{\text{Sup}} |\Delta(\vec{w})| = \underset{\vec{u}\in\vec{H}^1}{\text{Min}} \| \vec{\Delta}^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta}) \|_{\vec{L}^1},$$

and, since $\vec{H}^\infty \supset \vec{P}$:

$$\text{(A.10)} \qquad \underset{\substack{\vec{w}\in\vec{H}^\infty \\ \|\vec{w}\|_\infty \le 1}}{\text{Sup}} |\Delta(\vec{w})| \ge \underset{\vec{u}\in\vec{H}^1}{\text{Min}} \| \vec{\Delta}^*(\theta) + e^{i\theta}\vec{u}(e^{i\theta}) \|_{\vec{L}^1}$$

On the other hand, we obviously have, for any $\vec{w} \in \vec{\mathrm{H}}^{\infty}$:

(A.11) $$| \Delta(\vec{w}) | \leq \| \vec{w} \|_{\infty} \, \| \vec{\Delta}^*(\theta) + e^{i\theta} \vec{u}(e^{i\theta}) \|_{\vec{\mathrm{L}}^1}$$

Formula (A.1) then follows from eqs. (A.10) and (A.11).

Proof of eq. (A.8).

From the fact that $\vec{\mathrm{C}}$ is a dense subset of $\vec{\mathrm{L}}^1$, we infer the existence of a sequence $\{ \vec{\phi}_n \}$ in $\vec{\mathrm{C}}$ such that:

(A.12) $$\lim_{n \to \infty} \vec{\phi}_n = \vec{\phi}$$

in the $\vec{\mathrm{L}}^1$-norm. Then:

(A.13) $$\| \phi_n \|_{\vec{\mathrm{C}}^*} \equiv \sup_{\substack{\vec{f} \in \vec{\mathrm{C}} \\ \| \vec{f} \|_{\infty} \leq 1}} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \vec{\phi}_n^*(\theta) \vec{f}(\theta) \right| \leq \sup_{\substack{\vec{f} \in \vec{\mathrm{C}} \\ \| \vec{f} \|_{\infty} \leq 1}} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\phi}_n(\theta) | \, | \vec{f}(\theta) |$$
$$\leq \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\phi}_n(\theta) | = \| \vec{\phi}_n(\theta) \|_{\vec{\mathrm{L}}^1}$$

But, since $\vec{f}_\varepsilon(\theta) \equiv \vec{\phi}_n(\theta) / [\, | \vec{\phi}_n(\theta) | + \varepsilon \,] \in \vec{\mathrm{C}}$ and $\| \vec{f}_\varepsilon \| < 1$ for any $\varepsilon > 0$:

(A.14) $$\| \phi_n \|_{\vec{\mathrm{C}}^*} \geq \sup_{\varepsilon > 0} \left| \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \vec{\phi}_n^*(\theta) \vec{f}_\varepsilon(\theta) \right|$$
$$= \sup_{\varepsilon > 0} \left[\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\phi}_n(\theta) | - \varepsilon \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{| \vec{\phi}_n(\theta) |}{| \vec{\phi}_n(\theta) | + \varepsilon} \right] = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} | \vec{\phi}_n(\theta) | = \| \vec{\phi}_n(\theta) \|_{\mathrm{L}^1}$$

Comparing eqs. (A.13) and (A.14), we obtain:

(A.15) $$\| \phi_n \|_{\vec{\mathrm{C}}^*} = \| \vec{\phi}_n(\theta) \|_{\vec{\mathrm{L}}^1} \qquad \forall n$$

On the other hand, eq. (A.12) obviously implies:

(A.16) $$\lim_{n \to \infty} \| \vec{\phi}_n(\theta) \|_{\vec{\mathrm{L}}^1} = \| \vec{\phi}(\theta) \|_{\vec{\mathrm{L}}^1}$$

and also:

(A.17) $$\lim_{n \to \infty} \| \phi_n \|_{\vec{\mathrm{C}}^*} = \| \phi \|_{\vec{\mathrm{C}}^*}$$

because:

(A.18) $$\left| \frac{| \phi_n(\vec{f}) |}{\| \vec{f} \|_{\infty}} - \frac{| \phi(\vec{f}) |}{\| \vec{f} \|_{\infty}} \right| \leq \frac{| (\phi_n - \phi)(\vec{f}) |}{\| \vec{f} \|_{\infty}} \leq \| \vec{\phi}_n(\theta) - \vec{\phi}(\theta) \|_{\vec{\mathrm{L}}^1} .$$

Eqs. (A.15)-(A.17) yield the announced formula (A.8).

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[4] See e. g. N. Dunford and J. T. Schwartz, *Linear operators*, Interscience Publishers, N. Y., 1958, Chap. V.4.

[5] See e. g. ref. [3], Theorem 7.1.

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