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## Topological and Noether-conservation laws

by

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**ABSTRACT.** — It is shown that Noether current fields and topological current fields and the corresponding conservation laws associated with these current-fields relate to a common mathematical structure which is de Rham cohomology. Topological current fields, however, differ from Noether current fields in that they are shown to display additional features which are certain homotopic properties. Therefore, topological fields will be referred to as homotopic current fields.

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### 1. INTRODUCTION

There are conservation laws in physics, homotopic (or topological) conservation laws, which are not derived from any symmetries of a Lagrangian, but rather from the topology of the manifold of solutions (or field manifold) to a given eq. of motion. These new conservation laws depend on a well defined pattern of symmetry breaking effect. In fact, in a theory with spontaneous symmetry breakdown, the symmetry properties of the « vacuum » or ground state are as significant as those of the defining Lagrangian. When there are degenerate « vacua », there exist topological currents. In this case, the lowest energy solution of a given field equation does not share the full symmetry of the field eq.; This non-invariance of a ground state under a symmetry group  $G$  is attributed to the dynamics. The concept

of topological current or charge arises then as follows. Consider the two most important classes of solutions to a field equation:

(A) *Static solutions  $\phi$  or solitons* (solitary waves) [I] which minimize a given energy functional, and

(B) *Constant solutions*, which are identified with a vacuum state  $\phi_0$ .

Soliton solutions (A) enjoy the following properties: *i)* Solitons occur only if the vacuum state  $\phi_0$  is degenerate, i. e. if the theory exhibits spontaneous symmetry breaking, *ii)* Soliton solutions with finite energy tend at large distances to a vacuum solution  $\phi_0$  of type (B) on some « boundary sphere »  $S^n(r)$ , of large radius  $r \rightarrow \infty$  i. e. to the asymptotic vacuum value

$$(1) \quad \lim_{r \rightarrow \infty} \phi(r \vec{n}) = \phi_0 \in M_0 \quad (M_0 \text{ is a vacuum manifold, cf. sect. II})$$

*iii)* Soliton solutions interpolate degenerate vacua at infinity. *iv)* Solitons are classified by equivalence classes. In fact, if  $\phi, S^n \rightarrow M_0$ ,  $\phi \mapsto \phi_0$  is a certain soliton solution for a certain direction  $\vec{n} = \vec{r}/r$ , there is an equivalent soliton which is obtained by a certain rotation. Since these equivalence classes are homotopy classes, solitons will be labelled by certain *homotopy invariants*, i. e. topological charges which determine these homotopy classes. *v)* Soliton solutions are stable.

That soliton solutions are stable with respect to small arbitrary perturbations of the dynamics is displayed best for the (1 + 1)-dimensional space-time. There is a trivial conservation law, i. a. a conserved current

$$(2) \quad j_\mu(t, x) = \varepsilon_{\mu\nu} \partial_\nu \phi(t, x); \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad \varepsilon_{01} = 1,$$

where

$$(3) \quad \partial_\mu j_\mu = 0,$$

which accounts for this stability. In fact, the soliton solution  $\phi(t, x)$  is stable in that it is the lowest energy solution such that

$$(4) \quad Q[\phi] = \int_{-\infty}^{+\infty} dx j_0(t, x) = \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial x} dx = \int_{-\infty}^{+\infty} d\phi \\ = \phi(t, +\infty) - \phi(t, -\infty); \quad \text{i. e.}$$

$$(5) \quad Q[\phi] = [\phi(t, +\infty) - \phi(t, -\infty)] = N[\phi] \equiv N$$

is regarded as *topological quantum number*, or *soliton number*. It is conserved due to the spatial asymptotic behaviour of solutions  $\phi(t, x)$  of a field eq. of the Sine-Gordon type  $\phi_{xx} - \phi_{tt} = \sin \phi$ . Its constant solutions (B) are  $\phi = 2\pi N$ . So  $\phi(t, +\infty) - \phi(t, -\infty) = 2\pi N$  for some  $N \in \mathbb{Z}$ . This soliton number takes the value  $\phi(t, \infty) - \phi(t, -\infty) = 0$  for the ground state (vacuum sector),  $N = +1, +2, \dots$  for one, two... solitons and  $N = -1, -2, -\dots$  for one, two, ... anti-solitons of the soliton sector. Here soliton

solutions are by property *ii*) the constant solutions and (5) becomes, (5')  $N = 1/2\pi [\phi(\infty) - \phi(-\infty)]$ . Clearly the conservation law (3) does not correspond to any symmetry of the theory, i. e. this conservation law is topological in nature and the topological properties of the 2-dimensional space-time relate  $N \in \mathbb{Z}$  to the conserved current (2). To summarize: Topological conserved currents (2) display the following distinctive features:

- A topological current is conserved independently of any field eq.
- Topological currents are not associated with any symmetry of a Lagrangian.
- Topological currents are derived from a degenerate vacuum.
- The integral  $Q[\phi] = \int_{-\infty}^{+\infty} j_0 dx$  is nonzero if the Higgs-field  $\phi$  satisfies the boundary conditions  $\phi(\pm \infty) \in M_0$  (cf. eq. (1)).
- The component  $j_0$  of the current contains only canonical coordinates and no momenta.

As regards *Noether currents* (cf. ref. [2], sect. V) they are associated with a Lie group symmetry  $G$  of a Lagrangian  $L$ , i. e. there is a group of automorphisms  $G$  whose transformations leave the corresponding dynamical system invariant. That is,  $L$  is  $G$ -invariant if  $L(\psi) = L(\psi')$ , i. e.  $\delta L = 0$  if  $L$  remains unchanged under a transformation

$$\psi_\alpha(x) \rightarrow \psi'_\alpha(x) = \sum_\beta U_{\alpha\beta} \psi_\beta(x)$$

of the fields. This implies the existence of conserved currents  $j_\mu^i$ ,  $\partial_\mu j_\mu^i = 0$  and the existence of conserved charges  $Q^i = \int dx j_0^i(t, x)$ .

The aim of this paper is to exhibit that both, topological conservation laws and Noether conservation laws derive from a common type of topological fields, which are related to a unified mathematical structure which is de Rham cohomology [2], [3], [5].

## 2. HOMOTOPY AND TOPOLOGICAL CONSERVATION LAWS

A classification of solutions (solitons, vortices, etc.) to field eqs. involves homotopy classes  $[\phi]$  of maps  $\phi : S^n \rightarrow M_0$  of an  $n$ -sphere  $S^n$  into a certain vacuum manifold  $M_0$  which labels the different possible vacuum states. Accordingly, solitons, vortices, etc. carry a topological label, the homotopy invariant or topological charge of  $\phi$ . Solitons occur in connection with a degenerate vacuum, i. e. in theories with spontaneous symmetry break-

down. Therefore, a rigorous characterization of the mechanism underlying spontaneous symmetry breaking is given first.

DÉFINITION 1. — Let  $\phi = \phi(x)$ ,  $x \in M^4$  (the space-time manifold) be a Higgs-field and  $V = V(\phi) \geq 0$  be an effective potential.  $\phi(x)$  is said to be in a vacuum in a certain region of  $M^4$ , if for the covariant derivative of  $\phi$ ,  $\nabla_\mu \phi$ ,

$$(6) \quad \nabla_\mu \phi = 0,$$

$$(7) \quad V(\phi) = 0$$

and that moreover the field strengths of a Yang-Mills type field  $F_{\mu\nu}$ , related to  $\phi$  through a Lagrangian, vanish;  $F_{\mu\nu} = 0$ .

The Higgs-field  $\phi$  is assumed to transform under a continuous representation  $G \rightarrow \varphi(G)$  of a given gauge group  $G$ , which leaves  $V$  invariant, i. e.  $V(\varphi(g)\phi) = V(\phi)$ ,  $g \in G$ . The vacuum manifold  $M_0$  which minimizes the self-interaction  $V(\phi)$  of  $\phi$  is given to be a homogeneous space  $G/H_\phi$  [3], i. e.

$$(8) \quad M_0 = \varphi(G)\phi_0 = \{ \varphi(g)\phi_0 \mid g \in G \} = G/H_\phi$$

where  $\phi$ ,  $\phi_0 \in M_0$  are fixed and where

$$(9) \quad H_\phi = \{ g \in G \mid \varphi(g)\phi = \phi \}$$

is the isotropy subgroup of  $G$  at  $\phi$ . By virtue of the invariance property  $V(\varphi(g)\phi) = V(\phi)$ , if  $\phi_0$  satisfies  $V(\phi) = 0$  (eq. (7)), so does  $\varphi(g)\phi_0 \forall g \in G$ , i. e.  $\phi_0$  and  $\varphi(g)\phi_0$  lie on the same orbit  $O_{\phi_0} = \{ \varphi(g)\phi_0 \mid g \in G, \phi_0 \in M_0 \}$  which is  $M_0 = \varphi(G)\phi_0$  itself. Otherwise stated:  $G$  acts transitively on  $M_0$ , that is

$$(10) \quad (\forall \phi_1, \phi_2 \in M_0)(\exists g \in G) : \quad \phi_1 = \varphi(g)\phi_2.$$

Physically  $H := H_\phi$  is of prime importance in that  $H$  is the exact gauge symmetry of some system. Now spontaneous symmetry break-down is characterized as follows:

DÉFINITION 2. — A gauge symmetry  $G$  associated with a Lagrange field-theoretical model

$$(11) \quad L = L_1(A_\mu) + |\nabla_\mu \phi|^2 - V(\phi)$$

( $A_\mu$  is a Yang-Mills type potential) is said to be spontaneously broken iff there is a vacuum manifold  $M_0$  given by eqs. (8)-(9).

The following cases must be distinguished:

(12 a)  $G = H$  : The vacuum  $\phi_0$  is unique; the symmetry  $G$  is exact.

(12 b)  $\{ e \} \subset H \subset G$  : The symmetry  $G$  is partly broken.

(12 c)  $H = \{ e \}$  : The symmetry is totally broken.

The way how homotopic conservation laws arise can be apprehended in the case of a classification of soliton-like solutions or of a classification of

static vortices (cylindrically symmetric case, where a vortex is lying along the  $z$ -axis) relating to a model with Lagrangian (11) and gauge symmetry group  $G = U(1)$ , such that

$$(13) \quad \phi(x) \mapsto e^{i\alpha(x)}\phi(x) \quad x \in M^4$$

and

$$(14) \quad \overset{1}{\omega} = A_\mu dx^\mu \mapsto \overset{1}{\omega}' = \overset{1}{\omega} + d\alpha \quad (A_\mu \mapsto A_\mu + \partial_\mu \alpha)$$

$d\alpha, \overset{1}{\omega}, \overset{1}{\omega}' \in F^1(M^4)$  (vector space of 1-forms on  $M^4$ ) are 1-forms. (All notations are explained in sect. III below, as well as in ref. [3])  $G$  is completely broken with one scalar Higgs-field, so that by (12 c)  $H = \{e\}$  and

$$M_0 = G/\{e\} = G = S^1.$$

Soliton solutions and static vortices are then labelled by elements of the fundamental group of  $S^1$  [6]:

$$(15) \quad \pi_1(S^1) = \pi_1(M_0) = \mathbb{Z} \quad [5], [6].$$

In the case of vortex solutions one has: If  $\phi_0(0)$  minimizes the potential  $V(\phi)$ , then

$$(16) \quad \phi(\alpha) = U(\alpha)\phi_0(0)$$

minimizes  $V(\phi)$ , where  $U(\alpha) = e^{in\alpha} \in U(1)$  [5]. Hence we conclude:

*The existence of a topological quantum number associated with a field of type (16) derives from the multivaluedness of the phase  $n\alpha$ .*

Alternately, the topological quantum number  $n \in \mathbb{Z}$  can be obtained from a Homology classification of static vortices [5]. Let

$$(17) \quad (\overset{1}{\omega}, c_1); \quad \overset{1}{\omega} = d\vartheta = \frac{x \cdot dy - y \cdot dx}{x^2 + y^2} \in F^1(\mathbb{R}^2 - \{0\})$$

and  $c_1 \in \overset{0}{C}_1(\mathbb{R}^2 - \{0\})$ , the space of 1-cycles (cf. subsequent sect. III), and denote by

$$(18) \quad H^1(S^1) = F^1(S^1)/dF^0(S^1) = \{ad\vartheta \mid a \in \mathbb{R}\} \cong \mathbb{R}^1 \quad [3], [5]$$

(closed 1-forms modulo exact 1-forms on  $S^1$ ) the first de Rham group and by  $H_1(S^1) = H_1$  the first homology group of  $S^1$  (chapt. 10 of [3] and subsequent sect. III). Then, by de Rham's first theorem there exists a non-degenerate bilinear form  $\beta$  ([2], [3], [5])

$$(19) \quad \beta : H^1 \times H_1 \rightarrow \mathbb{R}; \quad (\overset{1}{\omega}, c_1) \mapsto \text{const.} \int_{c_1} \frac{x dy - y dx}{x^2 + y^2}$$

such that

$$(20) \quad w_0(\psi) = n = 1/2\pi \int_{S^1} \overset{1}{\omega}; \quad n \in \mathbb{Z}, c_1 = S^1.$$

Eq. (20) is the winding number of a smooth map  $\psi : S^1 \rightarrow \mathbb{R}^2 - \{0\}$  about 0. It measures how many times  $c_1 = S^1$  winds around the origine. Hence the topological quantum number  $n$  or vortex number corresponding to the field  $(\omega, c_1)$  is the winding number of the map  $\psi$  about 0 [5].

Topological quantum numbers derive also from higher homotopy groups. Let  $M_0 = G/H$  be a homogeneous space.  $G$  is simply connected by assumption. Then for the first homotopy group  $\pi_1(G) = 0$ . Since  $G$  is a Lie group and  $G_0$  the arcwise connected component of the identity, the factor group  $G/G_0$  can be identified with  $\pi_0(G)$ , i. e.  $\pi_0(G) = G/G_0$ . There is an isomorphism [6], [8], [9]

$$(21) \quad \pi_1(G/H) \cong \pi_0(H),$$

moreover  $\pi_2(G/H)$  and  $\pi_1(H)$  are isomorphic, i. e. if  $G$  is simply connected and  $\pi_2(G/H)$  is the second homotopy group of  $G/H$  then

$$(22) \quad \pi_2(G/H) \cong \pi_1(H) \quad [8], [9]$$

The isomorphism (20) then admits a characterization of the structure of topological quantum numbers in terms of the first homotopy group  $\pi_1(H)$  of the isotropy subgroup  $H$  of  $G$ , where  $H$  is the exact gauge symmetry group, in terms of the following classification:

- (23 a)  $\pi_1(H) = 0$ : No solutions of fields eqs. with non-trivial topological quantum numbers occur. Thus, symmetries as  $H = SU(2)$  or  $SU(3)$  don't admit topological quantum numbers.
- (23 b)  $G$  cannot be completely broken, since otherwise  $H = \{e\}$  (12 c),  $\pi_2(G/H) = \pi_2(G) = 0$  and no non-trivial solitons occur. ( $\pi_2(G)$  is the second homotopy group of  $G$ ).
- (23 c)  $G$  cannot be Abelian, since otherwise  $H$  is Abelian and for the second homotopy group  $\pi_2(G/H) = 0$  [8], [9], [6].

### 3. HOMOLOGY AND NOETHER CONSERVATION LAWS

III.1. Let  $M := M^n$  be an  $n$ -dimensional physical configuration space and denote by  $F^p(M)$ ,  $\overset{0}{F}^p(M)$  and  $dF^{p-1}(M)$ ,  $p = 0, 1, \dots, n$ , the spaces of differential forms of degree  $p$  ( $p$ -forms), closed  $p$ -forms and exact  $p$ -forms on  $M$  respectively [3].  $C_p(M)$  and  $\overset{0}{C}_p(M)$ ,  $p = 0, 1, \dots, n$ , stand for the spaces of differentiable  $p$ -chains and closed  $p$ -chains (or  $p$ -cycles), respectively [3];  $H^p(M)$  and  $H_p(M)$  denote the  $p$ -th de Rham cohomology and the  $p$ -th homology group of  $M$ , respectively [3].

A description of Noether conservation laws, i. e. conserved Noether currents and the conserved Noether observables associated with these currents can now be given in terms of a field theoretical framework of non-

local field quantities which generalize the field (17) of sect. II and which are defined as

$$(24) \quad (\overset{p}{\omega}, c_p), \overset{p}{\omega} \in F^p(M) (\overset{p}{\omega} \in F^p(M), \text{ or } \overset{p}{\omega} \in dF^{p-1}), \text{ or } \overset{p}{\omega} \in [\overset{p}{\omega}] \in H^p(M) \\ c_p \in C_p(M) (c_p \in \overset{0}{C}_p(M))$$

The corresponding integral laws (physical observables) associated with these field quantities are of the form [3], [5]

$$(25) \quad (\overset{p}{\omega}, c_p) \mapsto \int_{c_p} \overset{p}{\omega}.$$

A Noether current field of type (24) (for  $p = 3$ ) can now be given in terms of de Rham's first theorem ([3], chapt. 10) following which there is a non-degenerate bilinear map  $\beta$  (cf. eq. (19)), [2], [3], [5]:

$$(26) \quad \beta : H^3(M^4) \times H_3(M^4) \rightarrow \mathbb{R}; (\overset{3}{\omega}, c_3) \mapsto f(c_3, Y) = \int_{c_3} Y \lrcorner dx = \int_{c_3} \overset{3}{\omega}$$

$Y \in \mathcal{X}(M^4)$  is a current vector field on  $M^4$ ,  $f(c_3, Y)$  stands for a classical conserved observable,  $dx = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \in F^4(M^4)$  is the volume element 4-form on the orientable space-time manifold  $M^4$  [3]. The 3-differential form  $\overset{3}{\omega}$  in (26) is given as

$$(27) \quad \overset{3}{\omega} = Y \lrcorner dx \equiv i(Y)dx = j_0 dx^1 \wedge dx^2 \wedge dx^3 - j_1 dx^0 \wedge dx^2 \wedge dx^3 \\ + \dots - j_3 dx^0 \wedge dx^1 \wedge dx^2$$

where

$$i(Y) := \lrcorner : F^p \rightarrow F^{p-1}$$

is the inner product of a  $p$ -form by the vector field  $Y$  [3], and

$$(27') \quad j_\mu(x) = \frac{\partial L}{\partial \psi_{\alpha,\mu}} \eta_\alpha; \quad \eta_\alpha = \left. \frac{d}{dt} \psi_\alpha^i(x) \right|_{t=0};$$

$L = L(x^\mu, \psi_\alpha, \psi_{\alpha,\mu})$  is a Lagrangian (see eq. (32)).

The condition

$$(28) \quad d\overset{3}{\omega} = 0 \quad (d : F^p \rightarrow F^{p+1} \text{ is the exterior derivative [3]})$$

then expresses that (27) is a conserved current-form corresponding to the 4-current vector  $j_\mu$ . Relation (27) derives from Cartan's formula for the Lie-derivative [3], [4]:

$$(29) \quad L_Y(dx) = Y(dx) = Y \lrcorner d(dx) + d(Y \lrcorner dx) = d(Y \lrcorner dx)$$

and

$$Y \lrcorner dx = i(Y)dx = i(Y)(dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) \\ = i(Y)(dx^0) \wedge dx^1 \wedge dx^2 \wedge dx^3 + (-1)^1 dx^0 \wedge i(Y)(dx^1 \wedge dx^2 \wedge dx^3)$$

by the rules:  $i_Y(\alpha \wedge \beta) = i_Y\alpha \wedge \beta + (-1)^{\text{deg}\alpha} \alpha \wedge i_Y\beta$ ;  $i_Ydf = Y(f)$  [3]; hence

$$(29') \quad Y \lrcorner dx = Y^0 dx^1 \wedge dx^2 \wedge dx^3 - Y^1 dx^0 \wedge dx^2 \wedge dx^3 + Y^2 dx^0 \wedge dx^1 \wedge dx^3 - Y^3 dx^0 \wedge dx^1 \wedge dx^2,$$

since

$$i_Y(dx^1 \wedge dx^2 \wedge dx^3) = i_Y dx^1 \wedge dx^2 \wedge dx^3 + (-1)^1 dx^1 \wedge i_Y(dx^2 \wedge dx^3) \dots \text{etc.}$$

and

$$i_Y(dx^0) = dx^0(Y) = Y(x^0) = Y^0, \quad i_Y dx^1 = Y^1, \quad i_Y dx^2 = Y^2, \quad i_Y dx^3 = Y^3.$$

Thus (27') is the same form as (27) whenever  $Y^0 = j_0, Y^1 = j_1, Y^2 = j_2$  and  $Y^3 = j_3$ . From (27') one infers the following relation to hold:

$$(30) \quad d(Y \lrcorner dx) = \frac{\partial Y^0}{\partial x^0} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 - \frac{\partial Y^1}{\partial x^1} dx^1 \wedge dx^0 \wedge dx^2 \wedge dx^3 \pm \dots - \frac{\partial Y^3}{\partial x^3} dx^3 \wedge dx^0 \wedge dx^1 \wedge dx^2.$$

DÉFINITION 3. — The pair  $(\overset{3}{\omega}, c_3), \overset{3}{\omega} = Y \lrcorner dx \in \overset{0}{F}^3(M^4), c_3 \in C_3(M^4)$  is called a Noether current field. The quantity  $f = f(c_3, Y)$  is an observable which is said to be associated with  $(\overset{3}{\omega}, c_3)$ .

PROPOSITION 4. — Observables which are associated with Noether current fields are constants of motion.

Proof. — Let  $c_3, c'_3 \in \overset{0}{C}_3(M^4)$  be two 3-cycles that « cobound », i. e. are the boundaries of a 4-dimensional region:  $c_3 - c'_3 = \partial c_4$ ;  $\psi, \psi' \in \Gamma(E)$  denote 2 cross sections of a fibre bundle  $E(M^4)$  as given below, then

$$f(c_3, \psi, Y) - f'(c'_3, \psi', Y) = \int_{c_3} \overset{3}{\omega} - \int_{c'_3} \overset{3}{\omega} = \int_{c_3 - c'_3} \overset{3}{\omega} = \int_{\partial c_4} \overset{3}{\omega} = \int_{c_4} d\overset{3}{\omega} = 0 \Rightarrow f(c_3, \psi, Y) = f'(c'_3, \psi', Y) \quad \blacksquare$$

In particular we have

$$(31) \quad (\overset{3}{\omega}, c_3) \mapsto Q = \int_{\substack{c_3 \\ x^0 = \text{const.}}} j_0 dx^1 dx^2 dx^3$$

Noether currents fields are related to a variational principle and to some internal symmetry as follows: Let  $E(M^4)$  be a fibre bundle with base  $M^4$  [2], [3], [4] and  $J^1(E)$  the space of 1-jets associated with  $E$  (cf. sect. III.2 and [2], [4]); a Lagrangian is then a realvalued function on  $J^1(E(M^4))$  [2], [4]

$$(32) \quad L : J^1(E(M^4)) \rightarrow \mathbb{R}, (x^\mu, \psi_\alpha, \partial_\mu \psi_\alpha) \mapsto L(x^\mu, \psi_\alpha, \partial_\mu \psi_\alpha)$$

where  $\psi \in \Gamma(E)$  is a cross-section of  $E$ , determined by  $\psi(x) = (\psi_\alpha(x))$ . A Lagrange 4-form  $\overset{4}{\omega}$  on  $J^1(E)$  is obtained in terms of the diagram

$$(33) \quad \begin{array}{ccc} J^1(E(M^4)) & \xrightarrow{\tau} & E(M^4) \\ \pi^1 \downarrow & \swarrow \pi & \\ M^4 & & \end{array}$$

$\pi$  and  $\pi^1$  are projective mappings of  $E$  and  $J^1(E)$  onto  $M^4$ , respectively,

$$(34) \quad \overset{4}{\omega} = L\pi^{1*}(dx) \in F^4(J^1(E)).$$

DÉFINITION 6. — A vector field  $X \in \mathcal{X}(E(M^4))$  is an infinitesimal symmetry of  $L$  if its first order prolongation  $X^1 \in \mathcal{X}(J^1(E))$ ,

$$X^1 = X^\mu \frac{\partial}{\partial x^\mu} + X^\alpha(X, \psi) \frac{\partial}{\partial \psi_\alpha} + X^{\alpha\mu}(x, \psi, \psi_{,\mu}) \frac{\partial}{\partial \psi_{\alpha\mu}}$$

satisfies

$$(35) \quad X^1(\overset{4}{\omega}) = X^1(L\pi^{1*}dx) = 0 \quad [2], [3], [4].$$

Geometrically, vector fields  $X \in \mathcal{X}(E(M^4))$  where

$$(36) \quad X = \sum_{\mu} X^\mu(x) \frac{\partial}{\partial x^\mu} + \sum X^\alpha(x, \psi) \frac{\partial}{\partial \psi_\alpha},$$

are  $C^\alpha$ -vector fields on  $E$  which generate 1-parameter groups of transformations of  $E(M^4)$  that act on  $M^4$  and, as fiber space automorphisms, permute the fibers of  $E$ , i. e.  $\bar{\varphi}_t : E \rightarrow E$ , such that  $\varphi_t : M^4 \rightarrow M^4$  is a family of diffeomorphisms of  $M^4$  satisfying  $\pi \circ \varphi_t = \bar{\varphi}_t \circ \pi$ . Otherwise stated: The 1-parameter group of automorphisms  $\bar{\varphi}_t : E \rightarrow E$  that map fibres into themselves is a 1-parameter group of internal symmetries for the Lagrangian  $L$ , i. e.

$$(37) \quad \varphi_t^* L(j_1(\psi_t)) dx = L(j_1(\psi)) dx \in F^4(M^4) \quad [2]$$

whenever  $\{ \psi_t \mid t \in \mathbb{R} \}$  is a 1-parameter family of cross sections with  $\psi_0 = \psi$  and  $x \mapsto j_1(\psi)(x)$  a cross section of  $J^1(E)$ , i. e.  $j_1(\psi)$  is a 1-jet.

Remark 7. — One proves [2], that there exists a unique 4-form  $\theta$  on  $J^1(E)$ , such that the pull back  $j_1(\psi)^*$  to  $j_1(\psi)$  [3] yields (37):

$$(37') \quad j_1(\psi)^* \theta = L(j_1(\psi)) \omega, \quad \omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \equiv dx \quad \forall \psi \in \Gamma(E).$$

As regards the variational principle, it selects the extremal cross-sections

$\psi \in \Gamma(E)$  of the functional  $I_C(\psi) = \int_C L(j_1(\psi)) dx$  ( $C \subset M^4$  is a compact subset of  $M^4$ ).  $\psi$  is an extremum of  $I_C$  means  $d/dt I_C(\psi_t)|_{t=0} = 0$ , i. e. a cross section  $\psi$  is an extremal if it satisfies the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\mu} \left( j_1(\psi)^* \frac{\partial L}{\partial \psi_{\alpha,\mu}} \right) = j_1(\psi)^* \frac{\partial L}{\partial \psi_\alpha}$$

### 3.2. Noether theorem [2], [3].

Let  $(P, \overset{4}{\omega})$ ,  $P = J^1(E)$  and  $\overset{4}{\omega} = L\pi^{1*}dx$  is a Lagrangian form on  $P$ . If  $X^1 \in \mathcal{X}(J^1(E))$  is a  $C^\infty$ -vector field leaving  $\overset{4}{\omega}$  invariant, i. e.  $X^1(\overset{4}{\omega}) = 0$  by eq. (35), then we have

**PROPOSITION 7 (Noether).** — Every symmetry field  $X$  (def. 6) of the Lagrangian  $L$  determines a vector field  $Y$  which generates a conserved current  $\omega = Y \lrcorner dx$  (eq. (27)), that is

$$(38) \quad d(Y \lrcorner dx) = j_1(\psi)^* X^1(\omega) = 0 \quad (\text{eqs. (28) and (30)})$$

holds, where  $j_1(\psi) : M^4 \rightarrow J^1(E)$  is a 1-jet of the extremal  $\psi$ .

For a proof refer to [2] and [4].

### 3.3. Construction of the jet-bundle $J(E)$ [2], [4].

Let  $M^4$  and  $E$  as given previously,  $\pi : E \rightarrow M^4$  a projective mapping and consider the set

$$M^4 \times \Gamma(E) = \{ (x, \psi) \mid x \in U \subset M^4, \psi : U \subset M^4 \rightarrow E, \psi \in \Gamma(E) \}$$

where  $\Gamma(E)$  is the space of cross sections of  $E$  with  $\pi\psi(x) = x \in M^4$ . Consider the equivalence relation  $R$  defined by

$$(x, \psi) \equiv (x', \psi') \pmod{R} \Leftrightarrow D_x \psi = D_{x'} \psi'$$

( $D$  stands for « derivative »).

Let  $j : M^4 \times \Gamma(E) \rightarrow J^1(E) = M^4 \times \Gamma(E)/R$  be the canonical map of  $M^4 \times \Gamma(E)$  on the quotient  $M^4 \times \Gamma(E)$  by  $R$ . The set  $J^1(E)$  admits a natural structure of differentiable manifold such that the projection  $\tau : J^1(E) \rightarrow E$ ;  $\tau(j(x, \psi)) = \psi(x)$  (diagramme (33)) is differentiable and

$$\pi^1 = \pi \circ \tau : J^1(E) \rightarrow M^4$$

is a differentiable bundle [3] according to the diagramme (33). The elements of  $J^1(E)$

$$(39) \quad j_1(\psi) : M^4 \rightarrow J^1(E)$$

are called 1-jets, i. e. the mapping (39) is defined as follows:  $j_1(\psi)(x) =$  equivalence class to which the point  $(x, \psi)$  belongs.  $\pi^1 : J^1(E) \rightarrow M^4$  is the first jet prolongation of  $\pi : E \rightarrow M^4$  (cf. [2]).

## 4. HOMOTOPY-HOMOLOGY UNIFICATION SCHEME FOR CONSERVATION LAWS

The mathematical structure underlying both, Noether conservation laws and topological conservation laws relates to de Rham cohomology ([3], chapt. 10) in terms of field quantities  $(\omega^p, c_p)$  (24). In fact, Noether and topological charges derive from topological current fields  $(\omega^3, c_3)$ . In the case of Noether conservation laws,  $\omega \in F^3(M^4)$  ((27), (31)) is given by proposition 7; a topological current form  $\omega \in F^3(M^4)$  is constructed in section IV.1.

4.1. Construction of a topological current form.

Topological charges are invariants of homotopy classes  $[\phi]$  of differentiable maps  $\phi : S^3 \rightarrow M_0$  (8), sect. II.

Such a topological charge may be obtained from a Higgs-field  $\phi = (\phi^i)$  whenever the vacuum manifold  $M_0 = S^2_\phi$  (8) is chosen and a fieldstrength-form

$$(40) \quad F = F_{\mu\nu} dx^\mu \wedge dx^\nu = \varepsilon_{ijk} \phi^i \frac{\partial \phi^j}{\partial x^\mu} \frac{\partial \phi^k}{\partial x^\nu} dx^\mu \wedge dx^\nu \in F^2(M^4), \quad M^4 = S^3 \times \mathbb{R}$$

is given. Note, that such a set-up arises from a suitable Lagrangian model (11), with  $F^1_{\mu\nu} = \phi F^2_{\mu\nu} + \phi(\partial_\mu \phi \wedge \partial_\nu \phi)$  [12]. Since  $dF = 0$  (eq. (52), remark 9) there exists (locally) a 1-form  $\alpha \in F^1(M^4)$  such that  $F = d\alpha$ .

Define now a topological current 3-form on  $M^4 = S^3 \times \mathbb{R}$  by

$$(41) \quad \overset{3}{\omega} = \alpha \wedge F \in F^3(M^4)$$

where

$$(42) \quad \alpha = A_\mu dx^\mu \in F^1(M^4)$$

is the vector potential 1-form (14) satisfying

$$(43) \quad F = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu = d\alpha \in F^2(M^4), \quad F_{\mu\nu} = \phi(\partial_\mu \phi \wedge \partial_\nu \phi).$$

I. e. since  $dF = 0$  (remark 9) eq. (43) holds locally and even on  $S^3 \times \mathbb{R}$  and we have

**PROPOSITION 8.** — *The current 3-form (41) represents a conserved current, that is*

$$(44) \quad d(\alpha \wedge F) = 0; \quad \overset{3}{\omega} = \alpha \wedge F \in F^3(M^4); \quad d : F^p(M^4) \rightarrow F^{p+1}(M^4)$$

*Proof.* — The following rule holds for the exterior derivative  $d$  [3]:

$$(45) \quad d(\alpha \wedge F) = d\alpha \wedge F + (-1)^{\text{deg}\alpha} \alpha \wedge dF = F \wedge F - \alpha \wedge dF = F \wedge F$$

on account of (43), since  $dF = d(d\alpha) = 0$  by Poincaré's Lemma [3]. Now

$$(46) \quad F \wedge F = (\det(F_{\mu\nu}))^{1/2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \in F^4(M^4)$$

and

$$(47) \quad \det(F_{\mu\nu}) = F_{\mu\nu} *F^{\mu\nu} = I_1$$

where  $*F^{\mu\nu} = 1/2 \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$  is the adjoint tensor whose associated 2-form is  $*F = *F_{\mu\nu} dx^\mu \wedge dx^\nu$  [3]. So we must prove  $\det(F_{\mu\nu}) = 0$ . We evaluate the invariant  $I_1$ , by knowing, that

$$(48) \quad \begin{aligned} X = F_{10} &= -F^{10} = *F_{23} = *F^{23}; \\ Y = F_{20} &= -F^{20} = *F_{31} = *F^{31}; \quad Z = F_{30} = -F^{30} = *F_{12} = *F^{12}; \\ L = F_{23} &= F^{23} = -*F_{10} = *F^{10}; \\ M = F_{31} &= F^{31} = -*F_{20} = *F^{20}; \quad N = F_{12} = F^{12} = -*F_{30} = *F^{30} \end{aligned}$$

thus

$$(49) \quad I_1 = 2(LX + MY + NZ) = 2(F_{23}F_{10} + F_{31}F_{20} + F_{12}F_{30}) \\ = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma};$$

$\phi\phi = 1$  and  $\phi \frac{\partial\phi}{\partial x^\mu} = 0$ , so

$$(50) \quad F_{\mu\nu} F_{\rho\sigma} = \phi(\partial_\mu\phi \wedge \partial_\nu\phi)\phi(\partial_\rho\phi \wedge \partial_\sigma\phi) \\ = \begin{vmatrix} \partial_\mu\phi\partial_\rho\phi & \partial_\mu\phi\partial_\sigma\phi \\ \partial_\nu\phi\partial_\rho\phi & \partial_\nu\phi\partial_\sigma\phi \end{vmatrix} = \partial_\mu\phi\partial_\rho\phi\partial_\nu\phi\partial_\sigma\phi - \partial_\nu\phi\partial_\rho\phi\partial_\mu\phi\partial_\sigma\phi.$$

Combining (49) and (60) yields:

$$(51) \quad \left. \begin{aligned} F_{23}F_{10} &= \partial_2\phi\partial_1\phi\partial_3\phi\partial_0\phi - \partial_3\phi\partial_1\phi\partial_2\phi\partial_0\phi \\ F_{31}F_{20} &= \partial_3\phi\partial_2\phi\partial_1\phi\partial_0\phi - \partial_1\phi\partial_2\phi\partial_3\phi\partial_0\phi \\ F_{12}F_{30} &= \partial_1\phi\partial_3\phi\partial_2\phi\partial_0\phi - \partial_2\phi\partial_3\phi\partial_1\phi\partial_0\phi \end{aligned} \right\} \Rightarrow I_1 = \det(F_{\mu\nu})^{1/2} = 0 \quad \blacksquare$$

*Remark 9.* — The condition  $\det(F_{\mu\nu}) = 0$  (eq. (51)) amounts to  $\text{rank}(F) < 4$ , hence  $\text{rank}(F) = 2$  or  $0$ .  $F$  is thus a monomial  $F = \theta_\mu \wedge \theta_\nu$  ( $\mu, \nu$  fixed) and  $dF = 0$ . Moreover, the second invariant for the field  $F_{\mu\nu}$

$$I_2 = L^2 + M^2 + N^2 - X^2 - Y^2 - Z^2 \\ = F_{23}F_{23} + F_{31}F_{31} + F_{12}F_{12} - F_{10}F_{10} - F_{20}F_{20} - F_{30}F_{30}$$

vanishes as well. Hence, by virtue of  $I_1 = I_2 = 0$  (for some inertial frame) it follows that  $F$  is singular. Thus  $F$  and  $*F$  are both solutions of

$$(52 a) \quad dF = 0$$

and

$$(52 b) \quad d*F = 0$$

## 4.2. Construction of a topological conserved charge

Let

$$(53) \quad \phi : S^3 \rightarrow S_\phi^2$$

be a differentiable map and let

$$(54) \quad \overset{2}{\omega} = 1/2 \varepsilon_{ijk} \phi^i d\phi^j \wedge d\phi^k \\ = \phi^1 d\phi^2 \wedge d\phi^3 + \phi^2 d\phi^3 \wedge d\phi^1 + \phi^3 d\phi^1 \wedge d\phi^2 \quad \text{i. e.} \\ \overset{2}{\omega} = \sin \theta d\theta \wedge d\varphi \in F^2(\mathbb{R}^3 - \{0\})$$

be the volume-2-form on  $S^2$ , that is

$$\int_{S^2} \overset{2}{\omega} = 4\pi \quad \text{or} \quad \int_{S^2} \overset{2}{\omega'} = 1; \quad \overset{2}{\omega'} = \frac{\overset{2}{\omega}}{2\pi}$$

Since  $d\omega = 0$ , this entails  $d(\phi^*\omega) = \phi^*d\omega = 0$ . Set  $F = \phi^*\omega \in F^2(S^3)$

$$(55) \quad \Rightarrow \exists \alpha \in F^1(S^3) : F = d\alpha = \phi^*\omega$$

A physical interpretation of (54) and (55) can now be given as follows. Let

$$(55') \quad \alpha = A_i dx^i \in F^1(S^3), \quad i = 1, 2, 3$$

be a vector potential and let  $\phi = (\phi^i)$  be a Higgsfield-triplet. Then there is a field strength 2-form on  $S^3$  given by

$$(56) \quad F = d\alpha = \frac{1}{2} F_{em} dx^e \wedge dx^m = \phi^*\omega = 1/2 \varepsilon_{ijk} \phi^i \frac{\partial \phi^j}{\partial x^e} \frac{\partial \phi^k}{\partial x^m} dx^e \wedge dx^m \in F^2(S^3)$$

$$(56') \quad F_{em} = \phi(\partial_e \phi \wedge \partial_m \phi) = \varepsilon_{ijk} \phi^i \frac{\partial \phi^j}{\partial x^e} \frac{\partial \phi^k}{\partial x^m} \quad (\text{cf. eq. (43)})$$

*Proof.* — By (54) one obtains

$$\phi^*\omega = 1/2 \phi^*(\varepsilon_{ijk} \phi^i) \phi^* d\phi^j \wedge \phi^* d\phi^k$$

where

$$\begin{aligned} \phi^* d\phi^j &= \sum \frac{\partial \phi^j}{\partial x^e} dx^e \quad ; \quad \phi^* d\phi^k = \sum \frac{\partial \phi^k}{\partial x^m} dx^m \\ \Rightarrow \phi^*\omega &= 1/2 \varepsilon_{ijk} \phi^i(\phi(x)) \cdot \frac{\partial \phi^j}{\partial x^e} \frac{\partial \phi^k}{\partial x^m} dx^e \wedge dx^m \quad ; \quad x \in S^3 \quad \blacksquare \end{aligned}$$

Introduce now the number

$$(57) \quad \gamma_\phi = \int_{S^3} \alpha \wedge F = \int_{S^3} \alpha \wedge d\alpha.$$

This number enjoys the following properties [10]; [11]

(57 i)  $\gamma_\phi$  is independent of the choice of  $\alpha$  and is therefore referred to as *Hopf invariant of  $\phi$*  [10], [11].

(57 ii)  $\gamma_\phi$  is an integer  $n \in \mathbb{Z}$ .

(57 iii)  $\gamma_\phi$  is an invariant under deformations of  $\phi$ . That is let  $\phi : S^3 \times I \rightarrow S^2$ ;  $\phi(x, t) = \phi_t(x) (x \in S^3, t \in [0, 1] = I \subset \mathbb{R})$  be continuous. Then  $\phi_t : S^3 \rightarrow S^2$  is a homotopy with initial and terminal maps  $\phi_0$  and  $\phi_1$  and

$$(58) \quad \gamma_{\phi_1} = \int_{S^3} \alpha \wedge d\alpha = \int_{S^3} \alpha' \wedge d\alpha' = \gamma_{\phi_0}.$$

Given the properties i)-iii) we regard the Hopf-invariant (57) as a charge Q associated with the charge-density field

$$(59) \quad (\overset{3}{\omega}, c_3); \quad \overset{3}{\omega} = j_0 dx^1 \wedge dx^2 \wedge dx^3 = 4\pi(\alpha \wedge \phi^*\omega) \in F^3(S^3).$$

$\overset{3}{\omega} = j_0 dx^1 \wedge dx^2 \wedge dx^3$  is a charge density 3-form on  $S^3$  and  $c_3 = S^3 \in \overset{0}{C}_3(S^3)$  is the 3-cycle in  $S^3$  giving rise to (58) in terms of the map

$$(60) \quad (\overset{3}{\omega}, c_3) \mapsto Q = 4\pi\gamma_\phi = \int_{S^3} \overset{3}{\omega} = 4\pi \int_{S^3} \alpha \wedge d\alpha \quad (\text{cf. (25)}).$$

### 5. CONCLUSION

Noether currents and topological currents are current 3-fields as given by (38), proposition 7, and (44) proposition 8, respectively. Their common feature is that these currents-fields of type (24) relates them to de Rham cohomology, in terms of a non-degenerate bilinear form  $\beta$  (cf. eqs. (19) and (26)) :

$$\beta : H^3(M^4) \times H_3(M^4) \rightarrow \mathbb{R},$$

$$(\overset{3}{\omega}, c_3) \mapsto Q = \int_{\substack{c_3 \\ x^0 = \text{const.}}} j_0(x, t) dx^1 dx^2 dx^3 \quad [3].$$

( $H^3(M^4)$  and  $H_3(M^4)$  are the 3rd cohomology group and 3rd homology group of  $M^4$ , respectively [3], chapt. 10). The nature of the two types of charges, i. e. the topological charge

$$Q = \int j_0(x, t) dx^1 dx^2 dx^3 = 4\pi\gamma_\phi$$

((57)-(60),  $\overset{3}{\omega} = \alpha \wedge F = j_0 dx^1 \wedge dx^2 \wedge dx^3$ , and

$$Q = f(c_3, Y) = \int Y \lrcorner dx,$$

the Noether charge, differs, in that a topological charge displays in addition to its homological feature also a homotopic feature as enunciated by the property (57 iii)).

The charge density 3-form  $\overset{3}{\omega} = \rho dx^1 \wedge dx^2 \wedge dx^3$  (eqs. (27) and (57) defines an absolute integral invariant in the sense of E. Cartan [3]) for a certain differential system. That is, the integral

$$(61) \quad Q = \int_{c_3^{(0)}} \rho dx^1 dx^2 dx^3 = \int_{c_3^{(1)}} \rho dx^1 dx^2 dx^3$$

does not change when the 3-chain  $c_3$  is deformed along the tube of trajectories of the given differential system. That is (61) expresses that the charge  $Q$  of charged matter is preserved in the flow which is another form of expressing conservation of charge [3].

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