IZU VAISMAN

A coordinatewise formulation of geometric quantization


<http://www.numdam.org/item?id=AIHPA_1979__31_1_5_0>
A coordinatewise formulation of geometric quantization

by

Izu VAISMAN

Department of Mathematics,
University of Haifa, Israel

SUMMARY. — In this paper, geometric quantization via a complex integrable polarization is rediscussed. The main idea is to use extensively the local real-complex coordinates defined by the «integration» of the polarization. The main new result is the construction of a distinguished trivialization of the Kostant-Souriau line bundle, which allows a simple characterization of the wave functions. These will be square integrable and analytic in a part of the variables. Another new result is a pairing formula for two complex polarizations which are really transverse but have a common complex part. Metalinear structures are introduced only in an elementary form. The paper ends by applying the considered formulation of the geometric quantization to the harmonic oscillator, which simplifies Simms’ discussion in [5].

MOS Subject classification 1970: 81-00; 81 A 12, 81 A 81.

The aim of this paper is to give a formulation of the geometric quantization procedure which uses local coordinates with respect to a specially adapted atlas on the basic symplectic manifold.

Except for a number of aspects, this formulation is equivalent to the standard geometric quantization with respect to a complex integrable polarization but, in our opinion, it is simpler to handle with in concrete cases.

The main new result is the construction of a distinguished local trivialization...
zation of the Kostant-Souriau line bundle, allowing a definition of the wave functions which makes no direct use of differential operators. We can therefore consider square-integrable wave functions which is important in physical applications. Another new result is a pairing formula for two complex polarizations which are really transverse but have a common complex part.

We are referring to [6] for a standard exposition of geometric quantization and for bibliographical references which we fail to give here.

Except for the general wave functions, everything will be in the \(C^\infty\) category and this is without any further notice.

### 1. COMPLEX INTEGRABLE DISTRIBUTIONS AND ASSOCIATED HILBERT SPACES

Let \(M\) be a \(d\)-dimensional differentiable manifold, \(T(M)\) its tangent bundle and \(T'(M) = T(M) \otimes \mathbb{C}\). An \(m\)-dimensional complex distribution \(S\) on \(M\) is a field of \(m\)-subspaces of the fibers of \(T'(M)\). Then the field \(\tilde{S} = S + \bar{S}\) can be constructed pointwise and \(S\) is called (complex) integrable if: i) \(S\) is closed by brackets, ii) \(\tilde{S}\) is a regular distribution and, again, it is closed by brackets.

Our basic starting point is

**THEOREM (Nirenberg [4]).** — The distribution \(S\) is integrable iff every point \(x \in M\) has a coordinate neighbourhood with coordinates \((y^1, \ldots, y^d)\) such that \(S\) be spaned by the local vector fields

\[
\left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{m-h}}, \frac{\partial}{\partial y^{m-h+1}}\right) + i \left(\frac{\partial}{\partial y^{m+1}}, \ldots, \frac{\partial}{\partial y^m} + i \frac{\partial}{\partial y^{m+h}}\right) (i = \sqrt{-1})
\]

for some fixed integer \(h \geq 0\).

The atlas defined by these local coordinates will be called adapted to \(S\) and, henceforth, the integrable distributions \(S\) will always be considered via their adapted atlases.

We shall make the following index conventions:

\[
(1.2) \quad a, b, \ldots = 1, \ldots, m - h; \quad \alpha, \beta, \ldots = m - h + 1, \ldots, m; \\
\bar{a}, \bar{b}, \ldots = m + 1, \ldots, m + h; \quad u, v, \ldots = m + h + 1, \ldots, d; \\
j, k, \ldots = 1, \ldots, d.
\]

Formulas (1.1) suggest introducing

\[
(1.3) \quad z^a = y^a + iy^{a+h}, \quad \bar{z}^\bar{a} = \bar{z}\bar{a}
\]
after what \( S \) has the local bases

\[
(1.4) \quad \left\{ \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial z^\beta} \right\}.
\]

Now we have in the adapted atlas mixed (real-complex) local coordinates \((y^\alpha, z^\beta, \bar{z}^\gamma, y^\nu)\) and it is easy to see that every point of the intersection of two such coordinate neighbourhoods has an open neighbourhood where the coordinate transformation takes the form

\[
\begin{align*}
\tilde{y}^\alpha &= \tilde{y}^\alpha(y^\beta, z^\gamma, \bar{z}^\delta, y^\nu), \\
\tilde{z}^\gamma &= \tilde{z}^\gamma(z^\delta, y^\nu), \\
\tilde{y}^\mu &= \tilde{y}^\mu(y^\nu),
\end{align*}
\]

and, particularly, \( \tilde{z}^\gamma \) depend analytically on \( y^\nu \).

Clearly, Formulas (1.5) are characteristic for an adapted atlas. Except for the analyticity of \( \tilde{z}^\gamma \), these formulas also show that there are two real foliations related to \( S \). One is defined by the local equations \( y^\nu = \text{const.} \); we denote it by \( \Delta \) and call it the large foliation. The other one \( D \), the small foliation is a subfoliation of \( \Delta \) and is defined locally by \( y^\nu = \text{const.} \), \( z^\gamma = \text{const.} \), \( \bar{z}^\delta = \text{const.} \).

The obtained structure allows considering various important classes of functions and other geometric objects on \( M \) by the following rules: i) everything which depends locally on the coordinates \( z^\gamma, \bar{z}^\delta, y^\nu \) only is called \( D \)-foliate; ii) everything which depends locally on \( y^\nu \) only is called \( \Delta \)-foliate; iii) everything which depends locally on \( y^\nu \) and, analytically, on \( z^\gamma \) is called adapted. Particularly, we shall make an essential use of adapted functions and adapted bundles, where the last are characterized by the existence of a local trivialization with adapted transition functions.

The main aim of this Section is to show that one can associate Hilbert spaces to some integrable distributions \( S \). We do this by a usual pattern in geometric quantization [1].

We begin by imposing to \( S \) a supplementary condition which is equivalent to the existence of a so-called metalinear structure. Consider an intersection of two coordinate neighbourhoods \( U, \bar{U} \) of the adapted atlas of \( S \) with the transition functions (1.5). Put

\[
A_{U\bar{U}} = \det \left( \frac{\partial z^\gamma}{\partial \bar{z}^\delta} \right) \det \left( \frac{\partial y^\nu}{\partial \bar{y}^\mu} \right).
\]

Then the condition which we request is: (C) there is an adapted subatlas for which one of the values of \((A_{U\bar{U}})^\frac{1}{2}\) can be fixed continuously and such that

\[
(1.6) \quad (A_{U\bar{U}})^{\frac{1}{2}}(A_{\bar{U}U})^{\frac{1}{2}}(A_{\bar{U}\bar{U}})^{\frac{1}{2}} = 1.
\]
(This is the so-called cocycle condition and a change of the sub-atlas leads to a cohomologuous cocycle).

An S which satisfies (C) together with a fixed \((A_{U})^{\frac{1}{2}}\) will be called metalingar. The interested reader is referred to Appendix B of [6] for the discussion of the existence of a metalingar structure on S. The existence condition is the vanishing of some characteristic class in \(H^{2}(M, Z_{2})\). We take this opportunity to note that the existence of an integrable S imposes by itself some rather restrictive topological conditions, e.g., the vanishing of the high enough Pontryagin classes of the transverse bundles of the foliations D and \(\Delta\) [2].

Now, let S be an integrable metalingar distribution on M. Then, an adapted half-form of S is a geometric object (quantity [8]) having a single component \(\rho\) with respect to every local chart of the adapted atlas such that \(\rho : U \rightarrow C\) is an adapted function (locally \(\rho = \rho(z^{a}, y^{b})\) analytic in \(z^{a}\)) and the components for two charts are related by

\[
\tilde{\rho} = (A_{U})^{\frac{1}{2}} \rho.
\]

Obviously, the adapted half-forms are adapted sections of an adapted complex line bundle on M with transition functions \((A_{U})^{\frac{1}{2}}\), and we shall denote this line bundle by \(L(S)\).

It is interesting to note that these objects are acted on by diffeomorphisms \(\Phi : M \rightarrow M\) which preserve S. In fact, take two corresponding points \(x_{0}\) and \(y_{0} = \Phi(x_{0})\) in M and let be \((U; y^{a}, z^{a}, y^{b}), (U'; y'^{a}, z'^{a}, y'^{b})\) local adapted charts at \(x_{0}, y_{0}\). Then \(\Phi\) is given locally by

\[
y'^{a} = y'^{a}(y^{b}, z^{a}, z'^{a}, y^{b}), \quad z'^{a} = z'^{a}(z^{b}, y^{b}), \quad y'^{b} = y'^{b}(y^{b})
\]

and a continuous determination of

\[
B_{UU'} = \left[\det (\partial z^{a}/\partial z'^{b}) \det (\partial y^{b}/\partial y'^{a})\right]^{\frac{1}{2}}
\]
can be fixed by choosing it arbitrarily at \(y_{0}\). Furthermore, if compatibility conditions are required, the similar quantity is fixed for any two charts at \(x_{0}, y_{0}\) and, if M is connected, it is also fixed for charts at arbitrary points \(x, y = \Phi(x)\) by going from \(y_{0}\) to \(y\) through a chain of consecutively intersecting coordinate neighbourhoods. (If M is not connected, we must fix arbitrarily \(B\) at a point of every connected component of M.)

Now, a pull-back of half-forms by \(\Phi\) is defined by

\[
(\Phi_{*}\rho)(x) = B_{UU'} \rho_{U'}(y).
\]

It is an important fact that Formulas (1.7) and (1.9) do not involve derivatives of \(\rho\). Hence, we may also consider non-differentiable half-forms.

Before proceeding, let us impose one more condition for S. Namely, we shall ask that its small foliation D be strongly regular in the sense that the coset space \(N = M/D\) of the leaves of D is a Hausdorff manifold.
this case, \( N \) has an induced atlas with the local coordinates \((z^a, \bar{z}^a, y^\mu)\).

Then, the adapted half-forms of \( M \) are pull-backs of \( \text{half-forms} \) of \( N \). Moreover, if \( \rho \) and \( \rho' \) are two adapted half-forms, then the components \( \rho, \rho' \) define a \( \text{density} \) \([8]\) on \( N \), i.e. an object which can be integrated over \( N \).

In this context, a (non-necessarily differentiable) adapted half-form \( \rho \) on \( M \), whose components depend analytically on the \( z^a \) and for which \( \rho \rho' \)
is (Lebesgues) integrable over \( N \left( \int_N \rho \rho' < + \infty \right) \) is called a \( \text{square integrable adapted half-form} \) on \( M \).

For two such half-forms \( \rho, \rho' \), a \( \text{scalar product} \) can be defined by

\[
\langle \rho, \rho' \rangle = \int_N \rho \rho'
\]

and (since a corresponding version of the Schwartz inequality is obviously available) we see that the square integrable adapted half-forms of \( M \) generate a Hilbert space \( \mathcal{H} (M, S) \) which we shall call the \textit{adapted Hilbert space} of \( (M, S) \).

If \( \Phi: M \to M \) is a diffeomorphism which preserves \( S \), \( \Phi_* \) of \((1.9)\) acts as a unitary operator on the adapted Hilbert space.

It is also interesting to consider the pre-Hilbert subspace \( \mathcal{H}'(M, S) \) of \( \mathcal{H}(M, S) \) which consists of the differentiable adapted half-forms of \( M \) whose support projects to a compact subset of \( N \). Namely, there is an interesting action of some tangent vector fields of \( M \) on \( \mathcal{H}'(M, S) \).

Indeed, a real tangent vector field \( \xi \) on \( M \) is called \textit{adapted} if \( \exp (t \xi) \) preserve \( S \) and, in this case, the radicals needed for \((1.9)\) can be fixed by requiring continuity with respect to \( t \) and fixing them to be \( 1 \) for \( t = 0 \). Then, a \textit{Lie derivative} of half-forms \( \rho \) can be defined by the usual formula

\[
L_{\xi} \rho = \frac{d}{dt} \left[ \exp (t \xi) \right]_{t=0} \rho.
\]

and it can be calculated by the general method in \([10]\).

Namely, put

\[
\xi = \xi^\mu \frac{\partial}{\partial y^\mu} + \lambda^a \frac{\partial}{\partial z^a} + \mu^\bar{z} \frac{\partial}{\partial \bar{z}^a} + \eta^\mu \frac{\partial}{\partial y^\mu}.
\]

This is real iff \( \mu^\bar{z} = \bar{\lambda}^a \) and adapted iff \( [\xi, \partial/\partial y^\mu] \) and \( [\xi, \partial/\partial \bar{z}^a] \) belong to \( S \), i.e. iff \( \lambda^a \) are adapted and \( \eta^\mu \) are \( \Delta \)-foliate functions. Then one gets \([10]\)

\[
L_{\xi} \rho = \lambda^a \frac{\partial \rho}{\partial z^a} + \eta^\mu \frac{\partial \rho}{\partial y^\mu} + \rho \left( \frac{\partial \lambda^a}{\partial z^a} + \frac{\partial \eta^\mu}{\partial y^\mu} \right),
\]

which is again an (adapted) half-form.

Moreover, formally, the operator \( L_\xi \) can be defined by \((1.13)\) for complex vector fields \( \xi \) given by \((1.12)\) with adapted \( \lambda^a \) and \( \Delta \)-foliate \( \eta^\mu \), i.e. fields which satisfy the bracket conditions stated above. These will be called
almost adapted fields or, if both their real and imaginary parts are adapted, adapted fields. The last means $\eta^\omega$-foliate, $\lambda^\omega$-adapted and $\mu^\omega$-adapted.

It is by (1.13) that such fields act on $\mathcal{H}'(M, S)$ and this action is linear. Moreover, the general formula \[ L_{[\xi, \eta]} = L_{\xi}L_{\eta} - L_{\eta}L_{\xi} \]
shows that we have actually a representation of the Lie algebra of the almost adapted vector fields of $M$ on $\mathcal{H}'(M, S)$.

Furthermore, an adapted field $\xi$ projects to a well defined field $\bar{\xi}$ on $N$. The last defines similarly a Lie derivative of densities $\phi$ on $N$ which is given by [10]
\[ (1.14) \quad L_{\bar{\xi}}\phi = \lambda^{\bar{\xi}} \frac{\partial \phi}{\partial z^a} + \mu^{\bar{\xi}} \frac{\partial \phi}{\partial \bar{z}^a} + \eta^{\bar{\xi}} \frac{\partial \phi}{\partial y^u} + \left( \frac{1}{2} \frac{\partial \lambda^{\bar{\xi}}}{\partial z^a} + \frac{1}{2} \frac{\partial \mu^{\bar{\xi}}}{\partial \bar{z}^a} + \frac{\partial \eta^{\bar{\xi}}}{\partial y^u} \right) \phi \]
and, when applied to a product $\rho \bar{\rho}'$ of half-forms, yields
\[ (1.15) \quad L_{\bar{\xi}}(\rho \bar{\rho}') = (L_{\bar{\xi}}\rho) \bar{\rho}' + \rho(L_{\bar{\xi}}\bar{\rho}') , \]
where $\bar{\xi}$ is the complex conjugate field of $\xi$.

Now, by using the same proof like the one given in [8] for exact forms (see [9]), a variant of the Stokes formula can be obtained to the effect that
\[ (1.16) \quad \int_N L_{\bar{\xi}}\phi = 0 \]
for every tangent field $\bar{\xi}$ of $N$ and every density $\phi$.

Hence (1.15) yields
\[ (1.17) \quad \langle L_{\xi}\rho, \rho' \rangle + \langle \rho, L_{\bar{\xi}}\bar{\rho}' \rangle = 0 , \]
i. e., $- L_{\bar{\xi}}$ is the adjoint of $L_{\xi}$. If $\xi$ is real $L_{\bar{\xi}}$ is skew-Hermitian and if $\xi$ is imaginary $L_{\bar{\xi}}$ is a Hermitian operator on $\mathcal{H}'(M, S)$.

2. QUANTIZATION OF POLARIZED SYMPLECTIC MANIFOLDS

In this Section, the manifold $M$ of Section 1 will be a symplectic manifold with $d = 2n$ and with a fundamental 2-form $\Omega$ satisfying $d\Omega = 0$. We shall also assume that it satisfies the so-called integrality-condition [6], i. e. that $\Omega$ represents via de Rham's theorem a real image of an integral cohomology class.

Furthermore, $S$ will be a complex integrable $n$-dimensional distribution on $M$ endowed with a metalinear structure, $D$-strongly-regular and such that $\Omega(\xi, \eta) = 0$ for every pair $\xi, \eta \in S$ (i. e. $S$ is Lagrangian). Such an $S$ will
be called a *nice polarization* of \( M \) and the triple \((M, \Omega, S)\) is a *polarized symplectic manifold*.

The quantization problem is that of representing functions on \( M \) by linear (Hermitian) operators on an associated Hilbert space, compatibly with the Poisson bracket.

We shall represent \( S \), like in Section 1, by an adapted atlas and we shall use the same notation and the same index conventions, taking of course \( d = 2n \) and \( m = n \).

The fact that \( S \) is Lagrangian means

\[
\Omega \left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) = 0, \quad \Omega \left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial z^\beta} \right) = 0, \quad \Omega \left( \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^\beta} \right) = 0,
\]

and we deduce the basic fact that, with respect to adapted coordinates one has

\[
\Omega = A_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta + \theta_u \wedge dy^u,
\]

where \( A_{\alpha\beta} \) is a skew-Hermitian matrix and \( \theta_u = B_idy^i \) are some Pfaff forms on the respective coordinate neighbourhood.

It is known from the Kostant-Souriau prequantization theory that there is a complex line bundle \( K \) on \( M \) endowed with a Hermitian metric \( h \) and with a Hermitian connection \( \nabla \) such that \( 2\pi i\Omega \) be the curvature form of \( \nabla \). We shall fix such a \( K \) and add it to the configuration \((M, \Omega, S)\). The following main result can now be proven:

**THEOREM.** — *With the notation above, \( K \) is an adapted line bundle on \( M \) and its metric \( h \) is D-foliate.*

**Proof.** — Take an adapted coordinate neighbourhood \( U \) which is trivializing for \( K \) and a basic cross-section \( \sigma \) of \( K \) over \( U \). On \( U \), the connection \( \nabla \) is defined by its local connection form

\[
\alpha = C_ady^a + C_\beta dz^\beta + C_d\bar{z}^\alpha + C_d dy^d.
\]

For this we have \( d\alpha = 2\pi i\Omega \), which implies, in view of (2.2):

\[
\frac{\partial C_a}{\partial y^b} = \frac{\partial C_b}{\partial y^a}, \quad \frac{\partial C_\beta}{\partial z^\alpha} = \frac{\partial C_\alpha}{\partial z^\beta}, \quad \frac{\partial C_a}{\partial \bar{z}^\alpha} = \frac{\partial C_\alpha}{\partial \bar{z}^a}.
\]

Now, let us go over to the basic section \( \sigma' = b'\sigma \). Then \( \alpha \) is replaced by \( \alpha + d \ln b' \) and we want to chose \( b' \) such that

\[
\frac{\partial \ln b'}{\partial z^a} = - C_x.
\]

If \( U \) is taken sufficiently small, then, because of the second relation (2.4), the existence of such a \( b' \) follows from the classical Grothendick-Dolbeault lemma [3].

With respect to $\sigma'$, the connection $V$ is given by a form (2.3) where $C_\tau = 0$ and hence, by the last relation (2.4), we also have that the new $C_a$ are analytic in $z^a$.

Furthermore, we change again the basic section by $\sigma'' = b''\sigma'$, where $b''$ will be required to satisfy, with respect to the new coefficients $C$, the condition

$$\frac{\partial \ln b''}{\partial y^a} = -C_a.$$  

The first relation (2.4) assures, by a well known lemma of Poincaré that such a $b''$ exists and, in view of the last relation (2.4), it is analytic in $z^a$.

Now, with respect to $\sigma''$ the local connection form becomes

$$\beta = \beta_\tau dz^\tau + \beta_\alpha dy^\alpha$$  

and $d\beta = 2\pi i \Omega$ implies

$$\frac{\partial \beta_\alpha}{\partial y^\alpha} = 0$$  

i. e. $\beta_\alpha$ are D-foliate functions.

A basic local cross-section of $K$ which satisfies (2.5) will be called distinguished and we just proved that $K$ admits a local trivialization endowed with distinguished bases.

One can prove that the distinguished bases are given by the following geometric construction. Take a sufficiently small cubical adapted coordinate neighbourhood $U$ and fix in it arbitrarily (by some equations $y^a = \text{const.}$, $z^\tau = \text{const.}$) a slice $\Sigma_0$ transversal to the large foliation $\Delta$ and a differentiable basis $\sigma_0$ of $K/\Sigma_0$. Consider next the transversal slice $\Sigma'_0$ of the small foliation $D$ which contains $\Sigma_0$ and extend $\sigma_0$ to a basis $\sigma'_0$ of $\Sigma'_0$ which is parallel with respect to the fields $\partial / \partial z^\tau$. (The existence of $\sigma'_0$ is deducible from the Grothendieck-Dolbeault lemma.) Finally, translate $\sigma'_0$ parallelly along the slices of the small foliation to get the desired distinguished basis $\sigma''$. The last is a correct operation since $V$ clearly induces a flat connection on the restriction of $K$ to the leaves of the small foliation.

Now, if $\sigma_1$ and $\sigma_2$ are distinguished bases of $K/U_1$ and $K/U_2$ and if $\sigma_2 = f\sigma_1$ over $U_1 \cap U_2$, we must have for the corresponding connection forms $\beta_2 = \beta_1 + d\ln f$. Since $\beta_{1,2}$ are both of the form (2.5), we get $\partial f / \partial y^a = 0$, $\partial f / \partial z^a = 0$, i. e. $f$ is an adapted function. This ends the proof of the fact that $K$ is an adapted line bundle.

As for its metric $h$ we have

$$\frac{\partial}{\partial y^a} [h(\sigma, \sigma)] = h(V_{i;i}^\tau \sigma, \sigma) + h(\sigma, V_{i;i}^\tau \sigma) = 0,$$

if $\sigma$ is distinguished, since in this case $V_{i;i}^\tau \sigma = 0$ by (2.5). This is just the meaning of the fact that $h$ is a D-foliate metric.
The stated Theorem is thereby proven.

In view of this Theorem it is now meaningful to speak of adapted cross-sections of the line bundle $K \otimes L(S)$ ($L(S)$ defined by the half forms of $(M, S)$). These will be sections which have local expressions of the form $\phi(\sigma \otimes \delta)$ where $\sigma$ and $\delta$ are respectively distinguished local bases of $K$ and $L(S)$ and where $\phi$ is an adapted function. Moreover, we may admit non-differentiable functions $\phi$ which, however, are analytic with respect to the variables $z^a$.

Considering again the manifold $N = M / D$ we can try to define a scalar product for sections of $K \otimes L(S)$ by a natural extension of the formula

$$\langle s \otimes \rho, s' \otimes \rho' \rangle = \int_N h(s, s') \rho \bar{\rho'},$$

where $s, s'$ are adapted sections of $K$ and $\rho, \rho'$ adapted half-forms. The integrand of (2.7) is clearly a density on $N$ since the metric $h$ is $D$-foliate.

The scalar product (2.7) makes sense for what we shall call square integrable adapted sections of $K \otimes L(S)$, i.e. sections for which the scalar product of the section with itself exists and which, also, are analytic in the $z^a$.

Then, these sections generate a Hilbert space which we denote by $H(M, \Omega, S)$ and call it the adapted Hilbert space of the triple $(M, \Omega, S)$. The elements of $H$ will be called wave functions.

Particularly, we get an interesting pre-Hilbert subspace of the adapted space if we take the wave functions which are also differentiable with respect to all the variables and whose support projects to a compact subset of $N$. This subspace will be denoted by $H' (M, \Omega, S)$.

And now about operators.

Let $f : M \to R$ be a differentiable function, i.e. on observable on $M$. Then, we shall denote by $sg f$ its symplectic gradient defined by

$$(sg f) \wedge \Omega = df.$$ (2.8)

This allows defining the Kostant-Souriau prequantization operator $\hat{f}$ associated to $f$ which, following [6], we shall take as

$$\hat{f}(s) = \frac{1}{2\pi i} \nabla_{sg f} s + fs,$$ (2.9)

where $s$ is a differentiable section of $K$.

If we put locally $s = t\sigma$, where $\sigma$ is a distinguished basis of $K$, (2.9) becomes

$$\hat{f}(s) = \frac{1}{2\pi i} \{ [sg f](t) + \beta (sg f)t + 2\pi it \} \sigma,$$ (2.10)

where $\beta$ is the connection form (2.5).
Now it is natural to try extending \( \hat{f} \) to \( \mathcal{H}'(M, \Omega, S) \) by a definition of the type

\[
\hat{f}(s \otimes \rho) = \frac{1}{2\pi i} s \otimes L_{sg} \rho.
\]

This is not well defined for every \( f \), however. First, for \( L_{sg} \rho \) to be adapted we have to ask \( sg f \) to be an adapted field. Then, the first term of (2.10) is also adapted and we still have to ask that

\[
\Xi = \beta (sg f) + 2\pi if
\]

be an adapted function.

Nevertheless, we can see that this last condition is implied by the first one. Indeed, if

\[
sg f = \xi^a \frac{\partial}{\partial y^a} + \lambda^x \frac{\partial}{\partial z^x} + \lambda^y \frac{\partial}{\partial \bar{z}^y} + \eta^u \frac{\partial}{\partial y^u}
\]

we have by (2.8) and by \( \Omega = (1/2\pi i)d\beta \) together with (2.5), (2.6):

\[
\frac{\partial \beta_u}{\partial y^a} \eta^u + 2\pi i \frac{\partial f}{\partial y^a} = 0, \\
\frac{\partial \beta_x}{\partial \bar{z}^y} \lambda^x + \frac{\partial \beta_y}{\partial z^y} \eta^u + 2\pi i \frac{\partial f}{\partial \bar{z}^y} = 0.
\]

Under the hypotheses that the functions \( \lambda^x \) are adapted and \( y^u \) \( \Delta \)-foliate these conditions are just

\[
\frac{\partial \Xi}{\partial y^a} = 0, \quad \frac{\partial \Xi}{\partial \bar{z}^y} = 0,
\]

i. e. \( \Xi \) is an adapted function.

Hence, (2.11) yields a well defined operator

\[
\hat{f} : \mathcal{H}'(M, \Omega, S) \to \mathcal{H}'(M, \Omega, S)
\]

for every observable \( f \) for which \( sg f \) is an adapted vector field on \( M \). \( \hat{f} \) will be called the \textit{quantization} of \( f \), and it follows easily from (1.15) and (1.16) that this \( \hat{f} \) is a Hermitian operator.

A particularly important case is obtained by asking \( f \) itself to be an adapted function, i. e., actually, a \( \Delta \)-foliate function (since it is real). In this case, (2.10) and (2.11) yield

\[
\hat{f}(s \otimes \rho) = fs \otimes \rho,
\]

i. e. the quantization is simply multiplication by \( f \). Clearly, such a quantization can be extended to arbitrary wave functions of the whole adapted Hilbert space \( \mathcal{H}'(M, \Omega, S) \).
Note also the following commutation formula which follows from (2.9) and (2.11)

\[ \hat{f} \hat{g} - \hat{g} \hat{f} = \frac{1}{2\pi i} \{ f, g \}, \]

where \( \{ f, g \} \) is the Poisson bracket of the two functions.

Next since we should be interested in quantizing more general observables, new instruments must be considered. The main idea now used in geometric quantization is based on pairing the Hilbert spaces of wave functions of two different polarizations (see \([1], [6]\)). We shall present here a simple case when pairing is possible and which generalizes the case of two transverse real polarizations studied in \([1]\) (*).

Let \((M, \Omega)\) be a symplectic manifold and \(S_1, S_2\) two nice polarizations on \(M\). We call them complementary polarizations if every point \(x \in M\) has coordinate neighbourhoods \(\{ U_i; y^i_1, z^i_1, y^i_2 \}\) respectively adapted to \(S_i\) \((i = 1, 2)\) such that the following transition relations hold

\[ y^i_2 = y^i_1, \quad z^i_2 = z^i_1, \quad y^i_2 = y^i_1 \]

for some convenient ordering of the indices. Two such charts at \(x\) will also be called complementary.

It is clear now that those charts of the adapted atlas of \(S_1\) which admit a complementary chart define an atlas of \(M\) whose coordinate transformations are locally of the form

\[ \tilde{y}^a = \tilde{y}^a(y^b), \quad \tilde{z}^a = \tilde{z}^a(z^b), \quad \tilde{y}^a = \tilde{y}^a(y^b), \]

and which is (up to a permutation of the coordinates) a common adapted atlas of \(S_1\) and \(S_2\).

With respect to this atlas, the form \(\Omega\) becomes in view of (2.2) and of the analogon of (2.1) for \(S_2\):

\[ \Omega = A_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta + B_{\alpha \mu} dy^\alpha \wedge dy^\mu. \]

In this formula, every term is a well defined 2-form on \(M\). We call \(B = B_{\alpha \mu} dy^\alpha \wedge dy^\mu\) the kernel form of the pair \((S_1, S_2)\) and note that (2.14) imply the transformation law

\[ B_{\alpha \mu} = \tilde{B}_{\beta \nu} \frac{\partial \tilde{y}^\beta}{\partial y^\alpha} \frac{\partial \tilde{y}^\nu}{\partial y^\mu}. \]

Furthermore, suppose that \(S_1\) and \(S_2\) are transversally orientable with

respect to their large foliations, i.e. that the atlas we are working with can be assumed to satisfy the conditions
\[
\det \left( \frac{\partial y^\mu}{\partial y^\nu} \right) > 0, \quad \det \left( \frac{\partial y^\mu}{\partial y^c} \right) > 0.
\]

Then, if we denote by \( \mathcal{H}_i = \mathcal{H}(M, \Omega, S_i) \) \((i = 1, 2)\), a Hermitian pairing \((,): \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}\) can be defined by
\[
(2.17) \quad (s_1 \otimes \rho_1, s_2 \otimes \rho_2) = c \int_M |\det (B_{an})|^{\frac{1}{2}} h(s_1, s_2) \rho_1 \bar{\rho}_2.
\]
This is well defined since by (1.7) and (2.16) the integrand of (2.17) is a density on \( M \) provided that the metalinear structures of \( S_1 \) and \( S_2 \) be conveniently chosen. (Again, we do not discuss the convergence of the considered integral.) As for \( c \) in (2.17), it is a constant number which may be conveniently chosen in every concrete case.

The relation between pairing and quantization is based on the following idea [1, 6]. If \( f \) is a real observable on \( M \) then \( \exp(t \sg f) \) should be again a nice polarization. If \( \mathcal{H}(M, \Omega, S) \) can be paired in such a manner that the pairing define a unitary intertwining operator \( U_t: \mathcal{H}(M, \Omega, S_i) \to \mathcal{H}(M, \Omega, S) \) then a one-parameter group of unitary transformations representing something of the kind \( U_t \circ \exp(t \sg f) \) should be expected on \( \mathcal{H}(M, \Omega, S) \). Its « generator » will be the quantization \( \hat{f} \) of \( f \) (See examples in [1, 6]).

A similar idea should be considered in discussing the relation between the quantizations defined by two different polarizations.

Of course, there are many other problems to be discussed such as the so-called Bohr-Sommerfeld conditions, etc. [7].

We shall end this section by a supplementary remark concerning the integrality condition. Namely, suppose it is not satisfied for \( (M, \Omega) \) but it is satisfied for a lift \( (M', \Omega') \) to some covering manifold \( M' \) of \( M \). Then everything can be lifted to \( M' \) and we can quantize the observables on \( M \) by operators on the wave functions on \( M' \). E.g., if the second homotopy group \( \pi_2(M) = 0 \), the integrality condition will always be satisfied on the universal covering space \( M' \) of \( M \) which can lead in this case to a quantization.

3. CLASSICAL EXAMPLES.
THE HARMONIC OSCILLATOR

In the present Section we shall see how the formulation of the geometric quantization given in Section 2 works in some classical cases. We shall thereby compare this formulation with those already used in the literature (e.g. [6]).
The basic case is

\[ M = \mathbb{R}^{2n} = \{ (p_a, q^a) \mid a = 1, \ldots, n \} \]

\[ \Omega = dp_a \wedge dq^a \]

(with the summation convention, of course).

Various polarizations are then available, two of them being used most often. Namely, the real polarization \( S_1 \) defined by the transverse coordinates \( q^a \) (i.e., the complexification of the tangent distribution of the leaves of the foliation \( q^a = \text{const.} \)), and the real polarization \( S_2 \) with the transverse coordinates \( p_a \). These are clearly complementary polarizations.

\( S_1 \)-quantization is quite simple. Namely, \( \Omega \) is integral and exact, hence we can take a trivial line bundle \( K \) with the basic section 1, with respect to which the Hermitian metric is given by \( h(1, 1) = 1 \) and the connection is given by the connection form \( \beta = 2\pi ip_a dq^a \). (2.5) shows that 1 is actually a distinguished basis. Since, on the other side, there is an adapted atlas consisting of a single chart, we see that the wave functions are just square integrable complex valued functions \( \varphi(q^a) \). It is also easy to see that (2.11) yields quantizations for those observables which are at most linear with respect to the variables \( p_a \).

As for \( S_2 \), we have the same connection, but 1 is no more a distinguished basis. Using the method of the proof of the Theorem given in Section 2, we see that

\[ \sigma = e^{-2\pi ip_a q^a} \]

yields a distinguished basis of \( K \), since the connection form is now \( \beta + d \ln \sigma = -2\pi iq^a dp_a \). Hence the \( S_2 \)-wave functions are of the form

\[ \chi = e^{-2\pi ip_a q^a} \psi(p_a), \]

where \( \psi(p_a) \) is a square integrable function.

The coefficients of the kernel form \( B \) (which, in this case, equals \( \Omega \)) form the unit matrix, and by (2.17) we obtain the pairing formula

\[ (\varphi, \chi) = c \int_M e^{2\pi ip_a q^a} \varphi(q^a) \bar{\psi}(p_a), \]

which can be arranged to lead to the Fourier transform as intertwining operator [6].

A third interesting polarization can be obtained on \( M \) in the following manner. Put

\[ z^a = p_a - iq^a \quad (a = 1, \ldots, n). \]

Then we get

\[ \Omega = -\frac{i}{2} \sum_{a=1}^{n} dz^a \wedge d\bar{z}^a \]

and we see that \( \{ \partial/\partial \bar{z}_a \} \) span a nice polarization \( S_3 \) for which the leaves of the small foliation are the points of \( M \).

We take again as \( K \) the trivial bundle with the basis \( 1 \) and note that

\[
\omega = \frac{\pi}{2} (z^a d \bar{z}^a - \bar{z}^a d z^a)
\]

is a Hermitian connection for the metric \( h(1,1) = 1 \), whose curvature is \( 2\pi i \Omega \).

Then

\[
\tau = e^{-\pi i \sum_{a=1}^n z^a \partial_a}
\]

defines a distinguished basis with respect to which the connection form becomes \( \omega = - \pi \sum_{a=1}^n \bar{z}^a dz^a \).

It follows that the wave functions are functions of the form

\[
\Psi = e^{-\pi i \sum_{a=1}^n z^a \partial_a} \psi(z^a),
\]

where \( \psi \) are complex analytic functions in \( z^a \), and their scalar product is

\[
\langle \Psi_1, \Psi_2 \rangle = \int_M e^{-\pi i \sum_{a=1}^n z^a \partial_a} \psi_1 \bar{\psi}_2
\]

(to be compared with a formula of Bargman mentioned in [6, p. 109]).

Finally, we want to make a more complete discussion of an important physical example, which is that of a harmonic oscillator. This could be done as in [6] by means of the above polarization \( S_3 \), but we shall prefer to proceed like in Simms [5] since this provides a fuller illustration of the general schema of Section 2.

Following [5], the harmonic oscillator is defined by the symplectic manifold

\[
M = \mathbb{R}^{2n} - \{ 0 \}, \quad \Omega = h^{-1} \sum_{a=1}^n dp_a \wedge dq^a,
\]

where \( \mathbb{R}^{2n} = \{ (p_a, q^a) \} \) and \( h \) is the Planck constant.

Its Hamiltonian is the function

\[
H = \frac{1}{2m} \sum_{a=1}^n [(p_a)^2 + k^2(q^a)^2],
\]

where \( k = m\chi, m \) is the mass and \( \chi \) the frequency of the oscillator, and the problem is to get a representation of \( H \) by a quantum operator.
Consider the complex coordinates \[5\]
\[
z^a = \frac{1}{(2m)^{\frac{1}{2}}} (p_a - ikq^a).
\]
Then we have
\[
H = \sum_{a=1}^{n} z^a \bar{z}^a = r^2
\]
and we should expect a simple quantization of \(H\) by the help of a polarization for which \(r\) is a \(\Delta\)-transverse coordinate.

Now, since we want \(r\) to be a coordinate it is natural to try some kind of polar coordinates. Following \[5\], we put
\[
M = \bigcup_{j=1}^{n} U_j, \quad U_j = \{ z \in M / z^j \neq 0 \}
\]
and define on \(U_j\) the local coordinates \((t_j, u^k_j, r)\), where
\[
\begin{align*}
  z^j &= |z^j| e^{i t_j}, \quad u^k_j = z^k / z^j \quad (k \neq j), \\
  z^j &= r e^{i t_j} \left( 1 + \sum_{h \neq j} u^h_j \bar{u}^h_j \right)^{-\frac{1}{2}}, \\
  z^k &= r e^{i t_j} u^k_j \left( 1 + \sum_{h \neq j} u^h_j \bar{u}^h_j \right)^{-\frac{1}{2}}, \quad (i = \sqrt{-1}).
\end{align*}
\]
Hereafter we change our index convention and agree that \(h, j, k = 1, \ldots, n\).

Then, we get on \(U_j \cap U_h\)
\[
\begin{align*}
  u^h_j &= \frac{1}{u^h_k}, \quad u^k_j &= \frac{u^h_k}{u^h_h} \quad (k \neq j, h), \quad r = r,
\end{align*}
\]
and it follows that we have here an adapted atlas, while the corresponding polarization \(S\) is generated over \(U_j\) by \(\{ \partial / \partial t_j, \partial / \partial \bar{u}^h_j \} \) \((k \neq j)\). (The ambiguity in the definition of \(t_j\) in (3.15) results in translations of this coordinate by multiples of \(2\pi\), which does not influence \(S\).)

This \(S\) actually is a polarization since it follows from (3.10) and (3.12) that
\[
\Omega = -i\pi \sum_{a=1}^{n} dz^a \wedge d \bar{z}^a = -i\pi \left\{ \sum_{k \neq j} \Phi^2 d u^k_j \wedge d \bar{u}^k_j + \frac{\Phi^4}{r^2} \left( \sum_{h \neq j} u^h_j d \bar{u}^h_j \right) \wedge \left( \sum_{h \neq j} \bar{u}^h_j du^h_j \right) + 2\pi r d t_j + \frac{\Phi^2}{r^2} \sum_{k \neq j} \left( \bar{u}^k_j du^k_j - u^k_j d \bar{u}^k_j \right) \wedge dr \right\},
\]
where \( x = 1/(h\chi) \), \( \Phi = r \left( 1 + \sum_{k \neq j} u_j^k \overline{u}_j^k \right)^{-\frac{1}{2}} \), and this shows that the conditions corresponding to (2.1) are satisfied.

Moreover, \( S \) is a nice polarization. Indeed, the definition of the local coordinates \((t_j, u^k_j, r)\) yields a diffeomorphism

\[
R^{2n} - \{ 0 \} \approx \mathbb{CP}^{n-1} \times R_+ \times S^1
\]

where \( S^1 \) is the unit circle, \( R_+ \) is the set of the positive real numbers and \( \mathbb{CP}^{n-1} \) is the \((n - 1)\)-dimensional complex projective space. Now, the leaves of the small foliation \( D \) of the polarization \( S \) correspond to \( S^1 \) and it follows that \( S \) is \( D \)-strongly regular with the quotient manifold \( M/D \approx \mathbb{CP}^{n-1} \times R_+ \).

Finally, \( S \) admits metalinear structures. This follows by cohomology arguments which yield the precise result that there are essentially two such structures for \( n = 1 \) and one for \( n > 1 \) [5].

This ends the proof of the fact that \( S \) is nice.

We shall give in the sequel a straightforward elementary construction of the metalinear structures of \( S \).

For \( n = 1 \), only the transverse coordinate \( r \) is to be considered in the plane \( M = U_1 \) of the complex coordinate \( z^1 \) with deleted origin. In order to get both structures, let us put \( M = U_1^+ \cup U_1^- \), where

\[
U_1^+ = \{ z^1 = re^{it_1} | \ 0 < t_1 < 2\pi \}, \\
U_1^- = \{ z^1 = re^{it_1} | -\pi < t_1 < \pi \},
\]

and \( U_1^+ \cap U_1^- \) has two connected components: \( 0 < t_1 < \pi \) and \( \pi < t_1 < 2\pi \).

Now, the transition function for the coordinate \( r \) is always equal to 1, which is also its determinant, and we can take

\[
a) \sqrt{1} = 1 \text{ on the whole of } U_1; \\
(3.20) \\
b) \sqrt{1} = \begin{cases} 
1 & \text{on } 0 < t_1 < \pi, \\
-1 & \text{on } \pi < t_1 < 2\pi.
\end{cases}
\]

This clearly defines the two metalinear structures of \( S \) in the case \( n = 1 \).

For \( n > 1 \), a simple calculation based on (3.17) yields

\[
\text{det} \left( \frac{\partial u^k_j}{\partial u^k_h} \right) \text{det} \left( \frac{dr}{dr} \right) = (-1)^n(z^1)^n \\
(3.21)
\]

and it is obviously possible to fix the radical if \( n \) is even.

For an arbitrary \( n \), we put \( U_h = U_h^+ \cup U_h^- \), where the two subsets are defined by (3.20) with the index 1 replaced by \( h \), and fix arbitrarily a continuous value of \( (z^h)^1 \) on \( U_h^+ \) and on \( U_h^- \). But then we must also consider the intersections \( U_h^+ \cap U_h^- \) with the identity as a coordinate transformation.
and 1 as its determinant. For them $\sqrt{1}$ will be fixed on every connected component of $U_h^+ \cap U_h^-$ in such a manner that $\sqrt{1}(z^h)^{\frac{1}{2}} = \sqrt{1}(z^h)^{\frac{1}{2}}$ on the respective component. It is now simple to see that this provides a metilinear structure on S. (Of course, we shall also have to choose $\sqrt{(-1)^h}$, but this can be done arbitrarily for every index $h$.)

We shall discuss, next, adapted half-forms and wave functions.

We begin by noticing that there is a global basic half-form $\beta$ in each one of the cases above. Namely, for $n = 1$ we take $\beta = 1$ on $U_1$ in the case (3.20) a), and $\beta = 1$ on $U_1^+$, $\beta = -1$ on $U_1^-$ in the case (3.20) b). Then, every adapted half-form is of the type $\rho = F(r)\beta$.

For $n > 1$, the basic half-form $\beta$ is defined by taking $\beta_h = (-1)^{h/2}(z^h)^{n/2}$ on $U_h$ ($h = 1, \ldots, n$), where the radicals are the fixed ones. Hence, again every half-form is of the type $\rho = F\beta$ where $F$ is a globally defined function on $M$. But $\beta$ is not adapted and, therefore, the condition that $\rho$ be adapted must be added separately.

Let us go over now to the Kostant-Souriau line bundle $K$. Since $\Omega$ is exact, $K$ is to be taken trivial, i.e. $K = M \times \mathbb{C}$ with the initial basis 1 and the Hermitian metric $h(1, 1) = 1$.

In view of (3.18), it follows that the corresponding Hermitian connection is given by the connection form

$$
(3.22) \quad \omega = \pi \xi \sum_{a=1}^{n} (z^a \bar{z}^a - \bar{z}^a dz^a),
$$

which using (3.16) is represented in $U_j$ by

$$
(3.23) \quad \omega_j = - \pi \xi \left\{ \Phi^2 \sum_{h \neq j} (\bar{u}_j^h du_j^h - u_j^h d\bar{u}_j^h) + 2ir^2 dt_j \right\},
$$

with the same $\Phi$ like in (3.18).

Next, to go over to a distinguished basis of $K$ we put first $\omega_j$ under the equivalent form

$$
(3.24) \quad \omega_j = - 2\pi \xi r^2 \left\{ d \left( \ln \frac{\Phi}{r} \right) + idt_j + \Phi^2 \sum_{h \neq j} \bar{u}_j^h du_j^h \right\}.
$$

Then, if we define over $U_j$ the local section

$$
(3.25) \quad b_j = e^{2\pi \xi r^2 \ln z^j}
$$

(where determination is obtained by putting $U_j = U_j^+ \cup U_j^-$ as before) we get new connection forms $\bar{\omega}_j = \omega_j + d \ln b_j$, and these are

$$
(3.26) \quad \bar{\omega}_j = - 2\pi \xi \Phi^2 \sum_{h \neq j} \bar{u}_j^h du_j^h + 2\pi \xi (1 + 2 \ln z^j) dr .
$$

Hence the $b_j$ yield a distinguished basis of $K$ and an adapted section of this line bundle is a complex valued function $s$ on $M$ such that $s_j = s b_j^{-1}$ are adapted functions for every $j = 1, \ldots, n$.

It follows that a generating element of the adapted Hilbert space $\mathcal{H}(M, \Omega, S)$, which is of the form $s \otimes \rho$ is given by

$$s \otimes \rho / U_j = s_j \rho_j b_j^{-1} b_j (1 \otimes \beta),$$

where $s_j$ and $\rho_j$ are adapted functions. Hence a general adapted section $\gamma$ of $K \otimes L(S)$ is defined by its restrictions

$$\gamma / U_j = \gamma_j b_j^{-1} b_j (1 \otimes \beta),$$

where $\gamma_j$ is adapted on $U_j$.

More exactly, for the case $n = 1$, a) we have

$$\gamma / U_1 = \nu(r) e^{2\pi i x^2 i t_1} 1,$$

where $1$ is the basic section and we fix the argument $t_1$ on $U_1^+$ and $U_1^-$. For the case $n = 1$, b), we have

$$\gamma / U_1^+ = \nu(r) e^{2\pi i x^2 i t_1} 1,$$
$$\gamma / U_1^- = - \nu(r) e^{2\pi i x^2 i t_1} 1.$$

Both in (3.29) and in (3.30) $\nu(r)$ has to be a square integrable function. For the case $n \geq 2$, we have

$$\gamma / U_j = (-1)^{j/2} (z^j)^{-n/2} \gamma_j (u^k_j, r) e^{2\pi i x^2 \text{ln} z^j (1 \otimes \beta)}$$

$$= \nu_j (u^k_j, r) \Phi^{-n/2} e^{2\pi i x^2 \text{ln} \Phi} e^{\frac{4\pi i x^2 - n}{2} i t_j} (1 \otimes \beta),$$

where we must take separately the determination of $t_j$ on $U_j^+$ and $U_j^-$, and $\nu_j$ are square integrable functions which are analytic with respect to $u^k_j$.

Moreover, because of the ambiguity in the determination of $t_j$, the fact that $\gamma$ is a global section of $K \otimes L(S)$ implies:

i) in the case $n = 1$, a), $\nu(r) \neq 0$ only where $2\pi i x^2$ is an integer;

ii) in the case $n = 1$, b), $\nu(r) \neq 0$ only where $2\pi i x^2$ is the half of an odd integer;

iii) in the case $n \geq 2$, $\nu_j \neq 0$ only where $\frac{4\pi i x^2 - n}{2}$ is an integer.

Consider the function

$$\delta(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \neq 0, \end{cases}$$

Then, it follows from the discussion above that we can get wave functions if we take respectively, in (3.29), (3.30), (3.31) the following:

i) in (3.29), $\nu(r) = \nu'(r) \delta(2\pi i x^2 - K),$

Annales de l'Institut Henri Poincaré-Section A
ii) in (3.30), \[ v(r) = v'(r)\delta\left(2\pi\alpha^2 - K - \frac{1}{2}\right), \]

iii) in (3.31), \[ v_j = v'_j(u^h_j, r)\delta\left(\frac{4\pi\alpha^2 - n}{2} - K\right), \]

where the \( v' \) are square integrable and analytic in \( u^h_j \) and \( K = 0, 1, 2, \ldots \).

In agreement with Section 2, the quantization of the energy \( H \) given by (3.13) consist in multiplication of the wave functions by \( r^2 \), and it is easy to see that we get, correspondingly, the following eigenvalues of this operator:

i) \( n = 1, a) \):
\[ \lambda = \frac{K}{2\pi\alpha} = K\hbar, \]

ii) \( n = 1, b) \):
\[ \lambda = \left(K + \frac{1}{2}\right)\hbar, \]

iii) \( n \geq 2 \):
\[ \lambda = \left(K + \frac{n}{2}\right)\hbar. \]

where \( \hbar = \hbar/2\pi \).

These are the classical energy levels for the harmonic oscillator [5].

Finally, in the case \( n \geq 2 \), we have one more condition for \( \gamma \) to be a global section. Namely, the form of the functions \( v'_j \) at points where \( 4\pi\alpha^2 - n = 2K \) is determined by asking that \( \gamma/U_j = \gamma/U_k \) for every pair \( (j, k) \). It follows from (3.31) that this condition means

\[ v'_j \left(\frac{z^h_j}{\bar{z}^j}\right)(z^h)^K = v_k \left(\frac{z^h_k}{\bar{z}^k}\right)(z^k)^K, \]

where the functions \( v' \) are analytic in their \( n - 1 \) arguments.

By replacing in (3.32) the functions \( v' \) by corresponding Taylor developments we see that the equality cannot hold unless \( v' \) are polynomials of total degree \( \leq K \). (Particularly, this shows why we must take in this case also \( K \geq 0 \).)

Clearly, the number of linearly independent such polynomials gives us the multiplicity of the corresponding eigenvalue and this number is
\[ \binom{K + n - 1}{K}, \]
which is again in agreement with the classical results [5].

REFERENCES


(Manuscrit reçu le 4 mai 1979)