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Some time-dependent Hartree equations

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Section A :
Physique theorique.

Some time-dependent Hartree equations

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ABSTRACT. — We extend earlier results concerning the existence of global solutions of the time-dependent Hartree equation by allowing the non-linear term to be of a more singular nature.

§ 1. INTRODUCTION

We prove the existence of global solutions for the time-dependent Schrödinger equation

$$
\begin{aligned}
i\frac{\partial f}{\partial t} &= -1/2\Delta f(x) + \mathrm{U}(x)f(x) \\
&\quad + f(x)\int_{\mathbb{R}^n} \mathrm{V}(x-y)\,|f(y)|^2 dy
\end{aligned}
\tag{1.1}
$$

on $L^2(\mathbb{R}^n)$, where U, V are suitable potentials. For short range potentials including the Coulomb potential, this problem has been solved in [*1*, *2*], where the Hartree-Fock theory is also treated, while for $U=0$ and $V(x)=x^2$ the equation has been shown in [*3*] to be exactly soluble with solitary wave solutions. Our present analysis includes both of the above results as special cases.

Before commencing, we mention some other work on these equations by way of motivation. When V is bounded, (1.1) can be derived rigorously from a linear multibody Schrödinger equation in the mean field or classical limits [*5*, *7*, *8*, *14*]. These derivations conform to the analysis of Haag and Bannier [*6*], who show that the probabilistic interpretation of such non-linear Schrödinger equations must be more classical than quantum-mechanical in certain respects.

Related work on the eigenvalue problem

$$\mathrm{E}f(x) = -1/2\Delta f(x) + \mathrm{U}(x)f(x) + f(x)\int_{\mathbb{R}^n} \mathrm{V}(x-y)\,|\,f(y)\,|^2 dy$$

and on minimization of the non-linear functional

$$\mathscr{E}\{f\} = \int_{\mathbb{R}^n} \{\, 1/2\,|\,\nabla f(x)\,|^2 + \mathrm{U}(x)\,|\,f(x)\,|^2 \,\}\, dx + 1/2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{V}(x-y)\,|\,f(x)\,|^2\,|\,f(y)\,|^2 dxdy$$

may be found in [*3*, *11*, *12*] and references cited here. The minimum of $\mathscr{E}(f)$ has been justified as an approximation to the ground state energy of an atom strongly coupled to an external phonon field in [*3*, *13*].

§ 2. THE ABSTRACT EXISTENCE THEOREMS

The theory of this section is taken from two papers of Kato [*9*, *10*], but we need to make some modifications in order to deal with quadratic form perturbations. In spite of this we describe the theory at a Banach space level. We assume that $\mathscr{H}_1$, $\mathscr{H}_0$, $\mathscr{H}_{-1}$ are three Banach spaces, that $\mathscr{H}_1$ is densely and continuously embedded in $\mathscr{H}_0$, and that $\mathscr{H}_0$ is densely and continuously embedded in $\mathscr{H}_{-1}$. To avoid technical difficulties we also assume that $\mathscr{H}_1$ is reflexive and uniformly convex.

We assume that Z^t is a family of linear operators parametrized by $t \in [0, \mathrm{T}]$ and satisfying the following conditions.

(A1) For each $t \in [0, \mathrm{T}]$, Z^t is the generator of a one-parameter group on $\mathscr{H}_{-1}$ such that

$$\exp(\mathrm{Z}^t s)\mathscr{H}_i = \mathscr{H}_i$$

for all $s \in \mathbb{R}$ and $i = -1, 0, 1$.

(A2) There exist constants β_i such that

$$\|\exp(\mathrm{Z}^t s) f\|_i \leq e^{\beta_i |s|}\,\|f\|_i$$

for all $0 \leq t \leq \mathrm{T}$, $s \in \mathbb{R}$, $f \in \mathscr{H}_i$ and $i = -1, 0, 1$.

(A3) $\quad \mathscr{H}_1 \subseteq \mathrm{Dom}(\mathrm{Z}^t) \quad$ for all $\quad t \in [0, \mathrm{T}]$.

(A4) If $f \in \mathscr{H}_i$ and $0 \leq t \leq \mathrm{T}$ then

$$\lim_{s \to 0} \|\exp(\mathrm{Z}^t s) f - f\|_i = 0.$$

(A5) If $f \in \mathscr{H}_i$ and $0 \leq s, t \leq \mathrm{T}$ then

$$(\mathrm{Z}^t f - \mathrm{Z}^s f) \in \mathscr{H}_0$$

and

$$\lim_{s \to t} \{ \sup_{\|f\|_1 \le 1} \| Z^t f - Z^s f \|_0 \} = 0.$$

THEOREM 1. — If Z^t satisfy (A1-5) then there exists a unique family of bounded operators $U(t, s)$ on $\mathscr{H}_{-1}$ defined for $0 \le s \le t \le T$ and satisfying the following conditions.

(B1) If $0 \le r \le s \le t \le T$ then $U(s, s) = 1$ and

$$U(t, r) = U(t, s)\, U(s, r).$$

(B2)

$$U(t, s)\mathscr{H}_i \subseteq \mathscr{H}_i$$

for all $0 \le s \le t \le T$ and $i = -1, 0, 1$, and

$$\| U(t, s)f \|_i \le e^{\beta_i(t-s)} \| f \|_i$$

for all $f \in \mathscr{H}_i$.

(B3) $s, t \to U(t, s)$ is strongly jointly continuous on $\mathscr{H}_i$ for $i = -1, 0, 1$.

(B4) If $f \in \mathscr{H}_1$ and $0 \le s \le t \le T$ then

$$\frac{\partial}{\partial t} U(t, s)f = Z^t U(t, s)f$$

the derivative being computed in $\mathscr{H}_{-1}$.

(B5) If $f \in \mathscr{H}_1$ and $0 \le s \le t \le T$ then

$$\frac{\partial}{\partial s} U(t, s)f = - U(t, s)Z^s f$$

the derivative being computed in $\mathscr{H}_{-1}$.

Proof. — Since the basic strategy is that of Kato [9] ,we content ourselves with a few comments on the variations.

The fact that the generators Z^t satisfy the hypothesis of [9] Theorem 4.1 with respect to the pair of spaces $\mathscr{H}_{-1}$, $\mathscr{H}_1$ immediately proves (B1), (B2) for $i = -1$, (B3) for $i = -1$, and (B5). A more careful reading of the convergence proof and of (A5) shows that (B2) and (B3) are also valid for $i = 0$. The proof of (B2) for $i = 1$ follows from [9] Theorem 5.1. Since each Z^t is the generator of a one-parameter group, the family Z^t is reversible in the sense of [9] Remark 5.3, and (B3) follows for $i = 1$ by [9] Remark 5.4. We finally prove (B4) by quoting [9] Theorem 5.2.

LEMMA 2. — Let Z^t and $\tilde{Z}^t$ both satisfy the conditions (A1-5) for the same constants β_i. If $f \in \mathscr{H}_i$ and $0 \le s \le t \le T$ then

$$U(t, s)f - \tilde{U}(t, s)f = \int_s^t \tilde{U}(t, x)(Z^x - \tilde{Z}^x)U(x, s)f dx. \tag{2.1}$$

Hence

$$\| U(t, s)f - \tilde{U}(t, s)f \|_0 \leq (t - s)e^{\gamma(t-s)} \| f \|_1 \sup_{s \leq x \leq t} ||| Z^x - \tilde{Z}^x ||| \quad (2.2)$$

where $\gamma = \max(\beta_0, \beta_1)$ and

$$||| Y ||| = \sup \{ \| Yg \|_0 : \| g \|_1 \leq 1 \}.$$

Proof. — The proof of (2.1) is based upon the identity

$$\frac{\partial}{\partial x} \tilde{U}(t, x)U(x, s)f = \tilde{U}(t, x)(Z^x - \tilde{Z}^x)U(x, s)f$$

valid for all $f \in \mathcal{H}_1$ and $0 \leq s \leq x \leq t \leq T$. The estimate (2.2) follows by applying (B2).

We now turn to the non-linear theory. We assume that Z^f is a family of linear operators parametrized by $f \in \mathcal{H}_1$ and satisfying the following conditions.

(C1) For all $f \in \mathcal{H}_1$, Z^f is the generator of a one-parameter group on $\mathcal{H}_{-1}$ such that

$$\exp(Z^f s)\mathcal{H}_i = \mathcal{H}_i$$

for all $s \in \mathbb{R}$ and $i = -1, 0, 1$.

(C2) For all $R < \infty$ there exist constants β_i for $i = -1, 0, 1$ such that

$$\| \exp(Z^f s)g \|_i \leq e^{\beta_i |s|} \| g \|_i$$

if $g \in \mathcal{H}_i$, $s \in \mathbb{R}$ and $\| f \|_1 \leq R$.

(C3) $\mathcal{H}_1 \subseteq \mathrm{Dom}(Z^f)$ for all $f \in \mathcal{H}_1$.

(C4) If $g \in \mathcal{H}_i$ for $i = -1, 0, 1$ and $f \in \mathcal{H}_1$ then

$$\lim_{s \to 0} \| \exp(Z^f s)g - g \|_i = 0.$$

(C5) For all $R < \infty$ there is a constant $c < \infty$ such that if $\| f \|_1 \leq R$, $\| g \|_1 \leq R$, $h \in \mathcal{H}_1$ then

$$(Z^f h - Z^g h) \in \mathcal{H}_0$$

and

$$\| Z^f h - Z^g h \|_0 \leq c \| f - g \|_0 \| h \|_1.$$

THEOREM 3. — Let Z^f satisfy (C1-5) above and let $R < \infty$. Then there exists $T > 0$ such that all $a \in \mathcal{H}_1$ satisfying $\| a \|_1 \leq R/2$ are associated with a unique function $f : [0, T] \to \mathcal{H}_1$ with the following properties.

(D1) $f(0) = a$, f is norm continuous for the $\mathcal{H}_1$ norm, and $\| f(t) \|_1 \leq R$ for all $t \in [0, T]$.

(D2) $f(t)$ is differentiable for the $\mathcal{H}_{-1}$ norm and

$$f'(t) = Z^{f(t)} f(t)$$

for all $t \in [0, T]$.

Proof. — We modify slightly the proof of [*10*] Theorem 6. Given a and any $T > 0$, we define $\mathscr{E}$ to be the space of functions $f : [0, T] \to \mathscr{H}_1$ such that $f(0) = a$, $\| f(t) \|_1 \leq R$ for all $t \in [0, T]$ and f is continuous for the $\mathscr{H}_0$ norm. Since $\mathscr{H}_1$ is reflexive, $\mathscr{E}$ is a complete space with respect to the metric

$$d(f, g) = \sup \{ \| f(t) - g(t) \|_0 : 0 \leq t \leq T \}$$

For all $f \in \mathscr{E}$, $Z^{f(t)}$ satisfies (A1-5) so there is an associated propagator $U^f(t, s)$ satisfying (B1-5). If $\tilde{f}$ is defined by

$$\tilde{f}(t) = U^f(t, 0)a$$

then $\tilde{f}(0) = 0, f$ is continuous for the $\mathscr{H}_0$ norm

$$\| \tilde{f}(t) \|_1 \leq e^{\beta_1 t} R/2$$

for all $t \geq 0$. If T is small enough, depending only on R, β_1, then $\tilde{f} \in \mathscr{E}$. If $f, g \in \mathscr{E}$ then (2.2) implies that

$$\begin{aligned} \| \tilde{f}(t) - \tilde{g}(t) \|_0 &\leq 1/2Rte^{\gamma t} \sup_{0 \leq t \leq T} ||| Z^{f(t)} - Z^{g(t)} ||| \\ &\leq 1/2cRte^{\gamma t} d(f, g). \end{aligned}$$

If T is small enough. depending only on R, c, γ, we deduce that

$$d(\tilde{f}, \tilde{g}) \leq 1/2d(f, g)$$

so by the contraction mapping principle there is a unique $f \in \mathscr{E}$ such that $\tilde{f} = f$. For this f, (D1) and (D2) are consequences of (B3) and (B4). Conversely given a function f satisfying (D1) and (D2)

$$\frac{\partial}{\partial s} U^f(t, s) f(s) = 0$$

by (B5), so

$$f(t) = U^f(t, 0) f(0) = \tilde{f}(t).$$

for all $t \in [0, T]$.

§ 3. NON-LINEAR SCHRODINGER EQUATIONS

Let $\mathscr{H} = \mathscr{H}_0$ be a complex Hilbert space and let H be a semibounded self-adjoint operator on $\mathscr{H}$ such that $(H + c) \geq 1$. We put

$$\mathscr{H}_1 = \text{Dom}\,(H + c)^{1/2},$$

this being a Hilbert space for the inner product

$$\langle f, g \rangle_1 = \langle (H + c)^{1/2} f, (H + c)^{1/2} g \rangle$$

We also let $\mathcal{H}_{-1}$ be the Hilbert space completion of $\mathcal{H}$ with respect to the inner product

$$\langle f, g \rangle_{-1} = \langle (H + c)^{-1} f, g \rangle$$

We suppose that A^f are symmetric linear operators in $\mathcal{H}$, parametrized by $f \in \mathcal{H}_1$ and satisfying the following conditions.

(E1) The domain of A^f contains $\mathcal{H}_1$ for all $f \in \mathcal{H}_1$.

(E2) For all $R < \infty$ there exists $a < \infty$ such that if $\| f \|_1 \leq R$ and $g \in \mathcal{H}_1$ then

$$\| A^f g \| \leq a \| (H + c)^{1/2} g \|.$$

(E3) For all $R < \infty$ there exists $\beta_1 < \infty$ such that if $\| f \|_1 \leq R$ then

$$\pm i[A^f, H] \leq 2\beta_1 (H + c)$$

as a form inequality on Dom (H).

(E4) For all $R < \infty$ there exists $c < \infty$ such that if $\| f \|_1 \leq R$, $\| g \|_1 \leq R$, $h \in \mathcal{H}_1$ then

$$\| A^f h - A^g h \| \leq c \| f - g \| \| (H + c)^{1/2} h \|.$$

THEOREM 4. — If H, A^f satisfy (E1-4) then

$$Z^f = - i(H + A^f)$$

satisfy (C1-5) with $\beta_0 = 0$, $\beta_{-1} = \beta_1$.

Proof. — We start by observing that

$$\lim_{b \to \infty} \| A^f (b + H)^{-1} \| \leq \lim_{b \to \infty} \| A^f (H + c)^{-1/2} \| \| (H + c)^{1/2} (H + b)^{-1} \| = 0$$

so A^f is relatively bounded with respect to H, with relative bound zero. Therefore $H + A^f$ is self-adjoint with

$$\text{Dom} \mid H + A^f \mid^\alpha = \text{Dom} \mid H \mid^\alpha$$

for all $0 \leq \alpha \leq 1$. Putting $\alpha = 1/2$ we deduce that

$$\exp \{ - i(H + A^f)t \} \mathcal{H}_1 = \mathcal{H}_1$$

for all $t \in \mathbb{R}$. Define the one-parameter group V_t on $\mathcal{H}_{-1}$ by

$$V_t = U_{-t}^*$$

where U_t is the restriction of $\exp \{ - i(H + A^f)t \}$ to $\mathcal{H}_1$, and we are using the natural conjugate linear isomorphism of $\mathcal{H}_{-1}$ with $\mathcal{H}_1^*$. If $g \in \mathcal{H}_0$ and $h \in \mathcal{H}_1$ then

$$\langle V_t g, h \rangle = \langle g, U_{-t} h \rangle = \langle \exp \{ - i(H + A^f)t \} g, h \rangle$$

so V_t is an extension of $\exp \{ - i(H + A^f)t \}$ from $\mathcal{H}_0$ to $\mathcal{H}_{-1}$.

We next determine the generator Z^f of V_t. If $\mathscr{H}_2 = \text{Dom}(H)$ then

$$\lim_{t\to 0} \| t^{-1}(V_t g - g) + i(Hg + A^f g) \| = 0$$

for all $g \in \mathscr{H}_2$, so $\mathscr{H}_2 \subseteq \text{Dom}(Z^f)$ and

$$Z^f g = - i(H + A^f)g$$

for all $g \in \mathscr{H}_2$. Since $\mathscr{H}_2$ is invariant under V_t it is a core for Z^f. Thus Z^f is the closure of $(H + A^f)|_{\mathscr{H}_2}$, which equals $(H + A^f)$ considered as an operator from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$. This completes the proof of (C1), (C3), (C4).

If $g, h \in \mathscr{H}_2$ then

$$\langle g, h \rangle_1 = \langle (H + c)g, h \rangle = \langle g, (H + c)h \rangle.$$

Hence

$$\begin{aligned} \pm \frac{d}{dt} \| V_t g \|_1^2 &= \mp i \langle (H + A^f)V_t g, (H + c)V_t g \rangle \\ &\quad \pm i \langle (H + c)V_t g, (H + A^f)V_t g \rangle \\ &= \pm i \langle [A^f, H]V_t g, V_t g \rangle \\ &\leq 2\beta_1 \| V_t g \|_1^2. \end{aligned}$$

Hence

$$\| V_t g \|_1 \leq e^{\beta_1 |t|} \| g \|_1$$

for all $t \in \mathbb{R}$. The same holds for all $g \in \mathscr{H}_1$ by density arguments. The proof of the remaining parts of (C2) is now straightforward.

Since

$$Z^f h - Z^g h = - i(A^f h - A^g h)$$

for all $f, g, h \in \mathscr{H}_1$, we see finally that (E4) implies (C5).

We next indicate how (E3) may be verified in a concrete application. We put $\mathscr{H} = L^2(\mathbb{R}^n)$ and let $H = - 1/2\Delta + W$ as a form sum, W being a locally L^2 potential whose negative part has form bound less than one with respect to $- 1/2\Delta$.

PROPOSITION 5. — If A is a continuously differentiable potential on $\mathbb{R}^n$ such that

$$\| [\partial_r, A](H + c)^{-1/2} \| \leq \alpha$$

for all r, then there exists a constant β_1 such that

$$\pm i[A, H] \leq 2\beta_1(H + c).$$

Proof. — Starting from the identity

$$\begin{aligned} i[A, H] &= - \frac{i}{2} \left[A, \sum_r \partial_r^2 \right] \\ &= - \frac{i}{2} \sum_r \{ [A, \partial_r]\partial_r + \partial_r[A, \partial_r] \} \end{aligned}$$

we deduce that

$$\| (H + c)^{-1/2} i[A, H](H + c)^{-1/2} \| \leq \sum_r \| H + c)^{-1/2} \partial_r [A, \partial_r](H + c)^{-1/2} \| \leq \sum_r \| \partial_r (H + c)^{-1/2} \| \| [\partial_r, A](H + c)^{-1/2} \|.$$

Now there exists a constant γ such that

$$0 \leq -\partial_r^2 \leq -\Delta \leq \gamma^2 (H + c)$$

so

$$\| (H + c)^{-1/2} i[A, H](H + c)^{-1/2} \| \leq n\alpha\gamma$$

as required.

Returning to the abstract problem, we shall need to use the following regularization later.

THEOREM 6. — If H, A^f satisfy (E1-4) then

$$A_\varepsilon^f = e^{-\varepsilon(H+c)} A^{\exp\{-\varepsilon(H+c)\} f} e^{-\varepsilon(H+c)}$$

satisfies (E1-4) uniformly in $\varepsilon > 0$. Moreover let $\| a \|_1 \leq 1/2R$ and let $f_\varepsilon(t)$ be the solution of

$$\frac{d}{dt} f_\varepsilon(t) = -i \{ H f_\varepsilon(t) + A_\varepsilon^{f_\varepsilon(t)} f_\varepsilon(t) \}$$

for $0 \leq t \leq T$ with initial condition $f_\varepsilon(0) = a$, $f(t)$ being defined similarly. Then

$$\| f_\varepsilon(t) \|_1 \leq R$$

for all $0 \leq t \leq T$ and $\varepsilon > 0$, and

$$\operatorname{im}_{\varepsilon \to 0} \{ \sup_{0 \leq t \leq T} \| f_\varepsilon(t) - f(t) \|_0 \} = 0.$$

We omit the proof, which is very similar to that of [*10*].

LEMMA 10.1. — It is presumably also the case that

$$\lim_{\varepsilon \to 0} \{ \sup_{0 \leq t \leq T} \| f_\varepsilon(t) - f(t) \|_1 \} = 0$$

but we avoid using this result, whose proof would probably be rather delicate.

§ 4. LOCAL EXISTENCE FOR THE HARTREE EQUATION

We apply the above ideas to the Hartree equation

$$i \frac{\partial f}{\partial t} = -1/2 \Delta f(x) + U(x) f(x) + f(x) \int_{\mathbb{R}^n} V(x - y) | f(y) |^2 dy$$

on $L^2(\mathbb{R}^n)$ where $n \geq 3$. We assume that $V = V_1 + V_2$, where V_1 is semi-bounded and continuously differentiable while V_2 is bounded at infinity. Apart from a phase factor (which may be adjusted by a separate argument if desired) the equation may be written in the form

$$\frac{\partial f}{\partial t} = - i(Hf + A_1^f f + A_2^f f) \equiv Z^f f \tag{4.1}$$

where

$$H = - 1/2\Delta + U + V_1 \tag{4.2}$$

$$A_1^f(x) = \int_{\mathbb{R}^n} \{ V_1(x - y) - V_1(x) - V_1(y) \} \, |f(y)|^2 dy$$

$$A_2^f(x) = \int_{\mathbb{R}^n} V_2(y) \, |f(x - y)|^2 dy.$$

We assume that U, V_1, V_2 satisfy the following conditions.

(F1) $V_1(x) \geq 1$ for all $x \in \mathbb{R}^n$.

(F2) U has form bound less than one with respect to $- 1/2\Delta$. If

$$\pm U \leq a(- 1/2\Delta) + (c - 1) \tag{4.3}$$

where $0 < a < 1$ and $0 < c < \infty$ then the form sum (4.2) defines a self-adjoint operator whose form domain is equal to that of $(- 1/2\Delta + V_1)$. Moreover

$$(1 - a)(- 1/2\Delta + V_1) + 1 \leq H + c \leq (1 + a)(- 1/2\Delta + V_1) + 2c.$$

We put

$$\mathscr{H}_1 = \text{Dom } (H + c)^{1/2} = \text{Dom } (- \Delta^{1/2}) \cap \text{Dom } (V_1^{1/2}).$$

Our remaining conditions are as follows.

(F3) There exists a constant $c' < \infty$ such that

$$| V_1(x - y) - V_1(x) - V_1(y) | \leq c' V_1(x)^{1/2} V_1(y)^{1/2}$$

for all $x, y \in \mathbb{R}^n$.

(F4) V_1 is continuously differentiable and

$$| (\partial_r V_1)(x) | \leq c'' V_1(x)^{1/2}$$

for some constant c'' and all r.

(F5) $$V_2 \in L^\infty(\mathbb{R}^n) + L^{n/2}(\mathbb{R}^n).$$

THEOREM 7. — If U, V_1, V_2 satisfy (F1-5), then H and $A^f = A_1^f + A_2$ satisfy (E1-4). Hence (4.1) has unique local solutions in the sense of Theorem 3.

Proof. — We first treat A_1^f. By (F3)

$$A_1^f(x) \le c' V_1(x)^{1/2} \int_{\mathbb{R}^n} V_1(y)^{1/2} |f(y)|^2 dy$$

so

$$\begin{aligned} \| A_1^f (H + c)^{-1/2} \| &\le c' \| V_1^{1/2} (H + c)^{-1/2} \| \langle V_1^{1/2} f, f \rangle \\ &\le c' \| V_1^{1/2} (H + c)^{-1/2} \|^2 \| (H + c)^{1/2} f \| \| f \| \\ &\le c'(1 - a)^{-1} \| (H + c)^{1/2} f \| \| f \| \end{aligned}$$

and (E2) holds.

Similarly

$$\begin{aligned} A_1^g(x) - A_1^f(x) &= \int_{\mathbb{R}^n} \{ V_1(x - y) - V_1(x) - V_1(y) \} \{ g(y) - f(y) \} \overline{g(y)} dy \\ &= \int_{\mathbb{R}^n} \{ V_1(x - y) - V_1(x) - V_1(y) \} f(y) \{ g(y) - f(y) \}^- dy \end{aligned}$$

so

$$\begin{aligned} | A_1^g(x) - A_1^f(x) | &\le c' V_1(x)^{1/2} | \langle g - f, V_1^{1/2} g \rangle | \\ &\quad + c' V_1(x)^{1/2} | \langle V_1^{1/2} f, g - f \rangle | \end{aligned}$$

and

$$\begin{aligned} &\| (A_1^g - A_1^f)(H + c)^{-1/2} \| \\ &\qquad \le c'(1 - a)^{-1} \| g - f \| \{ \| (H + c)^{1/2} g \| + \| (H + c)^{1/2} f \| \} \end{aligned}$$

which implies (E4).

Now

$$\partial_r A_1^f(x) = \int_{\mathbb{R}^n} \{ \partial_r V_1(x - y) - \partial_r V_1(x) \} |f(y)|^2 dy$$

so (F4) implies that

$$| \partial_r A_1^f(x) | \le c'' \int_{\mathbb{R}^n} \{ V_1(x - y)^{1/2} + V_1(x)^{1/2} \} |f(y)|^2 dy$$

But (F3) leads to the estimate

$$V_1(x - y) \le (2 + c') V_1(x) V_1(y)$$

so

$$| \partial_r A_1^f(x) | \le k V_1(x)^{1/2} \langle V^{1/2} f, f \rangle .$$

We deduce that

$$\| [\partial_r, A_1^f](H + c)^{-1/2} \| \le k(1 - a)^{-1} \| (H + c)^{1/2} f \| \| f \| \qquad (4.4)$$

which implies (E3) by Proposition 5.

We treat the term A_2^f only when $V_2 \in L^{n/2}$, the case $V_2 \in L^\infty$ being similar but easier. In the following proof $\| \ \|_p$ denotes the usual L^p norm. We quote from ([*15*], p. 124) the Sobolev inequality

$$\| f \|_{2n/(n-2)} \le a \| (1 - \Delta)^{1/2} f \|_2 \le b \| (H + c)^{1/2} f \|_2$$

If $f \in \mathscr{H}_1$ then $|f|^2 \in L_{n/(n-2)}$ so $A_2^f = V_2 . |f|^2 \in L^\infty$ and

$$\| A_2^f \|_\infty \leq b^2 \| V_2 \|_{n/2} \| (H + c)^{1/2} \|_2^2$$

which implies (E1) and (E2).

If $f, g, h \in \mathscr{H}_1$ then $g\bar{f} \in L^{n/(n-1)}$ with

$$\| g\bar{f} \|_{n/(n-1)} \leq \| g \|_2 \| f \|_{2n/(n-1)}$$

Hence $V_2 .(g\bar{f}) \in L^n$ and $\{ V_2 .(g\bar{f}) \} h \in L^2$ with

$$\begin{aligned} \| \{ V_2 \cdot (g\bar{f}) \} h \|_2 &\leq b \| V_2 \cdot (g\bar{f}) \|_n \| (H + c)^{1/2} h \|_2 \\ &\leq b \| V_2 \|_{n/2} \| g\bar{f} \|_{n(n-1)} \| (H + c)^{1/2} h \|_2 \\ &\leq b^2 \| V_2 \|_{n/2} \| g \|_2 \| (H + c)^{1/2} f \|_2 \| (H + c)^{1/2} h \|_2 \quad (4.5) \end{aligned}$$

This estimate easily yields (E4).

We finally note that

$$\begin{aligned} | \partial_r A_2^f(x) | &= \left| \int_{\mathbb{R}^n} V_2(y) \{ \overline{f(x-y)} \partial_r f(x-y) + f(x-z) \overline{\partial_r f(x-y)} \} \, dy \right| \\ &\leq 2 \int_{\mathbb{R}^n} | V_2(x) f(y-y) \partial_r f(x-y) | \, dy. \end{aligned}$$

Use of (4.5) now leads to the estimate

$$\begin{aligned} \| [\partial_r, A_2^f](H + c)^{-1/2} \| &\leq 2b^2 \| V_2 \|_{n/2} \| \partial_r f \|_2 \| (H + c)^{1/2} f \|_2 \\ &\leq k \| (H + c)^{1/2} f \|_2^2. \quad (4.6) \end{aligned}$$

The truth of (E3) now follows by an application of Proposition 5.

We finally make some comments on the conditions we have imposed on V_1 and V_2. If $n = 3$ then

$$| x |^{-\alpha} \equiv V_2(x) \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$$

if and only if $0 < \alpha < 2$. Such potentials have more singular local behaviour than those considered in [*1*, *2*]; indeed they have nearly the worst local singularities which can be treated by quadratic form techniques even for linear Hamiltonians.

Since the nature of the conditions on V_1 is not immediately apparent, we include some results concerning the class $\mathscr{C}$ of continuously differentiable functions $W : \mathbb{R}^n \to [1, \infty)$ satisfying (F3) and (F4).

PROPOSITION 8. — The class $\mathscr{C}$ is closed under sums and space translations.

Proof. — If $V, W \in \mathscr{C}$ then

$$\begin{aligned} | (V + W)(x - y) - (V + W)(x) - (V + W)(y) | &\leq cV(x)^{1/2} V(Y)^{1/2} + c'W(x)^{1/2} W(Y)^{1/2} \\ &\leq (c + c')(V + W)(x)^{1/2} (V + W)(Y)^{1/2} \end{aligned}$$

Moreover

$$| \partial_r(V + W)(x) | \leq cV(x)^{1/2} + c'W(x)^{1/2}$$
$$\leq (c + c')(V + W)(x)^{1/2}.$$

Hence $(V + W) \in \mathscr{C}$. If $V \in \mathscr{C}$, $a \in \mathbb{R}^n$ and $W(x) = V(x + a)$ then

$$V(x) \leq bW(x)$$

for a certain a-dependent constant b. Hence

$$\begin{aligned} &| W(x-y)-W(x)-W(y) | \\ &\quad = | \{ V(x+a-y)-V(x+a)-V(y) \} - \{ V(y+a)-V(y)-V(a) \} - V(a) | \\ &\quad \leq cV(x+a)^{1/2}V(y)^{1/2} + cV(y)^{1/2}V(a)^{1/2} + V(a) \\ &\quad \leq cb^{1/2}W(x)^{1/2}W(y)^{1/2} + cb^{1/2}W(y)^{1/2}V(a)^{1/2} + V(a) \\ &\quad \leq kW(x)^{1/2}W(y)^{1/2}. \end{aligned}$$

Moreover

$$\begin{aligned} | \partial_r W(x) | &= | \partial_r V(x + a) | \\ &\leq cV(x - a)^{1/2} \\ &= cW(x)^{1/2} \end{aligned}$$

so $W \in \mathscr{C}$.

PROPOSITION 9. — If $\alpha \geq 1$ and $\beta \geq 0$ then

$$V_{\alpha\beta}(x) = \alpha + \beta x^2$$

lies in $\mathscr{C}$. If $W \in \mathscr{C}$ then $1 \leq W \leq V_{\alpha\beta}$ for some α, β.

Proof. — The first assertion is trivial. For the second we note that if $W \in \mathscr{C}$ then

$$\begin{aligned} | \partial_r \{ W(x)^{1/2} \} | &= | 1/2W(x)^{-1/2}(\partial_r W)(x) | \\ &\leq 1/2c'' \end{aligned}$$

so

$$| W(x)^{1/2} | \leq W(0)^{1/2} + 1/2nc'' \, | x \, ||$$

from which the assertion follows.

PROPOSITION 10. — If $V \in \mathscr{C}$ and $0 < \lambda < 1$ then $V^\lambda \in \mathscr{C}$.

Proof. — Since $\lambda - 1 + 1/2 \leq 1/2\lambda$ when $0 < \lambda < 1$ we see that

$$\begin{aligned} | (\partial_r V^\lambda)(x) | &= | \lambda V(x)^{\lambda-1}(\partial_r V)(x) | \\ &\leq \lambda c'' V(x)^{\lambda-1+1/2} \\ &\leq \lambda c'' V(x)^{\lambda/2} \end{aligned}$$

so V^λ satisfies (F4).

The proof of (F3) is harder. If $0 < x < \infty$ then

$$(1 + x)^\lambda \leq 1 + x^\lambda.$$

Hence

$$(\alpha + \beta)^\lambda \leq \alpha^\lambda + \beta^\lambda$$

for all $0 < \alpha, \beta < \infty$. If $0 < x < \infty$ then

$$1 + x^\lambda \leq (1 + x)^\lambda + x^{\lambda/2}$$

by separate arguments for $0 < x \leq 1$ and $1 < x < \infty$.

Therefore

$$\alpha^\lambda + \beta^\lambda \leq (\alpha + \beta)^\lambda + \alpha^{\lambda/2}\beta^{\lambda/2}$$

for all $0 < \alpha, \beta < \infty$. Combining these results we obtain

$$| \alpha^\lambda + \beta^\lambda - (\alpha + \beta)^\lambda | \leq \alpha^{\lambda/2}\beta^{\lambda/2}.$$

If $V \in \mathscr{C}$ and $\alpha = V(x)$, $\beta = V(y)$, $\gamma = V(x - y)$ then (F3) states that

$$| \gamma - \alpha - \beta | \leq c'\alpha^{1/2}\beta^{1/2}.$$

Hence

$$\gamma \leq \alpha + \beta + c'\alpha^{1/2}\beta^{1/2}$$

and

$$\gamma^\lambda \leq (\alpha + \beta)^\lambda + (c'\alpha^{1/2}\beta^{1/2})^\lambda.$$

Similarly

$$(\alpha + \beta)^\lambda \leq \gamma^\lambda + (c'\alpha^{1/2}\beta^{1/2})^\lambda$$

so

$$\begin{aligned} | \alpha^\lambda + \beta^\lambda - \gamma^\lambda | &\leq | \alpha^\lambda + \beta^\lambda - (\alpha + \beta)^\lambda | + | \gamma^\lambda - (\alpha + \beta)^\lambda | \\ &\leq \alpha^{\lambda/2}\beta^{\lambda/2} + (c'\alpha^{1/2}\beta^{1/2})^\lambda \\ &= k\alpha^{\lambda/2}\beta^{\lambda/2}. \end{aligned}$$

§ 5. GLOBAL EXISTENCE FOR THE HARTREE EQUATION

As is usual in this subject, our proof of global existence depends on the discovery of an invariant non-linear functional. Formal calculations indicate that if we make the extra-hypothesis

(F6) $$V_i(-x) = V_i(x)$$

for $i = 1, 2$ and all $x \in \mathbb{R}^n$, then the functional

$$\begin{aligned} \mathscr{E}\{f\} = &\int_{\mathbb{R}^n} \{ 1/2 \,|\, \nabla f(x) \,|^2 + U(x) \,|\, f(x) \,|^2 \} \, dx \\ &+ 1/2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} V(x - y) \,|\, f(x) \,|^2 \,|\, f(y) \,|^2 dx dy \end{aligned}$$

is invariant. We define $\mathscr{E}$ rigorously on $\mathscr{H}_1$ by the formula

$$\mathscr{E}\{f\} = \langle Hf, f \rangle + F_1(f, f, f, f) + F_2(f, f, f, f)$$

where the bounded multilinear functionals F_i on $X^4\mathscr{H}_1$ are defined by

$$\begin{aligned} &F_1(f_1, f_2, f_3, f_4) \\ &\quad = 1/2 \iint \{ V_1(x - y) - V_1(x) - V_1(y) \} f_1(x)\overline{f_2(x)}f_3(y)\overline{f_4(y)} dx dy \end{aligned}$$

and

$$F_2(f_1, f_2, f_3, f_4) = 1/2 \iint V_2(x-y) f_1(x) \overline{f_2(x)} f_3(y) \overline{f_4(y)} dx dy.$$

The proof that F_1 and F_2 are finite and bounded on $X^4\mathcal{H}_1$ is elementary. Algebraically the crucial element in the proof of the invariance of $\mathcal{E}$ is the observation that if $f, g \in \mathcal{H}_1$, and V_1, V_2 satisfy (F6) then

$$F_i(g, f, f, f) = F_i(f, f, g, f) = 1/2 \langle g, A_i^f f \rangle$$
$$F_i(f, g, f, f) = F_i(f, f, f, g) = 1/2 \langle A_i^f f, g \rangle$$

Technical problems force us to prove invariance first for the regularized evolution equation

$$\frac{\partial f}{\partial t} = - i(Hf + A_{1\varepsilon}^f f + A_{2\varepsilon}^f f) \qquad (5.1)$$

$A_{i\varepsilon}^f$ being defined as in Theorem 6. The invariant quantity is now

$$\mathcal{E}_\varepsilon \{ f \} = \langle Hf, f \rangle + F_{1\varepsilon}(f, f, f, f) + F_{2\varepsilon}(f, f, f, f)$$

where

$$F_{i\varepsilon}(f_1, f_2, f_3, f_4) = F_i(e^{-\varepsilon(H+c)} f_1, e^{-\varepsilon(H+c)} f_2, e^{-\varepsilon(H+c)} f_3, e^{-\varepsilon(H+c)} f_4).$$

LEMMA 11. — If $f(t)$ is a local solution of (5.1) constructed for $0 \leq t \leq T$ by the method of Theorem 3, then $\mathcal{E}_\varepsilon \{ f(t) \}$ is independent of t on $[0, T]$.

Proof. — Since $f(t)$ satisfies (5.1), the derivative existing in the $\mathcal{H}_{-1}$ norm, the function $g(t) = e^{iHt} f(t)$ satisfies

$$\frac{dg}{dt} = - i e^{iHt} (A_{1\varepsilon}^f f + A_{2\varepsilon}^f f).$$

Now the right-hand side of this equation is a continuous function of time with respect to the $\mathcal{H}_1$ norm, so the derivative of g also exists in the $\mathcal{H}_1$ norm. Hence

$$\begin{aligned} \frac{d}{dt} \langle Hf, f \rangle &= \frac{d}{dt} \langle Hg, g \rangle \\ &= \langle Hg, g' \rangle + \langle g', Hg \rangle \\ &= i \langle Hf, A_{1\varepsilon}^f f + A_{2\varepsilon}^f f \rangle - i \langle A_{1\varepsilon}^f f + A_2^f f, Hf \rangle \end{aligned}$$

Secondly since the derivative on the left-hand side of the equation

$$\frac{d}{dt} e^{-\varepsilon(H+c)} f(t) = e^{-\varepsilon(H+c)} f'(t)$$

exists in the $\mathscr{H}_1$ norm, we see that

$$\begin{aligned}
&\frac{d}{dt} F_{i\varepsilon}(f, f, f, f) \\
&\quad = \frac{d}{dt} F_i(e^{-\varepsilon(H+c)}f, e^{-\varepsilon(H+c)}f, e^{-\varepsilon(H+c)}f, e^{-\varepsilon(H+c)}f) \\
&\quad = F_{i\varepsilon}(f', f, f, f) + F_{i\varepsilon}(f, f', f, f) \\
&\qquad + F_{i\varepsilon}(f, f, f', f) + F_{i\varepsilon}(f, f, f, f') \\
&\quad = \langle f', A^f_{i\varepsilon} f \rangle + \langle A^f_{i\varepsilon} f, f' \rangle
\end{aligned}$$

so

$$\begin{aligned}
&\frac{d}{dt}\{ F_{1\varepsilon}(f, f, f, f) + F_{2\varepsilon}(f, f, f, f) \} \\
&\quad = -i\langle Hf + A^f_{1\varepsilon} f + A^f_{2\varepsilon} f, A^f_{1\varepsilon} f + A^f_{2\varepsilon} f \rangle \\
&\qquad + i\langle A^f_{1\varepsilon} f + A^f_{2\varepsilon} f, Hf + A^f_{1\varepsilon} f + A^f_{2\varepsilon} f \rangle \\
&\quad = -i\langle Hf, A^f_{1\varepsilon} f + A^f_{2\varepsilon} f \rangle + i\langle A^f_{1\varepsilon} f + A^f_{2\varepsilon} f, Hf \rangle.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\mathscr{E}\{f(t)\} = 0.$$

THEOREM 12. — If $f(t)$ is a local solution of

$$\frac{df}{dt} = -i(Hf + A^f_1 f + A^f_2 f) \tag{5.2}$$

constructed for $0 \leq t \leq T$ by the method of Theorem 3, then $\mathscr{E}\{f(t)\}$ is independent of t on $[0, T]$, as is $\| f(t) \|_0$.

Proof. — Let $f(0) = a$ and let $f_\varepsilon(t)$ be the local solution of (5.1) considered in Lemma 11. We first note that estimates of the type proved in Theorem 7 show that the multilinear functionals F_i are bounded if any *one* of the four copies of $\mathscr{H}_1$ is replaced by $\mathscr{H}_0$. It follows by Theorem 6 that

$$\lim_{\varepsilon\to 0} F_{i\varepsilon}(f_\varepsilon, f_\varepsilon, f_\varepsilon f_\varepsilon) = F_i(f, f, f, f)$$

for all $0 \leq t \leq T$. The lower semi-continuity of the quadratic form of H implies that

$$\langle Hf(t), f(t) \rangle \leq \limsup_{\varepsilon\to 0} \langle Hf_\varepsilon(t), f_\varepsilon(t) \rangle.$$

Hence

$$\begin{aligned}
\mathscr{E}\{f(t)\} &\leq \limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon\{f_\varepsilon(t)\} \\
&= \limsup_{\varepsilon\to 0} \mathscr{E}_\varepsilon\{f_\varepsilon(0)\} \\
&= \lim_{\varepsilon\to 0}\{\langle Ha, a\rangle + F_\varepsilon(a, a, a, a)\} \\
&= \langle H\alpha, a\rangle + F(a, a, a, a) \\
&= \mathscr{E}\{f(0)\}.
\end{aligned}$$

The proof is completed by noting that the reversibility of the time evolution allows one to deduce that

$$\mathscr{E}\{f(t)\} \geq \mathscr{E}\{f(0)\}$$

by a similar method. The last statement of the theorem follows from the fact that $\beta_0 = 0$.

COROLLARY 13. — If $f(t)$ and $f_\varepsilon(t)$ are the solutions of (5.2) and (5.1) respectively with $f(0) = f_\varepsilon(0)$ then

$$\lim_{\varepsilon \to 0} \| f_\varepsilon(t) - f(t) \|_1 = 0$$

for all $0 \leq t \leq T$.

Proof. — By the proofs of Lemma 11 and Theorem 12 we find that

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathscr{E}_\varepsilon\{f_\varepsilon(t)\} &= \lim_{\varepsilon\to 0} \mathscr{E}_\varepsilon\{f_\varepsilon(0)\} \\ &= \mathscr{E}\{f(0)\} = \mathscr{E}\{f(t)\} \end{aligned}$$

and

$$\lim_{\varepsilon\to 0} = F_{i\varepsilon}(f_\varepsilon, f_\varepsilon, f_\varepsilon, f_\varepsilon) = F_i(f, f, f, f)$$

Hence

$$\lim_{\varepsilon\to 0} \langle (H + c) f_\varepsilon(t), f_\varepsilon(t) \rangle = \langle (H + c) f(t), f(t) \rangle .$$

The result follows by combining this with

$$\lim_{\varepsilon\to 0} \| f_\varepsilon(t) - f(t) \|_0 = 0.$$

Since the functional $\mathscr{E}\{f\}$ is invariant under space translations, at least if $U = 0$, while H is not, one cannot expect to control $\| f \|_1$ solely in terms of $\mathscr{E}\{f\}$. It is, however, possible to control the kinetic energy.

LEMMA 14. — There exist constants h, $k > 0$ such that

$$\langle (1 - \Delta) f, f \rangle \leq h \mathscr{E}\{f\} + k$$

for all $f \in \mathscr{H}_1$ with $\| f \| = 1$.

Proof. — We start from the fact that there exists a constant c such that for any potential $W \in L^{n/2}$ and f, $g \in \mathscr{H}_1$

$$| \langle Wg, f \rangle | \leq c \| W \|_{n/2} \| (1 - \Delta)^{1/2} g \|_2 \| (1 - \Delta)^{1/2} f \|_2$$

Now for any $\varepsilon > 0$, there is a decomposition $V_2 = V_3 + V_4$ with $\| V_3 \|_{n/2} < c^{-1}\varepsilon$ and $V_4 \in L^\infty$. Since $\| f \|_0 = 1$ we see that $A_2^f = A_3^f + A_4^f$ with $\| A_3^f \|_{n/2} < c^{-1}\varepsilon$ and $\| A_4^f \|_\infty \leq \| V_4 \|_\infty$. Therefore

$$\begin{aligned} F_2(f, f, f, f) &= 1/2 \langle A_3^f f, f \rangle + 1/2 \langle A_4^f f, f \rangle \\ &< \varepsilon \langle (1 - \Delta) f, f \rangle + 1/2 \| V_4 \|_\infty . \end{aligned}$$

Secondly, it is immediate from (F1) and the definition of F_1 that

$$F_1(f,f,f,f) > - \langle V_1 f, f \rangle.$$

Application of (4.3) now leads to the estimate

$$\begin{aligned}\mathscr{E}\{f\} &= - 1\langle \Delta f, f\rangle + \langle Uf, f\rangle + \langle V_1 f, f\rangle \\ &\quad + F_1(f,f,f,f) + F_2(f,f,f,f) \\ &\geq - 1/2\langle \Delta f, f\rangle + \frac{a}{2}\langle \Delta f, f\rangle - (c-1) \\ &\quad - \varepsilon\langle (1-\Delta)f, f\rangle - 1/2 \| V_4 \|_\infty .\end{aligned}$$

The result follows by taking $\varepsilon > 0$ small enough so that

$$h = \left(1/2 - \frac{a}{2} - \varepsilon\right)^{-1} > 0.$$

LEMMA 15. — If $f(t)$ is a local solution of

$$\frac{df}{dt} = - i(Hf - A_1^f f + A_2^f f)$$

constructed by the method of Theorem 3, then there is a constant α depending only on $\mathscr{E}\{f(0)\}$ such that

$$\| f(t) \|_1 \leq e^{\alpha t} \| f(0) \|_1$$

for all $0 \leq t \leq T$.

Proof. — Let $f_\varepsilon(t)$ be the solution of the regularized equation with $f_\varepsilon(0) = f(0)$. By the proof of Lemma 11

$$\begin{aligned}\frac{d}{dt}&\langle (H + c)f_\varepsilon(t), f_\varepsilon(t)\rangle \\ &= i\langle Hf_\varepsilon, A_{1\varepsilon}^{f_\varepsilon} f_\varepsilon + A_{2\varepsilon}^{f_\varepsilon} f_\varepsilon\rangle - i\langle A_{1\varepsilon}^{f_\varepsilon} f_\varepsilon + A_{2\varepsilon}^{f_\varepsilon} f_\varepsilon, Hf_\varepsilon\rangle \\ &= \frac{i}{2}\sum_r \{ - \langle [\partial_r, A_1^{g_\varepsilon} + A_2^{g_\varepsilon}]g_\varepsilon, \partial_r g_\varepsilon\rangle + \langle \partial_r g_\varepsilon, [\partial_r, A_1^{g_\varepsilon} + A_2^{g_\varepsilon}]g_\varepsilon\rangle \}\end{aligned}$$

where

$$g_\varepsilon = e^{-\varepsilon(H+c)} f_\varepsilon.$$

The estimates (4.4) and (4.6) now imply that

$$\begin{aligned}\frac{d}{dt}\| f_\varepsilon(t) \|_1^2 &\leq c \sum_r \| g_\varepsilon(t) \|_1^2 \| \partial_r g_\varepsilon(t) \|_0 (\| \partial_r g_\varepsilon(t) \|_0 + \| g_\varepsilon(t) \|_0) \\ &\leq c' \| g_\varepsilon(t) \|_1^2 \langle (1-\Delta)g_\varepsilon(t), g_\varepsilon(t)\rangle\end{aligned}$$

so

$$\| f_\varepsilon(t) \|_1^2 - \| f_\varepsilon(s) \|_1^2 \leq c' \int_s^t \| g_\varepsilon(x) \|_1^2 \langle (1-\Delta)g_\varepsilon(x), g_\varepsilon(x)\rangle \, dx$$

for all $0 \leq s \leq t \leq T$. Applying Corollary 13, Lemma 14 and the dominated convergence theorem we obtain

$$\begin{aligned} \| f(t) \|_1^2 - \| f(s) \|_1^2 &\leq c' \int_s^t \| f(x) \|_1^2 \langle (1 - \Delta) f(x), f(x) \rangle \, dx \\ &\leq c' \int_s^t \| f(x) \|_1^2 \{ h\mathscr{E} \{ f(x) \} + k \} \, dx \\ &= e\alpha \int_s^t \| f(x) \|_1^2 dx \end{aligned}$$

where

$$\alpha = 1/2c' \{ h\mathscr{E} \{ f(0) \} + k \}$$

The result now follows by elementary calculus.

THEOREM 16. — For each $a \in \mathscr{H}_1$ with $\| a \| = 1$ there exists a unique $f : [0, \infty) \to \mathscr{H}_1$ which is continuous with respect to the $\mathscr{H}_1$ norm, differentiable with respect to the $\mathscr{H}_{-1}$ norm, and satisfies

$$\frac{df}{dt} = -i(\mathrm{H}f + \mathrm{A}_1^f f + \mathrm{A}_2^f f) \tag{5.3}$$

with initial condition $f(0) = a$.

Proof. — If f is such a solution on [0, S] then an application of Theorem 12 to small enough successive subintervals of [0, S] shows that $\mathscr{E} \{ f(s) \}$ and $\| f(s) \|_0$ are constant on [0, S]. A similar application of Lemma 15 now shows that

$$\| f(t) \|_1 \leq e^{\alpha t} \| f(0) \|_1$$

for all $t \in [0, S]$, where α depends only on $\mathscr{E} \{ a \}$. Since Theorem 3 provides local solutions for an interval of time which depends only on the $\mathscr{H}_1$ norm of the initial value, we see that the maximal solution of (5.3) has domain $[0, \infty)$.

THEOREM 17. — If $f_n(t)$ and $f(t)$ are solutions of (5.3) such that

$$\lim_{n \to \infty} \| f_n(t) - f(t) \|_1 = 0 \tag{5.4}$$

for $t = 0$, then (5.4) holds for all $0 \leq t < \infty$.

Proof. — The proof that

$$\lim_{n \to \infty} \| f_n(t) - f(t) \|_0 = 0$$

follows [10] Lemma 10.1. We then show as in Corollary 13 that

$$\lim_{n \to \infty} \langle (\mathrm{H} + c) f_n(t), f_n(t) \rangle = \langle (\mathrm{H} + c) f(t), f(t) \rangle$$

and deduce (5.4) by combining these results.