ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 31, nº 4 (1979), p. 355-362
<http://www.numdam.org/item?id=AIHPA_1979__31_4_355_0>

*Ann. Inst. Henri Poincaré* Section A :
Vol. XXXI, n° 4, 1979, p. 355-362. *Physique théorique.*

# On Kähler metrics in Lorentzian manifolds

by

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ABSTRACT. — Elementary complex differential geometry is adapted for a Lorentzian $(+ - - -)$ signature of the metric. It is shown that the only Kählerian solutions of the Einstein-Maxwell, and the vacuum with cosmological constant field equations are respectively the Bertotti-Robinson and the Nariai metric.

## INTRODUCTION

In the last few years, considerable interest arised in the applications of complex differential geometry in General Relativity. The twistor theory [*1*] and the Euclidean version of quantum gravity [*2*] are the best examples. The first theory deals from the very beginning with a complex metric for which the signature is meaningless and the second one deals with positive metrics for which all the complex differential geometry can be applied exactly as it was developed. It is well-known that the existence of a metric of hyperbolic signature imposes some restrictions on the topology of the manifold so it is interesting to search for a generalization of the concept of « Hermitian structure » in order to apply the concepts of complex geometry to « physical » (*i. e.* with a metric of signature $+ - - -$) manifolds. The first complete attempt to solve this problem was put forward by Flaherty [*3*]; however, as his primary aim was the study of Heaven spaces and to elucidate the geometrical meaning of the complex transformations from the Schwarzschild to the Kerr metric, he does not study in details the case of real Kählerian metrics solutions of Einstein equations.

In Euclidean quantum gravity, most of the relevant solutions [*4*] (called instantons) are indeed Kählerian real manifolds so we feel interesting to search for real Kählerian solutions of the « physical » Einstein equations in order to compare the properties. We find that in the Lorentzian case the Kähler requirement is so strong that a large class of solutions no longer exists and we find the unique solutions for vacuum, Einstein-Maxwell and vacuum with cosmological constant field equations. In order to meet the requirements of the existence of a Lorentzian metric and of a globall null-frame (which is important in the formalism we have adopted), we shal suppose that our manifold does have a spinorial structure in the sense of Geroch [5].

In section one, the Debever's formalism is briefly reminded together with the relevant definitions. In section two, we give the results, and the last section deals with the connections between Kählerian structures and products structures in Lorentzian manifolds.

## 1. THE DEBEVER FORMALISM AND THE RELEVANT DEFINITIONS

Let (M, $g$) be a 4-dimensional differentiable parallelizable manifold of class $C^{\omega}$ (real analytic) with a metric $g$ of signature $(+ - - -)$. Let $\theta^{(i)}$ $i = 0, 1, 2, 3$ be a global null-frame:

$$g = \theta^{(0)} \otimes \theta^{(3)} + \theta^{(3)} \otimes \theta^{(0)} - \theta^{(1)} \otimes \theta^{(2)} - \theta^{(2)} \otimes \theta^{(1)}$$
$$\overline{\theta^{(1)}} = \theta^{(2)}$$

Let us suppose that (M, $g$) is endowed with the usual Riemannian connection $\nabla$:

$$\nabla_X Y - \nabla_Y X = [X, Y]$$
$$\nabla_X g = O$$

where X, Y, Z are vector fields. We shall use the Debever formalism (see [6], [7], [8] for more details), denoting by $Z^{(\alpha)}$ and $\overline{Z^{(\alpha)}}$ respectively the bases of $C_3$ and $\overline{C_3}$, the two subspaces in which $\Lambda_2$ (the space of complex valued two forms) decomposes under duality:

$$*F_{ij} = \frac{i}{2} \varepsilon_{ijkm} F^{km} \quad , \quad **F = -F \quad , \quad \Lambda_2 = C_3 \oplus \overline{C_3}$$
$$C_3 = \{ F \in \Lambda_2 \,;\, *F = iF \}$$
$$\overline{C_3} = \{ F \in \Lambda_2 \,;\, *F = -iF \}$$

Here and in the following, $i, j, k, m = 0, 1, 2, 3$ will denote the null tetrad components of a geometric object and $\varepsilon$ is the completely antisymmetric tensor.

In $\Lambda_2$ is defined a « scalar product »·

$$(F, G) = \frac{1}{4} F_{ij}G^{ij} = \gamma^{\alpha\beta}(F_\alpha G_\beta + F_{\bar{\alpha}}G_{\bar{\beta}})$$

where

$$\gamma^{\alpha\beta} = \gamma_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\forall F \in \Lambda_2 \quad , \quad F = F_\alpha Z^{(\alpha)} + F_{\bar{\alpha}}\overline{Z^{(\alpha)}}$$

An explicit expression for the base of $C_3$ is:

$$\begin{aligned} Z^{(1)} &= 2\sqrt{2}\theta^{(0)} \wedge \theta^{(1)} \\ Z^{(2)} &= 2\sqrt{2}\theta^{(2)} \wedge \theta^{(3)} \\ Z^{(3)} &= 2(\theta^{(1)} \wedge \theta^{(2)} - \theta^{(0)} \wedge \theta^{(3)} \end{aligned}$$

In the Debever formalism the spinor coefficients of Newman-Penrose (see [3]) naturally arise as the components of the one-forms $\sigma_{(\alpha)}$ which are the « components » of the bivector valued connection form:

$$\omega(X) = (\nabla_X \theta_{(i)}) \otimes \theta^{(i)} = \frac{1}{2}(\sigma_{(\alpha)}(X)Z^{(\alpha)} + \overline{\sigma_{(\alpha)}(X)Z^{(\alpha)}}$$

DEFINITION 1. — An almost-Hermitian structure J on $(M, g)$ is a differentiable tensor field of rank (1, 1) such that:

$$\begin{aligned} J(J(X)) &= -X \\ g(J(X), J(Y)) &= g(X, Y) \end{aligned}$$

It is easy to see that the covariant tensor field

$$K(X, Y) = g(X, J(Y))$$

is antisymmetric and that the definition of J is equivalent to (see *e. g.* [8])

$$(K, K) = 1 \qquad \text{and} \qquad {}^*K = \pm iK$$

DEFINITION 2. — An almost-product—$(2 \times 2)$—structure P on $(M, g)$ is a differentiable tensor field of rank (1, 1) such that

$$\begin{aligned} P(P(X)) &= X \\ P_i{}^i &= 0 \\ g(P(X), P(Y)) &= g(X, Y) \end{aligned}$$

One can show that given J one has P:

$$P(X) = \pm J(\bar{J}(X))$$

where J is extended linearly to the complexified tangent space.

The reason for the use of the Debever's formalism is that (see [8]) there exists a base in which

$$K = iZ^{(3)} \qquad \text{and} \qquad P = \{ Z^{(3)}, \overline{Z^{(3)}} \}$$

where { , } is the Rainich product:

$$\{ F, G \} = -\frac{1}{2}(F_{ip}G^{p}{}_{j} + {}^{*}F_{ip}{}^{*}G_{j}{}^{p})\theta^{(i)} \otimes \theta^{(j)}.$$

DEFINITION 3. — An almost-Hermitian structure J is said almost-Kählerian if $dK = 0$.

The most interesting case is when the almost-Hermitian structure is integrable, in the sense that there exists a maximal atlas of local coordinates in which the real metric $g$ can be written

$$ds^2 = g_{\alpha\tilde{\beta}}dz^{\alpha}dz^{\tilde{\beta}} \qquad \begin{array}{l} \alpha = 0, 1 \\ \tilde{\beta} = \tilde{0}, \tilde{1} \end{array}$$

and the transformations between overlapping charts is given by the so-called « formally analytic » transformations:

$$\begin{aligned} z^{\alpha} &\to z^{\alpha'}(z^{\alpha}) \\ z^{\tilde{\beta}} &\to z^{\tilde{\beta}'}(z^{\tilde{\beta}}) \end{aligned}$$

and, for the metric to be real, one must have some conditions, *e. g.* for the Kähler case,

$$z^{0} = \overline{z^{0}} \quad , \quad z^{\tilde{0}} = \overline{z^{\tilde{0}}} \quad , \quad z^{\tilde{1}} = \overline{z^{1}}.$$

All these considerations arise from the fact that (see [3], [8]) a Lorentzian manifold does *not* admit a *real valued* almost Hermitian structure. One of the most interesting consequences of this fact is that the coordinates arising from our integrable almost Hermitian structure are *null* coordinates. When dealing with the problem of the integrability of an almost-structure it is well-known that the Nijenhuis tensor N defined as [9]

$$N(X, Y) = [J(X), J(Y)] - [X, Y] - J[X, J(Y)] - J[J(X), Y]$$

plays a central role.

The fundamental result is due to Frölicher [10] and adapted by Flaherty to Lorentzian manifolds.

PROPOSITION 1. — A $C^{\omega}$ almost-Hermitian structure J is integrable if and only if $N = 0$.

We point out that in the Lorentzian case it seems to be impossible to relaxe the $C^{\omega}$ requirement. By computing the tetrad components of N and of $dK$ in the tetrad in which $K = iZ^{(3)}$ we have:

PROPOSITION 2. — A $C^\omega$ almost-Kählerian structure J is integrable if and only if $\sigma_{(1)} = \sigma_{(2)} = 0$.

By proposition 2 and the Goldberg-Sachxs theorem (see [*3*]), we have:

COROLLARY. — A Kählerian manifold is of Petrov type D or conformally flat.

## 2. KAHLERIAN SOLUTIONS OF EINSTEIN EQUATIONS

It is a trivial matter to show from the second Cartan structural equations and Bianchi identities:

$$\Sigma_{(\alpha)} = d\sigma_{(\alpha)} - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\sigma^{(\beta)} \wedge \sigma^{(\gamma)}$$

$$d\Sigma_{(\alpha)} = \varepsilon_{\alpha\beta\gamma}\sigma^{(\beta)} \wedge \Sigma^{(\gamma)}$$

where $\Sigma_{(\alpha)}$ are the « components » of the curvature form:

$$\Omega(X, Y) = \frac{1}{2}(\Sigma_{(\alpha)}(X, Y)Z^{(\alpha)} + \overline{\Sigma_{(\alpha)}}(X, Y)\overline{Z^{(\alpha)}})$$

$$\Sigma_{(\alpha)} = \left(\frac{1}{4}C_{\alpha\beta} - \frac{1}{48}R\gamma_{\alpha\beta}\right)Z^{(\beta)} + \tau_{\alpha\bar{\beta}}\overline{Z^{(\beta)}}$$

with:

$$C_{\alpha\beta}Z^{(\alpha)}_{ij}Z^{(\alpha\beta)}_{km} = C_{ijkm} - i\overset{*}{C}_{ijkm}$$

$$\tau_{\alpha\bar{\beta}}Z^{(\alpha)}_{ip}\overline{Z^{(\beta)}}{}^{p}{}_{j} = R_{ij} - \frac{1}{4}Rg_{ij}$$

that the only vacuum Kählerian solution of Einstein equations is Minkowski space-time.

So it is interesting to look at the Einstein-Maxwell field equations. The result is:

PROPOSITION 3. — The most general real Kählerian solution of the source-free Einstein-Maxwell equations is the Bertotti-Robinson [*11*, *12*] universe.

This theorem is a consequence of the following lemmas:

LEMMA 1.1. — An Einstein-Maxwell source-free manifold with $\sigma_{(1)} = \sigma_{(2)} = 0$ is conformally flat.

In fact from the second Cartan equations written in the Debever-formalism we have that:

$$\Sigma_{(1)} = \Sigma_{(2)} = 0$$

Moreover in an Einstein-Maxwell manifold:

$$R = 0$$

By using $(Z^{(\alpha)}, \overline{Z^{(\beta)}}) = 0 \ \forall \alpha, \beta$ we get $C_{\alpha\beta} = 0$ because $C_{\alpha\beta} = \frac{1}{4}(\Sigma_{(\alpha)}, Z_{(\beta)})$ and $C_{33} = 2C_{12}$.

LEMMA 1.2. — An Einstein-Maxwell manifold with $\sigma_{(1)} = \sigma_{(2)} = 0$ can not have a null electromagnetic self-dual two-form.

In fact from the definition of $\Sigma_{(\alpha)}$ given in the previous lemma we get:

$$\tau_{\alpha\bar{\beta}} = (\Sigma_{(\alpha)}, \overline{Z_{(\beta)}})$$

that is:

$$\begin{aligned} &\tau_{1\bar{1}} = \tau_{1\bar{2}} = \tau_{1\bar{3}} = \tau_{2\bar{1}} = \tau_{2\bar{2}} = \tau_{2\bar{3}} = 0 \\ &\tau_{3\bar{1}} = \overline{\tau_{1\bar{3}}} = 0 \\ &\tau_{3\bar{2}} = \overline{\tau_{2\bar{3}}} = 0 \end{aligned}$$

Moreover in an Einstein-Maxwell manifold with a null electromagnetic self-dual two-form one has:

$$R_{ab}R^{ab} = 0$$

that is $\tau_{\alpha\bar{\beta}}\tau^{\alpha\bar{\beta}} = 0$, thus $\tau_{3\bar{3}} = 0$ and, in this case, the conformally flat manifold of the previous lemma is indeed Minkowski space-time.

LEMMA 1.3. — The most general conformally flat solution of the source-free Einstein-Maxwell equations for a non null electromagnetic field is the Bertotti-Robinson universe (in this metric $\sigma_{(1)} = \sigma_{(2)} = 0$).

See N. Tariq and R. McLenaghan [*13*].

In the null coordinates arising from the integrability of the almost-Kählerian structure (indeed the coordinates employed by Tariq and McLenaghan, as one can easily show) the Bertotti-Robinson metric reads;

$$ds^2 = 2(1 + az^0z^{\tilde{0}})^{-2}dz^0dz^{\tilde{0}} - 2(1 + az^1z^{\tilde{1}})^{-2}dz^1dz^{\tilde{1}}$$

with the electromagnetic self-dual two-form F given by:

$$F = \sqrt{a}[2(1 + az^0z^{\tilde{0}})^{-2}dz^0 \wedge dz^0 - 2(1 + az^1z^{\tilde{1}})^{-2}dz^1 \wedge dz^{\tilde{1}}]$$

where $a \in R^+$, and, obviously, the Kählerian structure tensor is, in its covariant form:

$$K = 2i[(1 + az^0z^{\tilde{0}})^{-2}dz^0 \wedge dz^{\tilde{0}} - (1 + az^1z^{\tilde{1}})^{-2}dz^1 \wedge dz^{\tilde{1}}]$$

We note, moreover, that the covariantly constant electromagnetic self-dual two-form is proportional to the Kähler form. We also point out the general fact that a covariantly constant non null electromagnetic self-dual two-form F induces (up to a trivial sign indetermination) an integrable Kählerian almost structure:

$$K = \frac{1}{\sqrt{(F, F)}} F$$

In fact in a canonical tetrad for F, we have $F = AZ^{(3)}$, and $\nabla F = 0$ if and only if $\sigma_{(1)} = \sigma_{(2)} = 0$.

This last remark shows, en passant, that the Bertotti-Robinson metric is the most general solution of the Einstein-Maxwell equations for a non null covariantly constant electromagnetic field. This theorem was first proved by Eardley [*14*].

By relaxing the Kählerian requirement $dK = 0$, a straightforward calculation gives the result that J is integrable if and only if the manifold admits two geodesic and shear-free null congruences, that is if and only if:

$$\sigma_{(1)} \wedge Z^{(2)} = 0 \qquad \sigma_{(2)} \wedge Z^{(1)} = 0.$$

We shall now consider the Einstein's equations with cosmological constant.

From the general form of the metric with $\sigma_{(1)} = \sigma_{(2)} = 0$ given by McLenaghan and Tariq, it is easy to show that in this case the general solution is given by:

$$ds^2 = 2\left(1 - \frac{\lambda}{2} z^0 z^{\tilde{0}}\right)^{-2} dz^0 dz^{\tilde{0}} - 2\left(1 + \frac{\lambda}{2} z^1 z^{\tilde{1}}\right)^{-2} dz^1 dz^{\tilde{1}} \qquad (1)$$

where $\lambda$ is the cosmological constant.

PROPOSITION 4. — The most general real Kählerian solution of the Einstein's equation with cosmological constant is the metric 1). This metric was first described by Nariai [*15*].

## 3. PRODUCT (2 × 2) STRUCTURES

By computing the tetrad components of the Nijenhuis tensor of P, it is easy to see that in the tetrad in which $P = \{ Z^{(3)}, \overline{Z^{(3)}} \}$ we have $N = 0$ if and only if $\psi = \overline{\psi}$ where $\psi$ is the one form defined by Debever [*16*]:

$$dZ^{(3)} = \psi \wedge Z^{(3)}.$$

So we have the following proposition which is due to the signature of the metric:

PROPOSITION 5. — An almost-Kählerian manifold has a (2 × 2) *integrable* product structure.

In fact in the tetrad in which the Kählerian almost-structure is written as $K = iZ^{(3)}$, $dK = 0$ implies $\psi = 0$ so the almost-product $(2 \times 2)$ structure $P = \{ K, \overline{K} \}$ is integrable.

The condition $\psi = 0$ is stronger than the requirement of the integrability $(\psi = \overline{\psi})$ and in this case the tensor P is also of vanishing divergence; P is then said a conservative structure [*16*] [*8*].

When P is integrable there exist local charts of real coordinates in which the metric tensor reads:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + g_{\tilde{\alpha}\tilde{\beta}} dx^{\tilde{\alpha}} dx^{\tilde{\beta}}$$

Moreover (see Yano [*9*]), if $\nabla P = 0$, P is said $(2 \times 2)$-locally decomposable product structure, and we have:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$$
$$g_{\tilde{\alpha}\tilde{\beta}} = g_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{\gamma}})$$

PROPOSITION 6. — P is a $(2 \times 2)$-locally-decomposable-product structure if and only if $\sigma_{(1)} = \sigma_{(2)} = 0$.

We point out that, by proposition 2, every Kählerian manifold is also a $(2 \times 2)$-locally-decomposable-product manifold and *vice versa*.

Moreover we have also that the most general $(2 \times 2)$-locally-decomposable solution of the Einstein-Maxwell source-free equations is the Bertotti-Robinson universe.

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*(Manuscrit reçu le 12 juin 1979).*