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Section A
Physique théorique.

On the wave equation in curved spacetime

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1. INTRODUCTION

It is well-known [*1*, *2*] that the linear wave operator $\Box_g$ has one advanced and one retarded « Green's function », that is, two elementary kernels, E^+ and E^-, distributions, globally defined on every globally hyperbolic manifold (V_{l+1}, g). The equation

$$\Box_g \phi = \rho \tag{1.1}$$

with ρ a C^∞ function of support compact toward the past [resp. the future] has one solution $\phi^+(x) = \langle E^-(x, x'), \rho(x') \rangle$ with support compact toward the past [resp. one solution $\phi^-(x) = \langle E^+(x, x'), \rho(x') \rangle$ with support compact toward the future].

It is important for many problems, in classical propagation and quantum theories, to know for which noncompact sources the solutions will be defined. Moreover, even for compact sources, the energy of the solutions, in a curved spacetime, will not in general stay bounded, and no scattering property can easily be obtained.

We study essentially in this paper the case where g tends at timelike infinity to a stationary metric, we prove some existence theorems in spaces of solutions with finite energy for all times, and also with finite s-order energy.

2. DEFINITIONS AND HYPOTHESES

Let V_{l+1} be a $l+1$ dimensional C^∞ manifold endowed with a globally hyperbolic [1, 2] C^∞ metric g, of signature $(-, + \dots +)$. The manifold V_{l+1} is then diffeomorphic to a product $\mathbb{R} \times S$ [3]. There exists a C^∞ time function τ on V_{l+1} such that

$$g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau < 0$$

everywhere. This function generates a foliation $\{ S_t = \tau^{-1}(t) \mid t \in \mathbb{R} \}$ and each hypersurface $S_t \cong S$ is a spacelike future Cauchy surface, having a compact intersection with the future of any compact set. We shall assume that the lapse function N, given by

$$N = (-g^{\mu\nu}\partial_\mu\tau\partial_\gamma\tau)^{1/2} \tag{2.1}$$

is uniformly bounded: $N \leq A$ for some positive real number A. If also $\inf N > 0$, the spacetime has infinite proper time extension.

We denote by X a timelike vectorfield on V_{l+1} of class C^∞ and set $g(X, X) = -\beta$. We suppose that X is uniformly timelike: there exists a positive real number b such that

$$0 < b \leq \beta \quad \text{on} \quad V_{l+1}.$$

Let n be the unit normal vectorfield to the foliation $\{ S_t \}$

$$n^\mu = -Ng^{\mu\nu}\partial_\nu\tau \qquad (g(n, n) = -1) \tag{2.2}$$

and let us set $g(X, n) = -\alpha$. We assume finally that the hypersurfaces S_t are uniformly spacelike for X, that is, α is uniformly bounded: there exists a $B > 0$ such that

$$\alpha^2 \leq B$$

Since it holds $|g(X, n)|^2 \geq g(X, X)$, we have:

$$B \geq \alpha^2 \geq \beta \geq b.$$

We define on V_{l+1} the positive definite metrics γ and Γ by:

$$\gamma_{\mu\nu} = \frac{2}{\alpha}\left(g_{\mu\nu} + n_\mu n_\nu + \frac{1}{\beta}X_\mu X_\nu\right) \tag{2.3}$$

$$\Gamma_{\mu\nu} = g_{\mu\nu} + k(k\beta - 1)^{-1}X_\mu X_\nu, \tag{2.4}$$

where k is a positive real number greater than $1/b$. The corresponding inverses are given by

$$(\gamma^{-1})^{\mu\nu} = (1/2)(X^\mu n^\nu + X^\nu n^\mu + \alpha g^{\mu\nu}) \tag{2.5}$$

$$(\Gamma^{-1})^{\mu\nu} = g^{\mu\nu} + kX^\mu X^\nu. \tag{2.6}$$

We denote by $|\,T\,|$ the norm relative to γ of a tensor T at a point of V_{l+1}. We note that:

$$|\,g\,|^2 = \frac{1}{2}\beta + \frac{(l-1)}{4}\alpha^2 \leq \frac{(l+1)}{4}B \tag{2.7}$$

$$|\,g^{-1}\,|^2 = \frac{8}{\beta} + \frac{4(l-1)}{\alpha^2} \leq 4(l+1)(1/b) \tag{2.8}$$

$$|\,X\,|^2 = 2\alpha \leq 2B^{1/2}. \tag{2.9}$$

LEMMA 1. — If u and v are two tensors of some type and $|\quad|$ is the norm at a point relative to a given positive definite metric, one has:

$$|\,u \otimes v\,| = |\,u\,|\,|\,v\,| \quad , \quad |\,u.v\,| \leq |\,u\,|\,|\,v\,|,$$

where $u.v$ denotes a tensor product contracted in some way. The proof is immediate in an orthonormal frame.

LEMMA 2. — The metric Γ and γ define on V_{l+1} uniformly equivalent norms.

Proof. — If ξ is a vectorfield and $|\quad|_\Gamma$ the norm relative to Γ, lemma 1 implies

$$|\,\xi\,|_\Gamma^2 = \Gamma_{\mu\nu}\xi^\mu\xi^\nu \leq |\,\Gamma\,|\,|\,\xi\,|^2$$

and

$$|\,\xi\,|^2 = \gamma_{\mu\nu}\xi^\mu\xi^\nu \leq |\,\gamma\,|_\Gamma\,|\,\xi\,|_\Gamma^2$$

and hence:

$$|\,\Gamma\,|^{-1}\,|\,\xi\,|_\Gamma^2 \leq |\,\xi\,|^2 \leq |\,\gamma\,|_\Gamma\,|\,\xi\,|_\Gamma^2$$

Further, we find:

$$\begin{aligned} |\,\Gamma\,|^2 &= \frac{\alpha^2}{4}\left[l - 1 + \frac{k^2\beta^2}{(k\beta-1)^2}\right] - \frac{1}{2}\frac{\beta}{(k\beta-1)} \\ &\leq \frac{B}{4}\left[l - 1 + \frac{k^2b^2}{(kb-1)^2}\right] - \frac{1}{2}\frac{b}{(kb-1)} \end{aligned}$$

and

$$\begin{aligned} |\,\gamma\,|_\Gamma^2 &= \frac{8}{\beta} + \frac{4(l-1)}{\alpha^2} - 8k + 4k^2\alpha^2 \\ &\leq \frac{[8 + 4(l-1)]}{b} - 8k + 4k^2B \end{aligned}$$

and the uniform equivalence is established.

3. FUNCTION SPACES

Let V_{l+1}^T denote the open submanifold of V_{l+1} defined by:

$$V_{l+1}^T = \tau^{-1}(-\infty, T), \; T \leq +\infty$$

(the case $T = +\infty$ denotes V_{l+1} itself).

DEFINITION 1. — The space $\mathscr{L}_s$ is the space of functions f on V_{l+1}^T such that:

a) f together with its generalized covariant derivatives $\nabla^p f$ of order $p \leq s$, are measurable tensor fields on V_{l+1}^T which have a restriction on S_t for almost all $t \in (-\infty, T)$, whose γ-norm is square integrable (with respect to the canonical measure of the metric $\bar{g}_t$ induced by g on S_t). We pose:

$$\| f \|_s^{S_t} = \left\{ \int_{S_t} \sum_{p=0}^{s} | \nabla^p f |^2 d\mu(\bar{g}_t) \right\}^{1/2}$$

b) The mapping $(-\infty, T) \to$ by $t \mapsto \| f \|_s^{S_t}$ is integrable on $(-\infty, T)$. We set:

$$\| f \|_{\mathscr{L}_s} = \int_{-\infty}^{T} \| f \|_s^{S_t} dt.$$

DEFINITION 2. — $\mathscr{P}_s$ is the subspace of $\mathscr{L}_s$ consisting of those functions for which for each $p \leq s$, $t^{s-p+1} \| f \|_p^{S_t} \in L^1 (-\infty, T)$. We set

$$\| f \|_{\mathscr{P}_s} = \int_{-\infty}^{T} \sum_{p=0}^{s} (1 + | t |^{s-p+1}) \| f \|_p^{S_t} dt.$$

DEFINITION 3. — E_s is the space of functions f on V_{l+1}^T which satisfy part *a*) of Def. 1 and, in addition, the map $(-\infty, T) \to \mathbb{R}$ by $t \mapsto \| f \|_s^{S_t}$ is continuous and bounded. We set:

$$\| f \|_{E_s} = \sup_{t \in (-\infty, T)} \{ \| f \|_s^{S_t} \}$$

DEFINITION 4. — E_s^* is the space of equivalence classes of distributions on V_{l+1}^T, $f \sim f +$ constant, such that $\nabla f \in E_{s-1}$. We set:

$$\| f \|_{E_s^*} = \| \nabla f \|_{E_{s-1}}$$

DEFINITION 5. — $\mathscr{L}_s^*$ is the space of equivalence classes of distributions on V_{l+1}^T, $f \sim f +$ constant, such that $\nabla f \in \mathscr{L}_{s-1}$. We set:

$$\| f \|_{\mathscr{L}_s} = \| \nabla f \|_{\mathscr{L}_{s-1}}.$$

LEMMA 3. — $\mathscr{L}_s$, $\mathscr{P}_s$, E_s, E_s^*, $\mathscr{L}_s^*$ are Banach spaces under the corresponding norms.

Proof. — For $\mathscr{L}_s$, $\mathscr{P}_s$, E_s the completeness results, as for classical Sobolev spaces, from the completeness of L^p and the continuity of the derivation in the sense of distributions. To prove the completeness of E_s^*, $\mathscr{L}_s^*$ one uses moreover the fact that the cohomology classes (and in particular the trivial class of exact forms) are continuous in $\mathscr{D}'$.

DEFINITION 6. — We denote by $\overset{\circ}{E}_s$, $\overset{\circ}{\mathscr{L}}_s$, $\overset{\circ}{\mathscr{P}}_s$ the closure in the norms of E_s, $\mathscr{L}_s$, $\mathscr{P}_s$ of the space C_0^∞ of C^∞ functions with support compact in space and towards the past, *i. e.* supp $f \cap S_t$ compact for each S_t and supp $f \subset \{ S_t \mid t \in [t_0, \infty) \}$. We shall say that (V_{l+1}, g, τ, X) is s-regular if $\overset{\circ}{E}_s = E_s$, $\overset{\circ}{\mathscr{L}}_s = \mathscr{L}_s$, $\overset{\circ}{\mathscr{P}}_s = \mathscr{P}_s$.

LEMMA 4. — Under the assumptions of § 2, (V_{l+1}, g, τ, X) is s-regular if

a) for each $t \in \mathbb{R}$, the metric $\overline{g_t}$ is a complete Riemannian metric with a non-zero injectivity radius and, if $s \geq 2$, has a Riemannian curvature bounded in the C^{s-2} norm;

b) the vector field X as well as the functions α and N are bounded in the C^s norm.

4. FUNDAMENTAL ENERGY INEQUALITY

We shall in the following suppose that the metric g tends, in a weak sense, to be stationary at timelike infinity. Namely:

Hypothesis 1. — The function $I \to \mathbb{R}$ defined by $t \to \sup_{S_t} | L_X g |$ is integrable. We denote:

$$\pi = L_X g, \qquad p(t) = \sup_{S_t} | \pi |, \qquad P = \int_{-\infty}^{\infty} p(t)dt \tag{4.1}$$

The energy-momentum tensor of a scalar field ϕ is:

$$T = g^{-1}(\nabla\phi) \otimes g^{-1}(\nabla\phi) - \frac{1}{2} g^{-1}(\nabla\phi, \nabla\phi) g^{-1}$$

that is, in components

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2} g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi \tag{4.2}$$

Its covariant divergence is equal to

$$\nabla T^{\mu\nu} = \partial^\mu\phi\Box_g\phi \tag{4.3}$$

The energy-momentum vector relative to a vectorfield X is given by:

$$P^\mu = T^\mu{}_\nu X^\nu \tag{4.4}$$

The energy density with respect to a foliation $\{ S_t \}$ with unit normal vectorfield n is given by:

$$\varepsilon = P_\mu n^\mu = (\gamma^{-1})^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = | \nabla\phi |^2 \tag{4.5}$$

and the total energy of ϕ on the hypersurface S_t is

$$E(t) = \int_{S_t} \epsilon \, d\mu(\bar{g}_t) = \int_{S_t} | \nabla\phi |^2 d\mu(\bar{g}_t) \tag{4.6}$$

The above expressions provide the motivation for introducing the metric γ in § 2. The covariant divergence of the energy-momentum vector is found to be:

$$\begin{aligned} \nabla_\mu P^\mu &= X^\nu \partial_\nu \phi \, \square_g \phi + T^{\mu\nu} \nabla_\mu X_\nu \\ &= \nabla_X \phi \, \square_g \phi + \frac{1}{2} T \cdot \pi \end{aligned} \tag{4.7}$$

If ϕ is a C^2 function of compact spacelike support (namely supp $\phi \cap S_t$ compact for each $t \in \mathbb{R}$), Gauss' theorem applies:

$$\int_{t_0}^{t} \int_{S_\tau} \nabla_\mu P^\mu d\mu(g) = \int_{S_t} P^\mu n_\mu d\mu(\bar{g}) - \int_{S_{t_0}} P^\mu n_\mu d\mu(\bar{g}_{t_0})$$

Thus, in view of (4.7), if ϕ satisfies (1.1) we obtain:

$$E(t) = E(t_0) + \int_{t_0}^{t} \int_{S_\tau} \left(\rho \nabla_X \phi + \frac{1}{2} T \cdot \pi \right) N d\mu(\bar{g}_\tau) d\tau, \tag{4.8}$$

where we have used the fact that

$$d\mu(g) = N d\mu(\bar{g}_\tau) d\tau \tag{4.9}$$

At each point of V_{l+1}, the following inequalities follow from lemma 1:

$$\begin{aligned} | \nabla_X \phi | &\leq | X | \, | \nabla\phi | = 2^{1/2} B^{1/4} | \nabla\phi | \\ | T | &\leq \frac{3}{2} | g^{-1} |^2 | \nabla\phi |^2 = 6(l+1)(1/b) | \nabla\phi |^2 \\ | T \cdot \pi | &\leq | T | \, | \pi | \end{aligned} \tag{4.10}$$

Hence with $p(\tau) = \sup_{S_\tau} | \pi |$, we obtain:

$$\int_{t_0}^{t} \int_{S_t} \frac{1}{2} T \cdot \pi N d\mu(\bar{g}_\tau) d\tau \leq C_1 \int_{t_0}^{t} p(\tau) E(\tau) d\tau, \tag{4.11}$$

where

$$C_1 = 3(l+1)(A/b) \tag{4.12}$$

The Schwarz inequality gives:

$$\int_{S_\tau} | \rho | \, | \nabla\phi | \, d\mu(\bar{g}_\tau) \leq f(\tau)(E(\tau))^{1/2},$$

where we have defined:

$$f(\tau) = | \rho |_{L^2}^{S_\tau} \tag{4.13}$$

Hence we obtain

$$\int_{t_0}^{t} \int_{S_\tau} \rho \nabla_X \phi N d\mu(\bar{g}\tau) d\tau \leq C_2 \int_{t_0}^{t} f(\tau)(E(\tau))^{1/2} d\tau, \tag{4.14}$$

where:

$$C_2 = 2^{1/2}B^{1/4}A \tag{4.15}$$

Finally, comparing (4.8) with (4.11) and (4.14) we arrive at the inequality:

$$E(t) \leq E(t_0) + C_1 \int_{t_0}^{t} p(\tau)E(\tau)d\tau + C_2 \int_{t_0}^{t} f(\tau)(E(\tau))^{1/2}d\tau \tag{4.16}$$

LEMMA 5. — If $x \geq 0$ is a continuous function of bounded support satisfying the integral inequality:

$$(x(t))^2 \leq \int_{-\infty}^{t} y(\tau)(x(\tau))^2 d\tau + \int_{-\infty}^{t} z(\tau)x(\tau)d\tau$$

with $y, z \geq 0$ and $y, z \in L^1(-\infty, T)$ then for each $t \in (-\infty, T)$:

$$x(t) \leq \frac{1}{2} M(t) \int_{-\infty}^{t} z(\tau)d\tau,$$

where

$$M(t) = \exp\left\{\frac{1}{2}\int_{-\infty}^{t} y(\tau)dt\right\} \leq M = \exp\frac{1}{2} \mid y \mid_{L^1(-\infty,T)}.$$

If in addition $tz \in L^1(-\infty, T)$ one has:

$$\int_{-\infty}^{t} x(\tau)d\tau \leq M \int_{-\infty}^{t} (t - \tau)z(\tau)d\tau$$

The proof is by solving the corresponding integral equality.

Applying lemma 5 to inequality (4.16) with $x = (E)^{1/2}$, $y = C_1p$, $z = C_2f$ we obtain the following lemma:

LEMMA 6. — Let (V_{l+1}, g) satisfy the assumptions of § 2 as well as hypothesis 1 and let $\square_g\phi = \rho$, $\phi \in C^2$ with support compact in space and towards the past. Then the energy of ϕ is bounded by:

$$(E(t))^{1/2} \leq \frac{1}{2} C_2 \exp\left(\frac{1}{2}C_1P\right) \int_{-\infty}^{t} \mid \rho \mid_{L^2}^{S} d\tau \leq \frac{1}{2} C_2 \left(\exp\frac{1}{2}C_1P\right) \parallel \rho \parallel_{\mathscr{L}_0}$$

and one has:

$$\int_{-\infty}^{t} (E(\tau))^{1/2}d\tau \leq \frac{1}{2}C_2 \exp\left(\frac{1}{2}C_1P\right)\int_{-\infty}^{t} (t-\tau) \mid \rho \mid_{L^2}^{S} d\tau$$
$$\leq \frac{1}{2}C_2 \exp\left(\frac{1}{2}C_1, P\right)(1 + \mid t \mid) \mid \rho \mid_{\mathscr{P}_0}.$$

We shall use lemma 6 to prove the existence of a weak solution of the equation $\square_g\phi = \rho$, under the weak hypothesis that $\rho \in \mathscr{L}_0$.

THEOREM 1 (weak existence theorem). — Under the assumptions concerning (V_{l+1}, g) of § 2, as well as hypothesis 1, the equation $\square_g \phi = \rho$, with $\rho \in \mathscr{L}_0$ has a solution $\phi \in E_1^*$ such that

$$\| \phi \|_{E_1^*} \leq \frac{1}{2} C_2 \exp\left(\frac{1}{2} C_1 P\right) \| \rho \|_{\mathscr{L}_0}$$

If in addition $\rho \in \mathscr{P}_0$, then $\phi \in \mathscr{L}_1^*(V_{l+1}^T)$ for any finite T, and

$$\| \phi \|_{\mathscr{L}_1^*} \leq \frac{1}{2} C_2 \exp\left(\frac{1}{2} C_1 P\right)(1 + | T |) \| \rho \|_{\mathscr{P}_0}$$

Proof. — If $\rho \in \mathscr{L}_0$ there is a Cauchy sequence $\{ \rho_n \} \in C_0^\infty$ which tends to ρ in the $\mathscr{L}_0$ norm. For each ρ_n the equation $\square_g \phi = \rho$ has one C^∞ solution ϕ_n, whose support is contained in the future $\mathscr{E}(K_n)$ of the support K_n of ρ_n. The intersection of each Cauchy surface S_t with $\mathscr{E}(K_n)$ is a compact set, an empty set if $t \leq t_n$, $K_n \subset \{ S_t \mid t > t_n \}$. We apply lemma 6 to the equation

$$\square_g(\phi_n - \phi_m) = \rho_n - \rho_m$$

and we obtain

$$\| \phi_n - \phi_m \|_{E_1^*} \leq \frac{1}{2} C_2 \exp\left(\frac{1}{2} C_1 P\right) \| \rho_n - \rho_m \|_{\mathscr{L}_0}$$

Thus $\{ \phi_n \}$ is a Cauchy sequence in E_1^* and therefore tends to a limit $\phi \in E_1^*$. Since $\nabla \phi_n$ tends then to $\nabla \phi$ in the space of distribution valued 1-forms, we have that $\square_g \phi = \rho$ in the sense of distributions. The rest of the theorem follows in the same way from lemma 6.

5. HIGHER ORDER ENERGY INEQUALITIES

Given a function ϕ and a vectorfield X we define the scalars

$$\underset{h}{\psi} = \nabla_X^h \phi, \qquad h = 0, 1, \ldots \tag{5.1}$$

and the i-covariant tensorfields

$$\underset{r,i}{\zeta} = \nabla^i \underset{r-i}{\psi} \quad , \quad i \leq r, \qquad \underset{r}{t} = \underset{r,r}{\zeta} \tag{5.2}$$

To the tensorfield $\underset{r,i}{\zeta}$ we assign the 2-contravariant tensorfield

$$\underset{r,i}{T^{\mu\nu}} = \nabla^\mu \underset{r,i}{\zeta} \cdot \nabla^\nu \underset{r,i}{\zeta} - \frac{1}{2} g^{\mu\nu} (\nabla_\rho \underset{r,i}{\zeta} \cdot \nabla^\rho \underset{r,i}{\zeta}), \tag{5.3}$$

the vectorfield

$$\underset{r,i}{P^\mu} = \underset{r,i}{T^\mu{}_\nu} X^\nu \tag{5.4}$$

and the scalar

$$\underset{r,i}{\theta} = \underset{r,i}{\mathrm{P}}^{\mu} n_{\mu} = (\gamma^{-1})^{\mu\nu} \nabla_{\mu} \underset{r,i}{\zeta} \cdot \nabla_{\nu} \underset{r,i}{\zeta} \tag{5.5}$$

In (5.3) and (5.5) the dot denotes the scalar product in the hyperbolic metric g for the unwritten indices. The quantities $\underset{r,i}{\theta}$ are not positive definite, except when $i = 0$. For this reason we introduce in addition the positive definite quantities

$$\underset{r,i}{\varepsilon} = (\Gamma^{-1})^{\alpha_i \beta_i} \dots (\Gamma^{-1})^{\alpha_1 \beta_1} (\gamma^{-1})^{\mu\nu} \nabla_{\mu} \underset{r,i}{\zeta}_{\alpha_i \dots \alpha_1} \nabla_{\nu} \underset{r,i}{\zeta}_{\beta_i \dots, \beta_1} \tag{5.6}$$

We also introduce the corresponding integrals over the hypersurfaces S_t

$$\underset{r,i}{\Theta}(t) = \int_{\mathrm{S}_t} \underset{r,i}{\theta} \, d\mu(\bar{g}_t) \tag{5.7}$$

and

$$\underset{r,i}{\mathrm{E}}(t) = \int_{\mathrm{S}_t} \underset{r,i}{\varepsilon} \, d\mu(\bar{g}_t) \tag{5.8}$$

and the total r-th order energy of ϕ on S_t

$$\underset{r}{\mathrm{E}}(t) = \sum_{k=0}^{r} \underset{k,k}{\mathrm{E}}, \tag{5.9}$$

the 0-th order energy being the physical energy of § 4. As a consequence of lemma 2, there are constants c' and c'' depending only on k, b and B such that

$$c' \underset{r,i}{\varepsilon} \leq | \nabla \underset{r,i}{\zeta} |^2 \leq c'' \underset{r,i}{\varepsilon} \qquad \text{and} \qquad c' \underset{r}{\mathrm{E}}(t) \leq \| \nabla \phi \|_r^{\mathrm{S}_t} \leq c'' \underset{r}{\mathrm{E}}(t) \tag{5.10}$$

The quantities $\underset{r,i}{\zeta}$ and $\underset{h}{\psi}$ satisfy the recursion relations:

$$\underset{r+1,i+1}{\zeta} = \nabla \underset{r,i}{\zeta} \quad , \quad \underset{h+1}{\psi} = \nabla_{\mathrm{X}} \underset{h}{\psi} \tag{5.11}$$

Further, we have the following commutations relations [5]:

$$\begin{aligned} [\square, \nabla_{\beta}] \underset{r,i}{\zeta}_{\alpha_i \dots \alpha_1} &= \mathrm{R}_{\beta}{}^{\gamma} \nabla_{\gamma} \underset{r,i}{\zeta}_{\alpha_i \dots \alpha_1} \\ &- 2 \sum_{1 \leqslant p \leqslant i} \mathrm{R}^{\rho}{}_{\beta \alpha_p}{}^{\gamma} \nabla_{\rho} \zeta_{\alpha_i \dots \gamma \dots \alpha_1} - \sum_{1 \leqslant p \leqslant i} (\nabla_{\rho} \mathrm{R}^{\rho}{}_{\beta \alpha_p}{}^{\gamma}) \underset{r,i}{\zeta}_{\alpha_i \dots \gamma \dots \alpha_1} \end{aligned} \tag{5.12}$$

and

$$[\square, \nabla_{\mathrm{X}}] \underset{h}{\psi} = \pi^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \underset{h}{\psi} + \left(\nabla_{\rho} \pi^{\mu\rho} - \frac{1}{2} \partial^{\mu} \operatorname{tr} \pi \right) \nabla_{\mu} \underset{h}{\psi} \tag{5.13}$$

In this section we shall assume:

Hypothesis 2$_r$. — The derivatives $\nabla^p \mathrm{X}$ are uniformly bounded in γ-norm for $p \leq r$. We denote

$$| \mathrm{X} |_{\mathrm{C}_b^r} = \sup_{\mathrm{V}_{t+1}} \max_{0 \leqslant p \leqslant r} | \nabla^p \mathrm{X} | = \mathrm{B}_r$$

Hypothesis 3$_r$. — The tensors $\nabla^i\pi$ and ∇^jR (R stands for the Riemann tensor of g) are bounded in γ-norm over each hypersurface S_t, for $i \leq r$, $j \leq r-1$. We denote

$$\underset{r-1}{q}(t) = |\pi|^{S_t}_{C^r_b} + |R|^{S_t}_{C^{r-1}_b} \tag{5.14}$$

We shall need the following lemmas:

LEMMA 7. — If $0 \leqslant i \leqslant l \leqslant r$, $\underset{r,i}{\zeta}$ can be expressed in the form:

$$\underset{r,i}{\zeta} = \sum_{k=1}^{l} \overset{l-k}{\underset{l,i}{L}} \cdot \underset{r-(l-k),k}{\zeta} \tag{5.15}$$

where $\overset{l-k}{\underset{l,i}{L}}$, a tensorfield of type $\binom{k}{i}$, is a sum of terms of the form:

$$\nabla^{h_1}X \dots \nabla^{h_{l-i}}X,$$

with

$$\sum_{s=1}^{l-i} h_s = l-k,$$

while:

$$\overset{\circ}{\underset{l,i}{L}}{}^{\beta_l\dots\beta_1}_{\alpha_i\dots\alpha_1} = \delta^{\beta_l}_{\alpha_i} \dots \delta^{\beta_{l-i+1}}_{\alpha_1} X^{\beta_{l-i}} \dots X^{\beta_1} \tag{5.16}$$

The proof is by induction on r using recursion relations (5.11). As a corollary of this lemma we have:

There exist constants c depending only on k, b, B_r such that:

$$|\overset{r}{\nabla}\underset{r,i}{\zeta}|^2 \leq c\underset{r\,r}{\varepsilon}, \quad (|\nabla\underset{r,i}{\zeta}|^{S_t}_{L^2})^2 \leq c\underset{r\,r}{E} \tag{5.17}$$

LEMMA 8. — If ϕ satisfies Eq. (1.1), we can express $\square\underset{r,i}{\zeta}$ in the form:

$$\square\underset{r,i}{\zeta} = \nabla^i\nabla^{r-i}_X\rho + \sum_{h=1}^{r+1} \overset{h}{\underset{r,i}{O}}\cdot \underset{h}{t} + \sum_{k=0}^{i-1} \overset{k}{\underset{r,i}{M}}\cdot \underset{r-k,i-k}{\zeta}, \tag{5.18}$$

where $\overset{k}{\underset{r,i}{M}}$, a tensorfield of type $\binom{i-k}{i}$ is linear in ∇^kR, while $\overset{h}{\underset{r,i}{O}}$, a tensorfield of type $\binom{h}{i}$, is a sum of terms of the form:

$$\nabla^{p_1}X \dots \nabla^{p_{r-i-1}}X\nabla^s\pi,$$

with

$$\sum_{l=1}^{r-i-1} p_l + s = r+1-h.$$

Proof. — Induction on r using commutation relation (5.12) gives

$$\square \underset{r,i}{\zeta} = \nabla^i \square \underset{r-i}{\psi} + \sum_{k=0}^{i-1} \overset{k}{\underset{r,i}{\mathrm{M}}} . \underset{r-k,i-k}{\zeta} , \tag{5.19}$$

where $\overset{k}{\underset{r,i}{\mathrm{M}}}$ has the specified properties. On the other hand, if ϕ satisfies Eq. (1.1), induction on h using commutation relation (5.13) yields:

$$\square \underset{h}{\psi} = \nabla_{\mathrm{X}}^h \rho + \sum_{k=1}^{h+1} \overset{k}{\underset{h}{\mathrm{N}}} \cdot \underset{k}{t}, \tag{5.20}$$

where $\overset{k}{\underset{h}{\mathrm{N}}}$, a tensorfield of type $\binom{k}{0}$ is a sum of terms of the form:

$$\nabla^{p_1}\mathrm{X} \ldots \nabla^{p_{h-1}}\mathrm{X}\nabla^s\pi,$$

with

$$\sum_{l=1}^{h-1} p_l + s = h + 1 - k$$

Substituting (5.20), after setting $h = r - i$, in (5.19) we recover (5.18) with

$$\overset{h}{\underset{r,i}{\mathrm{O}}} = \sum_{j=0}^{i} \binom{i}{j} \nabla^{i-j} \overset{h-j}{\underset{r-i}{\mathrm{N}}}$$

Finally, in view of Leibniz's rule, $\overset{h}{\underset{r,i}{\mathrm{O}}}$ has the required properties.

As a consequence of lemma 8, $| \square \underset{r,i}{\zeta} |_{\mathrm{L}^2}^{\mathrm{S}_t}$ is bounded by:

$$| \square \underset{r,i}{\zeta} |_{\mathrm{L}^2}^{\mathrm{S}_t} \leq c \left\{ \| \rho \|_r^{\mathrm{S}_t} + \left(| \pi |_{\mathrm{C}_b^0}^{\mathrm{S}_t} + | \mathrm{R} |_{\mathrm{C}_b^{i-1}}^{\mathrm{S}_t} \right) (\underset{r-1}{\mathrm{E}} (t))^{1/2} + | \pi |_{\mathrm{C}_b^0}^{\mathrm{S}_t} (\underset{r}{\mathrm{E}}(t))^{1/2} \right\} \tag{5.21}$$

where $\underset{r}{c}$ is a constant depending on k, b, B_r.

LEMMA 9. — For each $l \leq r$ there is a constant $\underset{r,l}{c}$ depending only on k, b, B_r such that:

$$\underset{r,l}{\mathrm{E}} \leq | \underset{r,l}{\Theta} | + \underset{r,l}{c} \left(\sum_{i=0}^{l-1} | \underset{r,i}{\Theta} | + \underset{r-1}{\mathrm{E}} \right) \tag{5.22}$$

Proof. — For $0 \leq h < r - i$, let us denote:

$$[\underset{r,i}{\zeta}]^h = \overset{\circ}{\underset{r-h,i}{\mathrm{L}}} \cdot \underset{r,r-h}{\zeta} \tag{5.23}$$

(cf. lemma 7). The operation $[\ ]^h$ amounts to taking the principal part of the included quantity with respect to $\underset{h}{\psi}$, namely the part containing the highest derivatives of $\underset{h}{\psi}$. Let also

$$\underset{r,i}{[\nabla\zeta]^h} = [\underset{r+1,i+1}{\zeta}]^h \tag{5.24}$$

(cf. (5.10)), and

$$\underset{r,i}{[\theta]^h} = (\gamma^{-1})^{\mu\nu}\underset{r,i}{[\nabla_\mu\zeta]^h}\cdot\underset{r,i}{[\nabla_\nu\zeta]^h} \tag{5.25}$$

We shall first show that for each $i < r$, $h < r - i$ there are constants $\underset{r}{c}$ depending only on k, b, B_r, such that

$$|\underset{r,i}{[\theta]^h}| \leq \underset{r}{c}\underset{r,i}{\varepsilon} + \underset{r-1}{\varepsilon} \tag{5.26}$$

From (5.25) we have:

$$|\underset{r,i}{[\theta]^h}| \leq c\,|\underset{r,i}{[\nabla\zeta]^h}|^2, \tag{5.27}$$

where c denotes a constant depending only on k, b and B. Further, using lemma 5, we express:

$$\underset{r,i}{[\nabla\zeta]^h} = \underset{r,i}{\nabla\zeta} - \sum_{k=1}^{r-h} \overset{r+1-h-k}{\underset{r+1-h,i+1}{L}}\cdot\nabla\underset{k+h-1,k-1}{\zeta}$$

Taking pointwise norms and using (5.17) (lemma 7) and the fact that, in virtue of lemma 7, $|\overset{r+1-h-k}{\underset{r+1-h,i+1}{L}}|$ is bounded by a constant depending only on k, b, $B_{r+1-h-k}$ we obtain

$$|\underset{r,i}{[\nabla\zeta]^h} \leq (c''\underset{r,i}{\varepsilon})^{1/2} + (\underset{r}{c}\underset{r-1}{\varepsilon})^{1/2}$$

which in turn, through (5.27), yields (5.26). Furthermore, since

$$|\underset{r,i}{[\Theta]^h}(t)| \leq \int_{S_t} |\underset{r,i}{[\theta]}|^h d\mu(\bar{g}_t),$$

we have:

$$|\underset{r,i}{[\Theta]^h}| \leq c(\underset{r}{E}\underset{r,i}{} + \underset{r-1}{E}) \tag{5.28}$$

In view of the definition of the metric Γ^{-1}: $(\Gamma^{-1})^{\alpha\beta} = g^{\alpha\beta} + kX^\alpha X^\beta$, the binominal theorem applied to the product $(\Gamma^{-1})^{\alpha_i\beta_i}\dots(\Gamma^{-1})^{\alpha_1\beta_1}$ in Eq. (5.6) allows us to express the quantities $\underset{r,l}{E}$ in the form:

$$\underset{r,l}{E} = \underset{r,l}{\Theta} + \sum_{i=0}^{l-1}\binom{l}{i}k^{l-i}\underset{r,i}{[\theta]^{r-l}} \tag{5.29}$$

To prove lemma 9 we use induction on 1. The lemma is true for $1 = 0$, since

$$\underset{r,0}{E} = \underset{r,0}{\Theta}$$

Let it be true for $\underset{r,i}{\mathrm{E}}$ with $0 \le i \le l-1$. From (5.29) and (5.28) we then have:

$$
\begin{aligned}
|\underset{r,l}{\mathrm{E}}| &\le |\underset{r,l}{\Theta}| + \sum_{i=0}^{l-1}\binom{l}{i}k^{l-i}\underset{r}{c}(\underset{r,i}{\mathrm{E}} + \underset{r-1}{\mathrm{E}}) \\
&\le |\underset{r,l}{\Theta}| + \sum_{i=0}^{l-1}\binom{l}{i}k^{l-i}\underset{r}{c}\left[|\underset{r,i}{\Theta}| + \underset{r,l-1}{c}\left(\sum_{j=0}^{i-2}|\underset{r,j}{\Theta}| + \underset{r-1}{\mathrm{E}}\right) + \underset{r-1}{\mathrm{E}}\right]
\end{aligned}
$$

which shows that we can define a constant $\underset{r,l}{c}$ as required such that (5.22) holds. In the following we shall use only the following consequence of lemma 9: there exists a constant $\underset{r}{c}$ depending on k, b, B_r such that

$$\underset{r}{\mathrm{E}} \le \underset{r}{c}\left(\sum_{i=0}^{r}|\underset{r,i}{\Theta}| + \underset{r-1}{\mathrm{E}}\right) \tag{5.30}$$

Let the tensorfield $\underset{r,i}{\zeta}$ be of class C^2 and having support compact in space and towards the past. Then by Gauss' theorem

$$\underset{r,i}{\Theta}(t) = \int_{-\infty}^{t}\int_{\mathrm{S}_\tau}\nabla_\mu \underset{r,i}{\mathrm{P}^\mu}\, d\mu(g)$$

and we have

$$
\begin{aligned}
\nabla_\mu \underset{r,i}{\mathrm{P}^\mu} &= \mathrm{X}_\mu(\nabla_\nu \underset{r,i}{\mathrm{T}^{\nu\mu}}) + \frac{1}{2}\pi\cdot \underset{r,i}{\mathrm{T}} \\
\nabla_\nu \underset{r,i}{\mathrm{T}^{\mu\nu}} &= \nabla^\mu \underset{r,i}{\zeta}\cdot\Box \underset{r,i}{\zeta} - \sum_{p=1}^{i}\mathrm{R}_{\nu}{}^{\mu}{}_{\alpha_p}{}^{\beta}\,\underset{r,i}{\zeta}_{\alpha_i\ldots\beta\ldots\alpha_1}\nabla^\nu \underset{r,i}{\zeta}^{\alpha_i\ldots\alpha_1}
\end{aligned}
$$

Applying lemma 1 to the definition of $\underset{r,i}{\mathrm{T}}$ (Eq. (5.3)) and then to the above expressions allows us to conclude that there are constants c depending only on k, b and B such that at each point of V_{l+1}

$$|\underset{r,i}{\mathrm{T}}| \le c\,|\nabla \underset{r,i}{\zeta}|^2$$

and

$$|\nabla_\mu \underset{r,i}{\mathrm{P}^\mu}| \le c(|\nabla \underset{r,i}{\zeta}|\,|\Box \underset{r,i}{\zeta}| + |\mathrm{R}|\,|\underset{r,i}{\zeta}|\,|\nabla \underset{r,i}{\zeta}| + |\pi|\,|\nabla \underset{r,i}{\zeta}|^2) \tag{5.32}$$

Consequently,

$$
\begin{aligned}
|\underset{r,i}{\Theta}(t)| &\le \mathrm{A}\int_{-\infty}^{t}\left\{\int_{\mathrm{S}_\tau}|\nabla_\mu \underset{r,i}{\mathrm{P}^\mu}|\, d\mu(\bar{g}_\tau)\right\}d\tau \\
&\le c\mathrm{A}\int_{-\infty}^{t}\left\{|\nabla \underset{r,i}{\zeta}|^{\mathrm{S}_\tau}_{\mathrm{L}^2}|\Box \underset{r,i}{\zeta}|^{\mathrm{S}_\tau}_{\mathrm{L}^2} + |\mathrm{R}|^{\mathrm{S}_\tau}_{\mathrm{C}^0_b}|\underset{r,i}{\zeta}|^{\mathrm{S}_\tau}_{\mathrm{L}^2}|\nabla \underset{r,i}{\zeta}|^{\mathrm{S}_\tau}_{\mathrm{L}^2}\right. \\
&\qquad \left. + |\pi|^{\mathrm{S}_\tau}_{\mathrm{C}^0_b}(|\nabla \underset{r,i}{\zeta}|^{\mathrm{S}_\tau}_{\mathrm{L}^2})^2\right\}d\tau.
\end{aligned}
\tag{5.33}
$$

where A is the bound on the lapse function N (§ 2) and we have applied the Schwarz inequality to the integrals over S_τ of the first two terms on the right in (5.32). Substituting in (5.33) the bounds

for $|\underset{r,i}{\zeta}|^{S_\tau}_{L^2}$ and $|\nabla\underset{r,i}{\zeta}|^{S_\tau}_{L^2}$ from (5.17) and for $|\square\underset{r,i}{\zeta}|^{S_\tau}_{L^2}$ from (5.21) we obtain

$$|\underset{r,i}{\Theta}(t)| \leq A\underset{r}{c}\int_{-\infty}^{t}\left\{|\pi|^{S_\tau}_{C^0_b}\underset{r}{E}(\tau)+\left[\left(|\pi|^{S_\tau}_{C^r_b}+|R|^{S_\tau}_{C^{r-1}_b}\right)\underset{r-1}{E}^{1/2}(\tau)+\|\rho\|^{S_\tau}_r\right]\underset{r}{E}^{1/2}(\tau)\right\}d\tau,$$

where $\underset{r}{c}$ is a constant depending on k, b, B_r.

Finally, substituting the above estimate for $|\underset{r,i}{\Theta}|$ in (5.30) (lemma 9) we obtain:

$$\underset{r}{E}(t) \leq \underset{r}{c^1}\underset{r-1}{E}(t) + Ac^2_r\int_{-\infty}^{t}\left\{p\underset{r}{E} + (\underset{r-1}{q}\underset{r-1}{E}^{1/2} + \underset{r}{f})\underset{r}{E}^{1/2}\right\}d^\tau \quad (5.34)$$

Here p is given by Eq. (4.1), $\underset{r-1}{q}$ by (5.14), $\underset{r}{f}(t) = \|\rho\|^{S_t}_r$ and $\underset{r}{c^1}$, $\underset{r}{c^2}$ are constants depending on k, b, B_r. We are now in position to prove:

LEMMA 10. — Let (V_{l+1}, g) satisfy the assumptions of § 2 as well as hypotheses 1, 2_s, 3_s and let $\square_g\phi = \rho$, $\phi \in C^{2+s}$ and of support compact in space and toward the past. Then for each $r \leq s$ the r-th order energy of ϕ satisfies the inequality

$$\underset{r}{E}(t) \leq A\underset{r}{c}\int_{-\infty}^{t}\left\{p\underset{r}{E} + (\underset{r-1}{q}\underset{r-1}{E}^{1/2} + \underset{r}{f})\underset{r}{E}^{1/2}\right\}d\tau, \quad (5.35)$$

where $\underset{r}{c}$ is a constant depending on k, b, B_r.

Proof. — We shall argue by induction. The lemma is valid for $r = 0$ (lemma 6). Let it be valid for $r - 1$. Then we have, *a fortiori*,

$$\underset{r-1}{E}(t) \leq A\underset{r-1}{c}\int_{-\infty}^{t}\left\{p\underset{r}{E} + (\underset{r-1}{q}\underset{r-1}{E}^{1/2} + \underset{r}{f})\underset{r}{E}^{1/2}\right\}d\tau$$

Substituting this estimate for $\underset{r-1}{E}(t)$ in the first term on the right in (5.34) we recover (5.35) with $\underset{r}{c} = \underset{r}{c^1}\underset{r-1}{c} + \underset{r}{c^2}$.

We shall now introduce

Hypothesis 4_r. — On V_{l+1} the tensor $\nabla^i\pi$ and $\nabla^j R$ are uniformly bounded in γ-norm for $i \leq r$ and $j \leq r - 1$. We denote

$$\underset{r-1}{Q} = \sup_{t\in\mathbb{R}}\underset{r-1}{q}(t) = |\pi|_{C^r_b} + |R|_{C^{r-1}_b} \quad (5.36)$$

LEMMA 11. — Let (V_{l+1}, g) satisfy the assumptions of § 2, as well as hypotheses 1, 2_s and 4_s and let $\square_g\phi = \rho$, $\phi \in C^{2+s}$ and of support compact in space and towards the past. Then for each $r \leq s$ the r-th order energy of ϕ is bounded by

$$\underset{r}{\mathrm{E}}^{1/2}(t) \leq \underset{r}{\mathrm{G}} \sum_{i=0}^{r} \underbrace{\int \cdots \int}_{r-i+1} \underset{i}{f}$$

$$= \underset{r}{\mathrm{G}} \left\{ \int_{-\infty}^{t} \underset{r}{f}(\tau)d\tau + \sum_{r=0}^{r-1} \int_{-\infty}^{t} \frac{(t-\tau)^{r-i}}{(r-i)\,!} \underset{i}{f}(\tau)d\tau \right\} \tag{5.37}$$

and we have

$$\int_{-\infty}^{t} \underset{r}{\mathrm{E}}^{1/2}(\tau)d\tau \leq \underset{r}{\mathrm{G}} \sum_{i=0}^{r} \underbrace{\int \cdots \int}_{r-i+2} \underset{i}{f} = \underset{r}{\mathrm{G}} \sum_{i=0}^{r} \int_{-\infty}^{t} \frac{(t-\tau)^{r-i+1}}{(r-i+1)\,!} \underset{i}{f}(\tau)d\tau, \tag{5.38}$$

where

$$\underset{r}{\mathrm{G}} = \underset{r}{\mathrm{M}} \left(1 + \sum_{i=0}^{r-1} \underset{r-1}{\mathrm{Q}} \underset{r-1}{\mathrm{M}} \cdots \underset{i}{\mathrm{Q}} \underset{i}{\mathrm{M}} \right) \tag{5.39}$$

and

$$\underset{r}{\mathrm{M}} = \frac{1}{2} \mathrm{A}\underset{r}{c} \exp \left(\frac{1}{2} \mathrm{A}\underset{r}{c} \int_{-\infty}^{\mathrm{T}} p(t)dt \right), \tag{5.40}$$

$$\underset{r}{\mathrm{Q}} = \sup_{-\infty < t < \mathrm{T}} \underset{r}{q}(t), \tag{5.41}$$

$\underset{r}{c}$ being the constant which appears in (5.35) and depends only on k, b, B_r.

Proof. — We first apply lemma 5 to (5.35) with the substitutions $\mathrm{X} = \underset{r}{\mathrm{E}}^{1/2}$, $y = \mathrm{A}\underset{r}{c}p$ and $z = \mathrm{A}\underset{r}{c}(\underset{r-1}{\mathrm{Q}}\; \underset{r-1}{\mathrm{E}}^{1/2} + \underset{r}{f})$ to obtain:

$$\underset{r}{\mathrm{E}}^{1/2}(t) \leq \underset{r}{\mathrm{M}} \left\{ \int_{-\infty}^{t} \underset{r}{f}(\tau)d\tau + \underset{r-1}{\mathrm{Q}} \int_{-\infty}^{t} \underset{r-1}{\mathrm{E}}^{1/2}(\tau)d\tau \right\} \tag{5.42}$$

We shall use induction .The lemma for $r = 0$ reduces to lemma 4. Let the lemma be valid for $r - 1$. Then we have:

$$\int_{-\infty}^{t} \underset{r-1}{\mathrm{E}}^{1/2}(\tau)d\tau \leq \underset{r-1}{\mathrm{G}} \sum_{i=0}^{r-1} \underbrace{\int \cdots \int}_{r-i+1} \underset{i}{f}$$

Substituting this in (5.42) we obtain

$$\underset{r}{\mathrm{E}}^{1/2} \leq \underset{r}{\mathrm{M}} \left\{ \int f_r + \underset{r-1}{\mathrm{Q}}\; \underset{r-1}{\mathrm{G}} \sum_{i=0}^{r-1} \underbrace{\int \cdots \int}_{r-i+1} \underset{i}{f} \right\}$$

$$\leq \underset{r}{\mathrm{M}}(1 + \underset{r-1}{\mathrm{Q}}\; \underset{r-1}{\mathrm{G}}) \sum_{i=0}^{r} \underbrace{\int \cdots \int}_{r-i+1} \underset{i}{f},$$

which is (5.37), since by (5.39)

$$\underset{r}{M}(1 + \underset{r-1}{Q}\ \underset{r-1}{G}) = \underset{r}{G}$$

and the induction is complete.

Now since for $k \geq 1$

$$(T - t)^k \leq 2^{k-1}(| T |^k + | t |^k)$$

we conclude from lemma 9, in view of (5.10), that on V_{l+1}^T for any finite T:

$$\| \phi \|_{E_{r+1}^*} \leq c G_r \{ \| \rho \|_{\mathscr{L}_r} + (1 + | T |^r \| \rho \|_{\mathscr{P}_{r-1}} \} \tag{5.43}$$

and

$$\| \phi \|_{\mathscr{L}_{r+1}^*} \leq c G_r (1 + | T |^{r+1}) \| \rho \|_{\mathscr{P}_r}, \tag{5.44}$$

where c denotes a constant depending on k, b and B only.

Lemma 11 gives us the following existence theorem:

THEOREM. — Let (V_{l+1}, g) be a globally hyperbolic manifold for which the assumptions of § 2 and of lemma 4 hold, as well as hypothese 1, 2_r and 4_r. Then if $\rho \in \mathscr{L}_r \cap \mathscr{P}_{r-1}$ the equation $\square_g \phi = \rho$ has a solution ϕ which belongs to $E_{r+1}^*(V_{l+1}^T)$ for any finite T and satisfies (5.43). If in addition $\rho \in \mathscr{P}_r$, then $\phi \in \mathscr{L}_{r+1}^*(V_{l+1}^T)$ and satisfies (5.44).

The proof is by approaching ρ by a sequence of functions $\{ \rho_n \}$ in C_0^∞ as in theorem 1.

Remark. — We can extend the above estimates for $\underset{r}{E}$ to the infinite future if we assume in addition that $\underset{r-1}{q}$ (cf. 5.14) is integrable on $(T, +\infty)$.

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