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A classical basis for quantum mechanics

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ABSTRACT. — It is shown that Schrödinger Quantum Mechanics can be derived from a classical two fluid variational principle.

INTRODUCTION

There is a widely held view that quantum mechanics cannot be satisfactorily derived from a classical basis and indeed there is a second and possibly equivalent contention that it cannot be derived from a classical variational principle. Elsewhere, this author (Gilson, 1978) has given details of a classical two fluid structure that leads via a thermodynamical route directly to the Schrödinger equation. In this paper, the second contention mentioned above concerning the variational basis of quantum mechanics will be shown to be false. The work in this paper does not depend on the first reference above.

A classical variational principle will be given and shown to lead to the correct equations for Schrödinger quantum theory. Equations and variational techniques related to super fluids (Zilsel, 1953; London, 1954; Landau, 1966; Tisza, 1947; Yourgrau, 1968) have been studied that are close to the line to be pursued here. However, these authors were not motivated towards finding a classical basis for quantum mechanics but rather towards solving the super fluid problem. There also seems to have been the tacitly held view that the two fluid classical like equations that they were using could only be approximate and temporary and would have to be replaced by a more fundamental and a more accurate quantum theory version. In fact, the classical like two fluid schemes that were used in super fluid investigations could not have led to quantum mechanics even if those workers had been motivated towards pursuing such a programme. This is because the classical basis on which the two fluid structure that does lead to quantum mechanics has to be built is wider than was employed in the super fluid theories and it has some unexpected and unusual characteristics. This author has shown (Gilson, 1978) that the classical basis for the one dimensional Schrödinger equation needs to contain negative mass and an extra degree of freedom. The success of the scheme to be demonstrated here would seem to vindicate Bohm's (1957) suggestion that below conventional quantum mechanics there are more fundamental classical like strata. An early one fluid interpretation of quantum mechanics was given by Madelung (1926) and the history of this area can be found in the book by Max Jammer (1974) which contains many references.

THE VARIATIONAL PRINCIPLE

We aim at deriving Schrödinger's one dimensional equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi, \tag{1}$$

for a general external potential $V(x)$ and from a classical variational principle. The two dimensional Lagrangian density will be taken to be

$$L = K_Q - (\mathscr{E}_C + \rho V_1), \tag{2}$$

where

$$\begin{aligned} K_Q &= K_C + K_{th} \\ &= \frac{\rho}{2}(v_1^2 - v_2^2) + \rho \frac{v}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \end{aligned} \tag{3}$$

$$\mathscr{E}_C = \frac{v^2}{2\rho}\left[\left(\frac{\partial \rho}{\partial x}\right)^2 - \left(\frac{\partial \rho}{\partial y}\right)^2\right] = \rho \varepsilon_C \tag{4}$$

and $V_1(x, y)$ is the real part of the usual one dimensional quantum external potential $V(x)$ analytically continued. The term,

$$K_C = \frac{\rho}{2}(v_1^2 - v_2^2), \tag{5}$$

is the classical kinetic energy density assuming also that the second velocity component v_2 carries a negative mass density $\rho_2 = - \rho$. The term,

$$K_{th} = \rho \frac{v}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \rho \frac{v}{2}\zeta \tag{6}$$

is a « thermal » enregy density which is of great importance. It relates the two dimensional vorticity $\zeta = | \nabla_\wedge \underline{v} |$ to an energy ε where $K_{th} = \rho\varepsilon$. The factor $(\nu/2)$ is a fundamental constant. Thus in this scheme the energy ε and vorticity are essentially the same thing. The term (4),

$$\mathscr{E}_C = \rho\varepsilon_C = \rho\varepsilon_C\left(\rho, \frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}\right)$$

is a combined internal energy density for both positive and negative mass flows.

We shall use the method of Lagrange multipliers with the following supplementary condition,

$$\frac{\partial\rho}{\partial t} = -\frac{1}{2}\left(\frac{\partial(\rho v_1)}{\partial x} + \frac{\partial(\rho v_2)}{\partial y}\right). \tag{7}$$

When the minimal condition applicable in the case of quantum mechanical flow holds, the continuity equation (7) reduces to the usual one dimensional quantum continuity equation,

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v_1). \tag{8}$$

Thus the variational principle takes the form,

$$\delta\int_{t_0}^{t}\int_V\left\{L - \alpha\left[\frac{\partial\rho}{\partial t} + \frac{1}{2}\left(\frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2)\right)\right]\right\}dxdydt \tag{9}$$

The independent variations will be taken in v_1, v_2 and ρ. We obtain the following three equations:

$$2v_1 = -\frac{\partial\alpha}{\partial x} - \nu\frac{\partial \ln\rho}{\partial y}, \tag{10}$$

$$2v_2 = \frac{\partial\alpha}{\partial y} - \nu\frac{\partial \ln\rho}{\partial x}, \tag{11}$$

and

$$V_1 - \frac{\partial\alpha}{\partial t} - \frac{1}{2}(v_1^2 - v_2^2) - \frac{1}{2}\left(\frac{\partial\alpha}{\partial x}v_1 + \frac{\partial\alpha}{\partial y}v_2\right) + \frac{\nu}{2}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - \varepsilon_C = 0. \tag{12}$$

From (10) and (11), we deduce that

$$2\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) = -\nu\left(\frac{\partial^2 \ln\rho}{\partial x^2} + \frac{\partial^2 \ln\rho}{\partial y^2}\right) \tag{13}$$

and

$$2\left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}\right) = -\nu\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right) \tag{14}$$

We are seeking a situation where it is possible for v_1 and v_2 to be the real and imaginary parts of a complex function. Thus fitting the form that the complex local quantum momentum has after analytic continuation into an imaginary y direction. That form is

$$v_1(x, y) + iv_2(x, y) = -\frac{i\hbar}{m}\frac{\partial \ln \Psi(x + iy, t)}{\partial(x + iy)}, \tag{15}$$

where $\Psi(x + iy, t)$ is the analytically continued wave function at time t. Of course, the variable y in (15) is identically zero in the usual analysis of the one dimensional Schrödinger wave equation. There only $v_1(x, 0) + iv_2(x, 0)$ appears. If then v_1 and v_2 are to fit this special category of function pairs, then the Cauchy Riemann equations must hold for v_1 and v_2. That is

$$\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} = 0, \tag{16}$$

and

$$\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} = 0. \tag{17}$$

Comparing (16) and (17) with (13) and (14), we see that the structure with which we are working appears to contain solutions of a more general nature than the restrictions (16) and (17) would seem to imply. However, let us see what consequences would result from restricting solutions of the variational principle by (16) and (17). Using (16) and (17) with (13) and (14), we get,

$$\nabla^2 \ln \rho = 0 \tag{18}$$

and

$$\nabla^2\alpha = 0 \tag{19}$$

and forming the complex combination $v_1 + iv_2$ from (10) and (11), we can express that combination in the form,

$$\begin{aligned} 2(v_1 + iv_2) &= \frac{\partial}{\partial x}(-\alpha - iv \ln \rho) - i\frac{\partial}{\partial y}(-\alpha - iv \ln \rho) \\ &= -\left(\frac{\partial \omega}{\partial x} - i\frac{\partial \omega}{\partial y}\right) \end{aligned} \tag{20}$$

where

$$\omega = \alpha + iv \ln \rho \tag{21}$$

Thus it appears that ω is a natural combination of α and $\ln \rho$ to give a complex potential for the flow field of a form which is compatible with quantum mechanics. Thus pursuing this line, we can rewrite (20) in the form

$$\begin{aligned} 2(v_1 + iv_2) &= -\left(\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial iy}\right) \\ &= -2\frac{\partial \omega}{\partial z} \end{aligned} \tag{22}$$

It should be noted that the complex potential ω here refers to the $(v_1, -v_2)$

flow field in contrast with the usual complex potential used in 2 dimensional hydrodynamics which refers to the (v_1, v_2) flow field. Thus comparisons with 2 dimensional hydrodynamics should be made with great care. For example, here we can have $\zeta_3 \neq 0$ and yet $\nabla^2\omega = 0$. That the function ω can be regarded as a function of a complex variable $z = x + iy$ is supported by (18) and (19). This all suggests that (10) and (11) form a partly redundant description of the two dimensional flow field and that therefore they can equally well be replaced by

$$v_1 = -\frac{\partial \alpha}{\partial x} = -\nu \frac{\partial \ln \rho}{\partial y}, \tag{23}$$

$$v_2 = \frac{\partial \alpha}{\partial y} = -\nu \frac{\partial \ln \rho}{\partial x}, \tag{24}$$

on using (22). By now substituting (17), (23) and (22) into (12), the whole matter is rapidly clarified because we obtain the result,

$$\frac{\partial \alpha}{\partial t} = \frac{v_1^2 - v_2^2}{2} + \nu \frac{\partial v_2}{\partial x} + V_1 \tag{25}$$

which the reader will recognise as the local quantum energy E, where $\nu = \hbar/2m$, (15) holds in the form (22) and

$$E = \frac{\partial \alpha}{\partial t} = Re \left\{ i\hbar \frac{\partial \ln \Psi}{\partial t} \right\} \tag{26}$$

when $y = 0$. (23) and (24) can be seen to be consistent with (17). Thus the classical fluid variational principle (9) leads to Schrödinger quantum mechanics. It is interesting to re-express the equations of fluid flow (23) and (24) by making use of (25). Upon differentiating (23) and (24) with respect to t and using (25), we obtain

$$\frac{\partial v_1}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{v_1^2 - v_2^2}{2} + \nu \frac{\partial v_2}{\partial x} + V_1\right) \tag{27}$$

$$\frac{\partial v_2}{\partial t} = +\frac{\partial}{\partial y}\left(\frac{v_1^2 - v_2^2}{2} + \nu \frac{\partial v_2}{\partial x} + V_1\right) \tag{28}$$

(27) and (28) can be rewritten, using (16) and (17) in the form

$$\frac{\partial v_1}{\partial t} - v_2\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\nu \frac{\partial v_2}{\partial x} + \frac{v_1^2 + v_2^2}{2} + V_1\right) \tag{29}$$

$$\frac{\partial v_2}{\partial t} + v_1\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = -\frac{\partial}{\partial y}\left(-\nu \frac{\partial v_2}{\partial x} + \frac{v_1^2 + v_2^2}{2} - V_1\right) \tag{30}$$

Thus if we denote $\rho \dfrac{\partial v_2}{\partial x}$ by P, we have

$$\frac{P}{\rho} = \nu \frac{\partial v_2}{\partial x} \tag{31}$$

and by (17)

$$\frac{P}{(-\rho)} = -\nu \frac{\partial v_2}{\partial x} = \nu \frac{\partial v_1}{\partial y} . \tag{32}$$

Then using (6), (29) and (30) assume the form

$$\frac{\partial v_1}{\partial t} - v_2 \zeta_3 = -\frac{\partial}{\partial x}\left(\frac{P}{\rho} + \frac{v^2}{2} + V_1\right), \tag{33}$$

and

$$\frac{\partial v_2}{\partial t} + v_1 \zeta_3 = -\frac{\partial}{\partial y}\left(\frac{P}{(-\rho)} + \frac{v^2}{2} - V_1\right). \tag{34}$$

These are the standard Eulerian equations for two dimensional motion with vorticity. However, clearly one of the dimensions refers to the positive mass movement $(+\rho)$ and the other dimension refers to the negative mass movement $(-\rho)$, both under an appropriately signed external potential $\pm V_1$. From (6) and (31), we have,

$$\varepsilon = \frac{P}{\rho} = \frac{\nu}{2}\zeta_3 = kT \tag{35}$$

where k is Boltzmann's constant and T is a temperature.

Thus in this scheme all the internal energy $\varepsilon = kT$ is pure vorticity.

CONCLUSIONS

The demonstration that quantum mechanics can be based on a classical fluid structure is of considerable philosophical importance as it turns upside down some very extensively and firmly held views about the nature of quantum mechanics. In particular, the view that superfluid theory can only logically be deduced from a quantum mechanical basis is clearly unsound. In fact, it appears from this work that fluids are the basis of quantum mechanics and possibly, these basic fluids are types of « superfluid ». No doubt, as fluid theory with it's potentially rich substructure is further explored in relation to quantum processes, technical developments will arise which go beyond the now orthodox Schrödinger wave function analysis

REFERENCES

[1] D. Bohm, *Causality and Chance in Modern Physics*, Routledge, 1957.

[2] J. G. Gilson, The fluid process underlying quantum mechanics, *Acta Physica Hungaricae*, t. **44** (4), 1978, p. 333.

[3] Max JAMMER, *The Philosophy of Quantum Mechanics*, John Wiley and Sons, 1974.
[4] L. D. LANDAU, Fluid Mechanics. Vol. 6. *Course of Theoretical Physics*, Pergamon Press, 1966.
[5] J. LONDON, *Superfluids*. Vol. II, John Wiley and Sons, 1954, p. 126.
[6] E. MADELUNG, Quantentheorie in Hydrodynamischer Form, *Zeitschrift für Physik*, t. **40**, 1926, p. 332.
[7] L. TISZA, *Phys. Rev.*, t. **72**, 1947, p. 838.
[8] W. YOURGRAU, *Variational Principles in Dynamics and Quantum Theory*, Third ed. Pitman and Sons Ltd., London, 1968.
[9] P. R. ZILSEL, *Phys. Rev.*, t. **92**, 1953, p. 1106.

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