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Superpositions of states and a representation theorem

by

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ABSTRACT. — A quantum logic (L, P) is considered, where P is a set of pure states. The set $\mathscr{L}(P)$ of all subsets of P closed under superpositions is studied. It is shown that $\mathscr{L}(P)$ is isomorphic to the set of all linear subspaces of a vector space. In case that each state in P has a carrier, an orthocomplementation can be defined in a subset $\mathscr{F}(P)$ of $\mathscr{L}(P)$. An imbedding theorem for the logic L into the logic L(H) of a Hilbert space H is then proved.

1. DEFINITIONS AND NOTATION

Let L be a partially ordered set with the first and last elements 1 and 0, respectively, and with the orthocomplementation $a \mapsto a^{\perp} : L \to L$. Let the latlice sum $\bigvee_i a_i$ exist in L for any sequence $\{ a_i \} \subset L$ such that $a_i \leq a_j^{\perp}$, $i \neq j$, $i, j = 1, 2, \ldots$ The elements $a, b \in L$ are said to be orthogonal ($a \perp b$) if $a \leq b^{\perp}$ and they are said to be compatible ($a \leftrightarrow b$) if there exist elements a_1, b_1, c in L, mutually orthogonal and such that $a = a_1 \vee c$, $b = b_1 \vee c$. A map $m : L \to [0, 1]$ is a state on L if *i)* $m(1) = 1$, *ii)* $m(\vee a_i) = \Sigma m(a_i)$ for any sequence of mutually orthogonal elements in L. The state m is pure if it cannot be written in the form $m = cm_1 + (1 - c)m_2$, where $0 < c < 1$ and m_1, m_2 are distinct states. Let P be a set of pure states on L. For $a \in L$, $m \in P$, define $P_a = \{ m \in P : m(a) = 1 \}$, $L_m = \{ a \in L : m(a) = 1 \}$. We shall suppose that *i)* $P_a \subset P_b$ implies $a \leq b$ (a, $b \in L$) and *ii)* $L_{m_1} \subset L_{m_2}$

implies $m_1 = m_2$. From *i*) it follows that L is orthomodular, i. e. $a \leq b$ ($a, b \in L$) implies $b = a \vee (b \wedge a^{\perp})$ and that to any $a \in L$, $a \neq 0$, there is $m \in P$ such that $m(a) = 1$ [*4*]. We shall suppose, in addition, that if $a, b, c \in L$ are mutually compatible, then $a \leftrightarrow b \vee c$. The pair (L, P), which satisfies all the suppositions mentioned above, is called a quantum logic.

A state $m \in P$ is a superposition of the states $p, q \in P$ if $p(a) = 0$ and $q(a) = 0$ imply $m(a) = 0$ (or, alternatively, if $p(a) = 1$ and $q(a) = 1$ imply $m(a) = 1$) [*12*]. A set $S \subset P$ is said to be closed under superpositions if it contains every superposition of any pair of its elements. If $S \subset P$ is not closed under superpositions, let $\Lambda(S)$ denote the smallest subset of P, closed under superpositions and containing S. The set $S \subset P$ is a sector if *i*) $S = \Lambda(S)$, *ii*) to any $p, q \in S$, $p \neq q$, there is $s \in S$, $s \neq p, q$ such that $s \in \Lambda\{p, q\}$, *iii*) if $q \in P$, $q \notin S$ then $\Lambda\{s, q\} = \{s, q\}$ for any $s \in S$. We say that the superposition principle holds in (L, P) if for any $p, q \in P$, $p \neq q$, there is $r \in P$, $r \neq p, q$ such that $r \in \Lambda(\{p, q\})$ [*9*].

Let C be the set of all elements of L which are compatible with all the other elements, i. e. $C = \{a \in L : a \leftrightarrow b \text{ for any } b \in L\}$. C is called the centre of L. It was shown that C is a Boolean sub-σ-algebra of L. If p is a pure state and $c \in C$, then $p(c) = 1$ or $p(c) = 0$ [*11, 12*]. A logic L is called irreducible if its centre C is trivial, i. e. $C = \{0, 1\}$. It was shown that if the superposition principle holds on (L, P), then L is irreducible [*9*].

For $S \subset P$ and $a \in L$, let us write $S(a) = i$ if $m(a) = i$ for all $m \in S$, where $i = 0.1$. Let $\overline{S} = \{m \in P : S(a) = 1 \text{ imply } m(a) = 1\}$. Gudder [*6*] introduced the following postulate (minimal superposition postulate, MSP): if S is any finite subset of P and $m \in \overline{S}$ is such that $m \notin \overline{Q}$ for any subset $Q \subset S$, $Q \neq S$ (i. e. m is a minimal superposition), then $\{m, S_1\}^- \cap \overline{S}_2 \neq \emptyset$ for any $S_1, S_2 \subset P$ such that $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$.

Let us denote by $\mathscr{L}(P)$ the set of all subsets $S \subset P$ such that $\Lambda(S) = S$.

2. STRUCTURE OF THE SET $\mathscr{L}$(P)

In the sequel we shall suppose that (L, P) is a quantum logic and that the MSP holds in P, P being a set of pure states on L.

We recall that the map $S \mapsto \Lambda(S)$ has the following properties [*9*]:

i) $S_1 \subset S_2$ implies $\Lambda(S_1) \subset \Lambda(S_2)$,

ii) if $S_\alpha \subset P$, $\alpha \in A$, then $\bigcap_{\alpha \in A} \Lambda(S_\alpha)$ is closed under superpositions, and

$$\Lambda\Big(\bigcap_\alpha S_\alpha\Big) \subset \bigcap_\alpha \Lambda(S_\alpha),$$

iii) if $S_\alpha \subset P$, $\alpha \in A$, then $\bigcup_\alpha \Lambda(S_\alpha) \subset \Lambda\Big(\bigcup_\alpha S_\alpha\Big)$.

In addition, if the MSP holds, then by [*10*]:

iv) $\Lambda(S) = \overline{S}$ for any finite subset S of P,

v) $p \in \Lambda(\{ r, q \})$ implies $r \in \Lambda(\{ p, q \})$ for any distinct states $p, q, r \in P$.

Let $\mathscr{L}(P) = \{ S : S \subset P, \Lambda(S) = S \}$. $\mathscr{L}(P)$ is a partially ordered set by the set inclusion.

For $S_\alpha \in \mathscr{L}(P)$, $\alpha \in A$, let us set

$$\bigwedge_{\alpha \in A} S_\alpha = \Lambda\Big(\bigcap_{\alpha \in A} S_\alpha\Big), \text{ and } \bigvee_{\alpha \in A} S_\alpha = \Lambda\Big(\bigcup_{\alpha \in A} S_\alpha\Big).$$

LEMMA 1. — For $S_\alpha \in \mathscr{L}(P)$, $\alpha \in A$, $\bigwedge_\alpha S_\alpha = \bigcap_\alpha S_\alpha$.

Proof. — By *ii)*, $\Lambda\Big(\bigcap_\alpha S_\alpha\Big) \subset \bigcap_\alpha \Lambda(S_\alpha) = \bigcap_\alpha S_\alpha$. On the other hand, $\bigcap_\alpha S_\alpha \subset \Lambda\Big(\bigcap_\alpha S_\alpha\Big)$, i. e. $\bigwedge_\alpha S_\alpha = \bigcap_\alpha S_\alpha$.

LEMMA 2. — For $S_1, S_2 \in \mathscr{L}(P)$,

$$S_1 \vee S_2 = \{ p \in P : p \in \Lambda \{ r, q \}, r \in S_1, q \in S_2 \}.$$

Proof. — Let us set $S = \{ p \in P : p \in \Lambda \{ r, q \}, r \in S_1, q \in S_2 \}$. Clearly, $S_1 \cup S_2 \subset S$ and $r \in S_1$, $q \in S_2$ imply $\Lambda \{ r, q \} \subset \Lambda(S_1 \cup S_2)$. We see that $S \subset \Lambda(S_1 \cup S_2) = S_1 \vee S_2$. We shall complete the proof by showing that $S = \Lambda(S)$. Let $p_1, p_2 \in S$. Then there are $r_1, r_2 \in S_1$ and $q_1, q_2 \in S_2$ such that $p_1 \in \Lambda \{ r_1, q_1 \}$, $p_2 \in \Lambda \{ r_2, q_2 \}$. Let $p \in \Lambda \{ p_1, p_2 \}$. Then, clearly, $p \in \Lambda \{ r_1, q_1, r_2, q_2 \} = \{ r_1, q_1, r_2, q_2 \}^-$. The following cases can occure: *i)* $p \in \Lambda \{ r_1, r_2 \}$, *ii)* $p \in \Lambda \{ q_1, q_2 \}$, *iii)* $p \in \Lambda \{ r_i, q_j \}$ $(i, j = 1, 2)$, *iv)* no of *i)*, *ii)*, *iii)* comes true.

It is straightforward that in the cases *i)*, *ii)*, *iii)* $p \in S$. Let us consider the case *iv)*. If $p \in \Lambda \{ r_1, q_1, r_2 \}$, then by MSP, $\Lambda \{ r_1, r_2 \} \cap \Lambda \{ p, q_1 \} \neq \emptyset$. Let $m \in \Lambda \{ r_1, r_2 \} \cap \Lambda \{ p, q_1 \}$. Then $m \in S_1$, $p \in \Lambda \{ m, q_1 \}$, $q_1 \in S_2$ imply that $p \in S$. Analogical reasoning can be done in all cases in which there is a set $Q \subset \{ r_1, r_2, q_1, q_2 \}$ such that $p \in \Lambda(Q)$. Now let $p \in \Lambda \{ r_1, r_2, q_1, q_2 \}$ be a minimal superposition. Then by MSP, there is

$$m \in \Lambda \{ r_1, r_2 \} \cap \Lambda \{ p, q_1, q_2 \}.$$

This implies $m \in S_1$, $m \in \Lambda \{ p, q_1, q_2 \}$. The following cases can occure· (*a*) $m \in \Lambda \{ p, q_1 \}$ (or, analogically, $m \in \Lambda \{ p, q_2 \}$), which implies $p \in \Lambda \{ m, q_1 \}$ (or $p \in \Lambda \{ m, q_2 \}$), i. e. $p \in S$. *b)* $m \in \Lambda \{ q_1, q_2 \}$. Then $q_1 \in \Lambda \{ m, q_2 \}$, but $m \in \Lambda \{ r_1, r_2 \}$ implies $q_1 \in \Lambda \{ r_1, r_2, q_2 \}$. Hence, $\Lambda \{ r_1, r_2, q_1, q_2 \} \subset \Lambda \{ r_1, r_2, q_2 \}$, i. e. $p \in \Lambda \{ r_1, r_2, q_2 \}$, which is the preceding case. *c)* $m \in \Lambda \{ p, q_1, q_2 \}$ is a minimal superposition. Then, by MSP, there is $n \in \Lambda\{q_1, q_2\} \cap \Lambda\{m, p\}$. $n \in \Lambda\{q_1, q_2\}$ implies $n \in S_2$ and $n \in \Lambda\{m, p\}$ implies $p \in \Lambda\{m, n\}$, $m \in S_1$, $n \in S_2$, hence $p \in S$. This completes the proof.

LEMMA 3. — For any $Q \subset P$, $\Lambda(Q) = \cup\{\Lambda(T) : T$ is a finite subset of $Q\}$.

Proof. — Let us set $B = \cup\{\Lambda(T) : T$ is a finite subset of $Q\}$. Clearly, $Q \subset B \subset \Lambda(Q)$. We show that B is closed under superpositions. Indeed, let $p_1, p_2 \in B$, then there are $T_1, T_2 \subset Q$, finite subsets, such that $p_1 \in \Lambda(T_1)$ and $p_2 \in \Lambda(T_2)$. But then $p_1, p_2 \in \Lambda(T_1 \cup T_2)$, hence

$$\Lambda\{p_1, p_2\} \subset \Lambda(T_1 \cup T_2) \subset B.$$

From this it follows that $\Lambda(B) = B$, hence $\Lambda(Q) = B$.

LEMMA 4. — If $\Phi \subset \mathscr{L}(P)$ is an ordered subset (by inclusion) then the set $B = \cup\{T : T \in \Phi\} \in \mathscr{L}(P)$.

Proof. — We have to show that $\Lambda(B) = B$. Let $p_1, p_2 \in B$, then there are $T_1, T_2 \in \Phi$ such that $p_1 \in T_1$, $p_2 \in T_2$. There holds $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$. Let $T_1 \subseteq T_2$, then $p_1, p_2 \in T_2$ implies that $\Lambda\{p_1, p_2\} \subset T_2$, hence $\Lambda\{p_1, p_2\} \subset B$.

THEOREM 1. — The lattice $\mathscr{L}(P)$ has the following properties:

i) it is modular,

ii) it is atomistic and its atoms are the singleton subsets of P,

iii) it has the covering property,

iv) if ω is an atom in $\mathscr{L}(P)$ and A is a set of atoms in $\mathscr{L}(P)$ such that $\omega \in \Lambda(A)$, then there exists a finite subset $\{\omega_1, \omega_2, \ldots, \omega_n\} \subset A$ such that $\omega \in \Lambda\{\omega_1, \ldots, \omega_n\}$,

v) to any $S \in \mathscr{L}(P)$ there is $T \in \mathscr{L}(P)$ such that $S \wedge T = \emptyset$ and $S \vee T = P$.

Proof. — *i)* Let $S_1, S_2, S_3 \in \mathscr{L}(P)$, $S_1 \subseteq S_3$. Clearly,

$$(S_1 \vee S_2) \wedge S_3 \supseteq S_1 \vee (S_2 \wedge S_3).$$

Let $p \in (S_1 \vee S_2) \wedge S_3$. Then $p \in S_1 \vee S_2$ implies $p \in \Lambda\{q_1, q_2\}$, $q_1 \in S_1$, $q_2 \in S_2$ (Lemma 2). Then

$$q_1 \in \Lambda\{p, q_2\} \subset S_3 \vee S_2 \quad , \quad q_2 \in \Lambda\{p, q_1\} \subset S_3 \vee S_1.$$

Hence, $q_1 \in (S_3 \vee S_2) \wedge S_1$, $q_2 \in (S_3 \vee S_1) \wedge S_2$, so that $p \in \Lambda\{q_1, q_2\}$ implies

$$p \in [(S_3 \vee S_1) \wedge S_2] \vee [(S_3 \vee S_2) \wedge S_1] = (S_3 \wedge S_2) \vee [(S_3 \vee S_2) \wedge S_1] \subset S_1 \vee (S_2 \wedge S_3).$$

ii) Evidently, the singleton sets $\{ s \}$, $s \in \mathrm{P}$, are atoms in $\mathscr{L}(\mathrm{P})$. If $\mathrm{S} \in \mathscr{L}(\mathrm{P})$, then $\mathrm{S} = \Lambda \{ s : s \in \mathrm{S} \} = \bigvee_{s \in \mathrm{S}} \{ s \}$.

iii) We have to show that for any S, $\mathrm{Q} \in \mathscr{L}(\mathrm{P})$ and $s \in \mathrm{P}$ ($s \notin \mathrm{S}$), $\mathrm{S} \subset \mathrm{Q} \subset \mathrm{S} \vee \{ s \}$ implies $\mathrm{Q} = \mathrm{S}$ or $\mathrm{Q} = \mathrm{S} \vee \{ s \}$. Let $\mathrm{Q} \neq \mathrm{S}$. Then there is $r \in \mathrm{Q}$, $r \notin \mathrm{S}$. From $\mathrm{Q} \subset \mathrm{S} \vee \{ s \}$ it follows $r \in \mathrm{S} \vee \{ s \}$, i. e. there is $p \in \mathrm{S}$ such that $r \in \Lambda \{ p, s \}$ (Lemma 2). From this it follows that $s \in \Lambda \{ r, p \} \subset \mathrm{Q}$. Then $\mathrm{S} \subset \mathrm{Q}$, $s \in \mathrm{Q}$ imply $\mathrm{S} \vee \{ s \} \subset \mathrm{Q}$, i. e. $\mathrm{S} \vee \{ s \} = \mathrm{Q}$.

iv) By Lemma 3, $\Lambda(\mathrm{A}) = \cup \{ \Lambda(\mathrm{S}) : \mathrm{S}$ finite subset of $\mathrm{A} \}$. Hence, for any $\omega \in \Lambda(\mathrm{A})$, there is a finite subset $\mathrm{S} = \{ s_1, \ldots, s_n \} \subset \mathrm{A}$ such that $\omega \in \Lambda(\mathrm{S})$.

v) Let Θ be the set of all $\mathrm{W} \in \mathscr{L}(\mathrm{P})$ such that $\mathrm{S} \wedge \mathrm{W} = \emptyset$. Θ contains $\emptyset$, therefore it is non-empty. If Φ is any ordered set of elements of Θ, let J be the set-theoretic sum of all elements in Φ. By Lemma 4, $\mathrm{J} \in \mathscr{L}(\mathrm{P})$; and, clearly $\mathrm{S} \wedge \mathrm{J} = \emptyset$. From this it follows that $\mathrm{J} \in \Theta$. By Zorn's lemma there is a maximal element $\mathrm{T} \in \Theta$. Now let us consider the element $\mathrm{S} \vee \mathrm{T}$. Let $s \in \mathrm{P}$, $s \notin \mathrm{T}$. Then $\mathrm{T} \subset \Lambda(\mathrm{T} \cup \{ s \})$, and by the maximality of T, $\mathrm{S} \wedge \Lambda(\mathrm{T} \vee \{ s \}) \neq \emptyset$. Let $p \in \mathrm{S} \wedge (\mathrm{T} \vee \{ s \})$. By Lemma 2 then there is $t \in \mathrm{T}$ such that $p \in \Lambda \{ t, s \}$. Then $s \in \Lambda \{ p, t \}$, and from $p \in \mathrm{S}$ and $t \in \mathrm{T}$ it follows that $s \in \mathrm{S} \vee \mathrm{T}$, hence $\mathrm{S} \vee \mathrm{T} = \mathrm{P}$.

We shall say that the states $s_1, \ldots, s_n \in \mathrm{P}$ are independent if $s_i \notin \Lambda \{ s_j : j \neq i \}$, $i, j = 1, 2, \ldots, n$.

If $s_1, \ldots, s_n$ are independent states and q is a state such that

$$s_1 \in \Lambda \{ q, s_2, \ldots, s_n \} \quad \text{then} \quad q \in \Lambda \{ s_1, \ldots, s_n \}.$$

Indeed, there is a minimal subset

$$\mathrm{I} \subset \{ 2, \ldots, n \} \quad \text{such that} \quad s_1 \in \Lambda \{ q, s_i : i \in \mathrm{I} \}.$$

From the MSP we obtain

$$\{ q \} \wedge \Lambda \{ s_1, s_i : i \in \mathrm{I} \} \neq \emptyset,$$

hence

$$q \in \Lambda \{ s_1, s_i : i \in \mathrm{I} \} \subset \Lambda \{ s_1, s_2, \ldots, s_n \}.$$

By permutation of the index set $1, 2, \ldots, n$ we obtain that $s_i \in \Lambda \{ q, s_j : j \neq i \}$ implies $q \in \Lambda \{ s_1, \ldots, s_n \}$.

We say that a finite set of states $\{ s_1, \ldots, s_n \}$ is a basis for $\mathrm{S} \in \mathscr{L}(\mathrm{P})$ if $s_1, s_2, \ldots, s_n$ are independent and $\mathrm{S} = \Lambda \{ s_1, \ldots, s_n \}$. It can be shown by the same method as in [*6*] that if $\{ s_1, \ldots, s_n \}$ and $\{ p_1, \ldots, p_k \}$ are bases for S then $n = k$. If $\mathrm{S} \in \mathscr{L}(\mathrm{P})$ has a basis $\{ s_1, \ldots, s_n \}$ then n is called the dimension of S and is denoted by $d(\mathrm{S}) = n$. If S has a basis, we say that S is finite dimensional. Recall that a dimension function on a lattice K is a real valued function on K with the properties:

i) $d(\emptyset) = 0$, $d(a) \geq 0$ for all $a \in \mathrm{K}$,

ii) if $a \leq b$ and $a \neq b$, then $d(a) < d(b)$, *iii)* $d(a \vee b) + d(a \wedge b) = d(a) + d(b)$ for all $a, b \in K$.

The following proposition can be proved analogically as Theorem 3.10 in [*6*].

PROPOSITION 1. — Let $S \in \mathscr{L}(P)$ be finite dimensional. Then d is a dimension function on $[\emptyset, S] = \{ T \in \mathscr{L}(P) : T \subseteq S \}$.

PROPOSITION 2. — Let $S \in \mathscr{L}(P)$ be finite dimensional. Then $[\emptyset, S]$ is a complemented modular lattice.

Proof. — It follows from Theorem 1.

We can define in the set $\mathscr{L}(P)$, as in a projective geometry, the notions of lines and planes. An element $S \in \mathscr{L}(P)$ is a line if $d(S) = 2$, and it is a plane if $d(S) = 3$. If $s_1, s_2 \in P$ are distinct states, then $d(\wedge \{ s_1, s_2 \}) = 2$ and hence $\wedge \{ s_1, s_2 \}$ is a line. If S_1 and S_2 are distinct lines and $S_1 \wedge S_2 \neq \emptyset$ then $d(S_1 \wedge S_2) = 1$. In this case the identity

$$d(S_1 \vee S_2) = d(S_1) + d(S_2) - d(S_1 \wedge S_2)$$

shows that $S_1 \vee S_2$ is a plane. This yields a new formulation of the SP: the superposition principle holds if and only if every line in $\mathscr{L}(P)$ has at least three distinct points lying on it. In this case $[\emptyset, S]$ is a geometry for any finite $S \in \mathscr{L}(P)$ [*12*, Th. 2.15, p. 30).

THEOREM 2. — Let (L, P) be a quantum logic such that the superposition principle (SP) and the minimal superposition principle (MSP) hold and let there exist at least four independent states in P. Then there exist a division ring K and a vector space V over K, such that the set $\mathscr{L}(P)$ is isomorphic to the lattice $\mathscr{L}(V)$ of all linear subspaces of V (in the sense that there exists a bijection between $\mathscr{L}(P)$ and $\mathscr{L}(V)$ that preserves their order structure). $\mathscr{L}(V)$ is the set of all linear subspaces of V ordered under set-theoretical inclusion and meet and join operations are defined by

$$\vee M_i = \Sigma M_i \quad , \quad M_i \in \mathscr{L}(V) \quad , \quad i = 1, 2, \ldots$$
$$\wedge M_i = \cap M_i \quad , \quad M_i \in \mathscr{L}(V) \quad , \quad i = 1, 2, \ldots$$

Proof. — Proof of this theorem follows from Theorem 1 and Theorem in [*1*, Ch. VII, § 6, p. 375].

In [*10*], there is shown that the set P can be written as the union of sectors if and only if $\wedge \{ p, q, r \} \neq \wedge \{ p, q \} \cup \wedge \{ q, r \}$ for any distinct states $p, q, r \in P$ such that $p \approx q$, $q \approx r$, $r \notin \wedge \{ p, q \}$, where $p \approx q$ means that there is a state $u \in P$, $u \neq p, q$ such that $u \in \wedge \{ p, q \}$. Now we shall show that this condition is always fulfilled.

THEOREM 3. — Let (L, P) be a quantum logic such that the MSP holds. Let p, q, r be distinct states in P such that $p \approx q$, $q \approx r$ and $r \notin \wedge \{ p, q \}$.

Then $\Lambda\{p, q, r\} \neq \Lambda\{p, q\} \cup \Lambda\{q, r\}$, so that P can be written as the union of sectors.

Proof. — From $p \approx q$ and $q \approx r$ it follows that there are $s_1 \in \Lambda\{p, q\}$, $s_1 \neq p, q$ and $s_2 \in \Lambda\{q, r\}$, $s_2 \neq q, r$. As

$$\Lambda\{s_1, s_2\} \vee \Lambda\{p, r\} \subset \Lambda\{p, q, r\} \quad , \quad d(\Lambda\{s_1, s_2\} \vee \Lambda\{p, r\}) \leq 3.$$

The relation $d(a \wedge b) = d(a) + d(b) - d(a \vee b)$ then implies that $d(\Lambda\{s_1, s_2\} \wedge \Lambda\{p, r\}) \geq 1$. But if $\Lambda\{s_1, s_2\} = \Lambda\{p, r\}$, then $s_1 \in \Lambda\{p, r\} \wedge \Lambda\{p, q\}$ implies $s_1 = p$, a contradiction. Hence, $d(\Lambda\{s_1, s_2\} \wedge \Lambda\{p, r\}) = 1$. Let $\Lambda\{s_1, s_2\} \wedge \Lambda\{p, r\} = \{t\}$. We shall show that $t \notin \Lambda\{p, q\}$, $t \notin \Lambda\{q, r\}$. Indeed, if $t \in \Lambda\{p, q\}$, then $q \in \Lambda\{t, p\}$, but $t \in \Lambda\{p, r\}$ implies $q \in \Lambda\{p, r\}$, a contradiction. Analogically we show that $t \notin \Lambda\{q, r\}$. Hence, we found $t \in \Lambda\{p, q, r\}$, $t \notin \Lambda\{p, q\}$, $t \notin \Lambda\{q, r\}$.

We shall call the elements of $\mathscr{L}(P)$ the subspaces of P.

3. CLOSED SUBSPACES OF P

Let us set $\mathscr{F}(P) = \{S \subset P : S = \overline{S}\}$. Clearly, $\Lambda(\overline{S}) = \overline{S}$, so that $\mathscr{F}(P) \subset \mathscr{L}(P)$. The map $S \mapsto \overline{S}$ is a closure operation in the sense of Birkhoff [3], so that the set $\mathscr{F}(P)$ becomes a complete lattice whose join and meet operations are given by

$$\bigvee_j S_j = \Big(\bigcup_j S_j\Big)^{-} \quad \text{and} \quad \bigwedge_j S_j = \bigcap_j S_j \ [5].$$

The proposition $a \in L$ is said to be a carrier of a state m, if

i) $m(a) = 1$,

ii) $b \not\perp a$ implies $m(b) > 0$.

Notice that the carrier of a state $m \in P$, whenever it exists, is uniquely determined by m, since it is the smallest element of the set L_m. The carrier of m, if it exists, will be denoted by carr m.

In the following we shall suppose that each state $p \in P$ has the carrier.

LEMMA 5. — If carr p is the carrier of the state $p \in P$, then $q(\text{carr } p) < 1$ for every pure state $q \neq p$, $q \in P$.

Proof. — Suppose $q(\text{carr } p) = 1$ for some $q \neq p$. Then $p(a) = 1$ implies $q(a) = 1$, $a \in L$, so that $L_p \subset L_q$. But then $q = p$, a contradiction.

PROPOSITION 3. — *i*) The logic L is atomistic and the correspondence carr : $p \mapsto \text{carr } p$, $p \in P$, is a one-to-one mapping of the set P onto the set of all atoms of the logic L.

ii) For every non-zero proposition $a \in L$ one has $a = \vee\{\text{carr } p : p \in P_a\}$.

Proof. — See [7].

We shall say that two states m_1, m_2 are mutually orthogonal and write $m_1 \perp m_2$ if for some proposition $a \in L$ one has $m_1(a) = 1$ and $m_2(a) = 0$ [5]. For any $S \subset P$, define $S^\perp$ to be the set of all pure states $p \in P$ such that $p \perp S$ (i. e. $p \perp q$ for all $q \in S$). Obviously $S \subset S^{\perp\perp}$. For the empty set $\emptyset$ we put $\emptyset^\perp = P$.

PROPOSITION 4. — For every non-empty subset $T \subset P$ one has $T^{\perp\perp} = \overline{T}$.

Proof. — See [7].

It can be easily seen that the map $\perp : T \mapsto T^\perp$ is an orthocomplementation in the set $\mathscr{F}(P)$ of all closed subspaces of P.

THEOREM 4. — For every $a \in L$, the set $P_a = \{ s \in P : s(a) = 1 \}$ belongs to $\mathscr{F}(P)$ and the mapping $a \mapsto P_a$ is an orthoinjection of the logic L into the set $\mathscr{F}(P)$.

Proof. — See [7].

THEOREM 5. — Let (L, P) be a quantum logic such that SP and MSP hold, and let there be at least four independent states in P. In addition, let each state $p \in P$ have the carrier carr $p \in L$. Then there exist a division ring K, an involutorial antiautomorphism $* : \lambda \to \lambda^*$ of K, a vector space V over K and a Hermitian form f, such that $\mathscr{F}(P)$ and $\mathscr{L}_f(V)$ are isomorphic (i. e. there exist, between them, a bijection which preserves order and orthocomplementation), where $\mathscr{L}_f(V)$ is the set of all subspaces of V, closed with respect to the form f.

Proof. — By Theorem 2 there exist a division ring K and a vector space V over K, such that the set $\mathscr{L}(P)$ of all subspaces of P is isomorphic to the lattice $\mathscr{L}(V)$ of all linear subspaces of V. If the set $\mathscr{L}(P)$ is finite dimensional, then V is finite dimensional. In this case $\mathscr{L}(P) = \mathscr{F}(P)$. Since $\mathscr{F}(P)$ is orthocomplemented, $\mathscr{L}(V)$ has an orthocomplementation induced by the one of $\mathscr{F}(P)$; then Theorem of Birkhoff and von Neumann [12] ensures the existence of a pair $(*, f)$, such that

$$M^\perp = M^0 = \{ v \in V : f(v, w) = 0, \text{ for all } w \in M \} \quad , \quad M \in \mathscr{L}(V).$$

Every subspace of V is closed with respect to the form f, and $\mathscr{L}(V)$ and $\mathscr{L}_f(V)$ coincide, so that the isomorphism between $\mathscr{L}(P)$ and $\mathscr{L}(V)$ preserves orthocomplementation.

Consider now the case of infinite dimension. We give a sketch of the proof, as in [2]. For the details see [8]. Let us denote by ω the isomorphism between $\mathscr{L}(P)$ and $\mathscr{L}(V)$. For every finite dimensional subspace M of V there exists a finite $T \in \mathscr{L}(P)$, such that $\omega(T) = M$. ω is an isomorphism between

$[\emptyset, T] = \{ S \in \mathscr{L}(P) : S \subset T \}$ and $\mathscr{L}(M)$, the mapping $S \mapsto S^{\perp} \wedge T$ is an orthocomplementation of $[\emptyset, T]$, hence $\mathscr{L}(M)$ has an orthocomplementation induced by the one of $[\emptyset, T]$.

Let M_0 be a fixed 4-dimensional subspace of V. Since $\mathscr{L}(M_0)$ is orthocomplemented, there exist, by the theorem of Birkhoff and von Neumann, an involutorial antiautomorphism $\lambda \mapsto \lambda^*$ and a Hermitian form f_0 on M_0, such that for $\omega(S) \in \mathscr{L}(M_0)$,

$$\omega(S)^{\perp} \wedge M_0 = \{ w \in M_0 : f_0(v, w) = 0 \text{ for all } v \in \omega(S) \}.$$

For every finite dimensional subspace M of V containing M_0, there exists a pair $(\bar{*}, f_M)$ such that, for all $\omega(S) \in \mathscr{L}(M)$,

$$\omega(S)^{\perp} \wedge M = \{ w \in M : f_M(v, w) = 0 \text{ for all } v \in \omega(S) \}.$$

Owing to the unicity of the pair $(\bar{*}, f_0)$ in M_0, there exists a $\gamma \in K$ such that $\lambda^{\bar{*}} = \gamma^{-1}\lambda^*\gamma$ and $f_M(v, w) = f_0(v, w)\gamma$ for every $v, w \in M_0$. Then, substituting $(\bar{*}, f_M)$ by $(*, f_M\gamma^{-1})$, we get a unique Hermitian form f_M (with respect to $*$) which induces the orthocomplementation of $\mathscr{L}(M)$ and $f_M = f_0$ on M_0 If $M_0 \subset M_1 \subset M_2$, then $f_{M_1} = f_{M_2}$ on M_1.

For every pair $v, w \in V$ define

$$f(v, w) = f_{M_0 + Kv + Kw}(v, w).$$

It can be shown that the function f so defined is a Hermitian form on V, that the image of the mapping ω is just $\mathscr{L}_f(V)$ and that ω preserves order and orthocomplementation between $\mathscr{F}(P)$ and $\mathscr{L}_f(V)$.

COROLLARY. — There exists an orthoinjection of the logic L into the set $\mathscr{L}_f(V)$.

Proof. — By theorem 4, the mapping $j : a \mapsto P_a$ is an orthoinjection of L into the set $\mathscr{F}(P)$. Then, by Theorem 5, the mapping $\omega \circ j$ is an orthoinjection of L into $\mathscr{L}_f(V)$.

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