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SU(n) meronlike configurations and Gribov ambiguities

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ABSTRACT. — The study of meron and meronlike configurations, started in a previous paper (I), is generalized to SU(n) gauge groups for a particular class of ansatz. A quite simple canonical formulation is found to be possible. General equations are given and the SU(4) case is discussed in detail as an example. A parallel discussion of Gribov ambiguities is given for SU(n).

1. INTRODUCTION

In Ref. 1 (referred to hereafter as I) we studied merons and meronlike configurations for SU(3) and SU(4) gauge groups. In this paper, selecting a class of ansatz, we generalize the results to SU(n) for arbitrary n. As before, we study, for flat Euclidean metric, certain types of singular solutions of the form

$$A_\mu = \frac{1}{2} i(\partial_\mu U)U^{-1} \tag{1.1}$$

where U is a gauge transformation operator for SU(n). Our detailed discussion in (I) of the effect of a certain class of singular gauge transformations on A_μ thus defined will hold for all n. Thus in section 2 we will define a « neutral gauge » (with $A_0 = 0$), useful in deriving the equations of motion; which can again be shown, as in (I), to correspond to zero topological charge. We will not repeat in this paper the discussions concerning this point. We will separately derive a general formula for the topological charge (for our ansatz) corresponding to the « meron gauge » (1.1). A general formula for the action will also be given.

Our ansatz, given in Section 2, exploits systematically the maximal representation of an SU(2) subgroup (maximal isospin, for example) and the generators of the Cartan subalgebra in SU(n). Thus we utilize the generators of isospin $J = \frac{n-1}{2}$ for SU(n). Lesser values of J already arise for SU($n - 1$) and so on and hence the maximal value is the essentially new possibility that arises on the passage from SU($n - 1$) to SU(n). There are of course other possibilities, which we exclude from our present study. Thus, for example, our general results for SU(n) include the SU(2) and SU(3) examples treated in (I) but *not* the SU(4) ones. As one moves up from SU(3) to SU(4), among others, two new features arise, namely

(1) there are two mutually commuting SU(2) subgroups in SU(4) algebra and

(2) the maximal isospin moves from 1 to $\frac{3}{2}$.

In (I) we exploited the possibility (1) while here we will study in detail (2). The number of such possibilities proliferates as n increases. In this paper we concentrate on one class of ansatz for which the results can be generalized fairly simply and in a canonical fashion to arbitrary n.

We note here a general feature that will be seen to arise from our equations of motion. For all n, a particular class of solutions will be found to be such that A_μ is expressed finally in terms of

$$u_j = \frac{1}{\lambda_j} u \qquad (j = J - 1, \ldots, -J) \tag{1.2}$$

where the constant $\lambda_j \geq 1$ and

$$(\partial_t^2 + \partial_r^2)u + \frac{u}{r^2}(1 - u^2) = 0 . \tag{1.3}$$

This situation discussed in detail for $n = 4$ can easily seen to be a general one. (In fact for our class of solutions $\lambda_j = (a_j/b_j)^{1/2}$ where $a_j < b_j$ and both are positive integers.)

Equations (1.3) arises for SU(2) with the only possibility $\lambda_{-1/2} = 1$. So our starting point can be *any* solution of (1.3), single meron, meron-antimeron or possible multimeron solutions. As an illustrative example (permitting simple, explicit calculations) we will use in the following sections only the single meron solution namely

$$u = \frac{t}{\sqrt{t^2 + r^2}} \tag{1.4}$$

though we might also have used the meron-antimeron solution of (1.3). The solution (1.4) will be injected into (1.2) to construct explicitely different simple possibilities. The point we illustrate is the following one. For all $\lambda_j = 1$ we get back, apart from an overall factor, depending on J, essentially the SU(2) case where the action

$$\mathrm{S} \simeq \int_0^\infty \frac{dx}{x} \tag{1.5}$$

diverges logarithmically. But where there are one or more λ_j's > 1 the leading divergence will be found to be of the type

$$\simeq \int_0^\infty \frac{dx}{x} \int_0^\pi \frac{d\alpha}{\sin^2 \alpha}. \tag{1.6}$$

This feature was noted in (I). Here, this very particular class of divergent solutions can be seen to be present systematically for all n (the generalization of our discussion for $n = 4$ being straightforward). Thus the generalization of the typical SU(2) boundary condition

$$u = 1 \qquad \text{for} \quad r = 0 \tag{1.7}$$

to

$$u_j = \frac{1}{\lambda_j} \leq 1 \qquad \text{for} \quad r = 0 \qquad (j = \mathrm{J} - 1, \ldots, -\mathrm{J}) \tag{1.8}$$

is always seen to be accompanied by a very definite type of change in the divergence (namely (1.6) instead of (1.5)) though a variety of choices λ_j becomes possible as n increases. Thus we establish, for the general case, the presence of this simple class in the hierarchy of divergent solutions. We discuss also the corresponding charges which will be found to depend on the values of $\cos^{-1}\left(\frac{1}{\lambda_j}\right)$.

We also give a parallel treatment of SU(n) Gribov ambiguities by using a related ansatz. The analogies and the differences with the meron solutions are thus exhibited. We conclude with a few brief remarks on the limit $n \to \infty$.

2. A CLASS OF SU(n) MERON ANSATZ

Definitions.

Let J_1, J_2, J_3 be the generators of the maximal representation of SU(2) in SU(n). Thus for SU(4) one has, for example, the representation corresponding to isospin 3/2 and more generally $J = \frac{n-1}{2}$. The matrix elements are taken to be the standard ones, namely

$$\begin{aligned} &(J_3)_{jk} = (j)\delta_{jk} \qquad (j = J, J-1, \ldots, -J) \\ &(J_1 + iJ_2)_{jk} = [(J+j)(J-j+1)]^{1/2}\delta_{j-1,k} = (J_1 - iJ_2)_{kj}\,. \end{aligned} \tag{2.1}$$

Let C_l ($l = 1, 2, \ldots, n-1$) be the generators of the Cartan subalgebra, diagonal matrices of zero trace. Then

$$e^{-i(\beta_l C_l)} = \operatorname{diag}\left\{ e^{-i\alpha_J}, e^{-i\alpha_{J-1}}, \ldots, e^{-i\alpha_{-J}} \right\} \tag{2.2}$$

where
$$\sum_{j=-J}^{J} \alpha_j = 0$$

the α_j's being suitable linear combinations of the β_l's.

We will assume that $\alpha_j = \alpha_j(r, t)$ (2.3) i. e. they are functions of the time t and r $\left(= \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}\right)$ only.

The ansatz.

Let

$$\begin{aligned} V &= e^{-i\varphi J_3} e^{-i\theta J_2} \\ U &= e^{-i2V(\beta.C)V^{-1}} = Ve^{-i2(\beta.C)}V^{-1} \end{aligned} \tag{2.4}$$

where θ, φ are the spherical angles and $\beta.C \equiv \beta_l C_l$ ($l = 1, 2, \ldots, n-1$). We adopt the ansatz

$$A_\mu = \frac{1}{2}(i\partial_\mu U)U^{-1}\,. \tag{2.5}$$

We define this choice as the « meron gauge ».

The neutral gauge and equations of motion.

In order to derive the equations of motion conveniently we pass (from 2.5) to what we define to be our « neutral gauge » by transforming with the operator

$$e^{i\beta.C}V^{-1}\,. \tag{2.6}$$

This gives

$$A_0\,(\equiv A_t) = 0$$
$$A_r = 0$$
$$A_\theta = -\frac{1}{2}\,[e^{-i\beta.C}J_2 e^{i\beta.C} + e^{i\beta.C}J_2 e^{-i\beta.C}] \tag{2.7}$$
$$A_\varphi = \frac{1}{2}\,[e^{-i\beta.C}(J_1 \sin\theta - J_3\cos\theta)e^{i\beta.C} + e^{i\beta.C}(J_1\sin\theta - J_3\cos\theta)e^{-i\beta.C}]\,.$$

Using (2.1), (2.2) and (2.7) one obtains the matrix elements

$$\begin{aligned}(A_\theta)_{jk} &= -\,(J_2)_{jk}\cos(\alpha_j - \alpha_k)\\ (A_\varphi)_{jk} &= (J_1\sin\theta - J_3\cos\theta)_{jk}\cos(\alpha_j - \alpha_k)\\ &= \sin\theta(J_1)_{jk}\cos(\alpha_j - \alpha_k) - (j\cos\theta)\delta_{jk}\,.\end{aligned} \tag{2.8}$$

Now from the definition

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu] \tag{2.9}$$

one gets using (2.7) and (2.8)

$$\begin{aligned}&F_{tr} = 0\\ &F_{t\theta} = \partial_t A_\theta\,, \qquad F_{t\varphi} = \partial_t A_\varphi\\ &F_{r\theta} = \partial_r A_\theta\,, \qquad F_{r\varphi} = \partial_r A_\varphi\end{aligned} \tag{2.10}$$

i. e.

$$(F_{t\theta})_{jk} = \;[(J_2)_{jk}\sin(\alpha_j - \alpha_k)](\dot\alpha_j - \dot\alpha_k) \tag{2.11}$$

and so on. (We always use the notations $\partial_t\alpha_k \equiv \dot\alpha_k$ and $\partial_r\alpha_k \equiv \alpha'_k$.) $F_{\theta\varphi}$ turns out to be *diagonal* and one gets finally

$$\begin{aligned}(F_{\theta\varphi})_{jk} = \left(\frac{1}{4}\sin\theta\delta_{jk}\right)[(2j) - (J+j)(J-j+1)\cos 2(\alpha_j - \alpha_{j-1})&\\ + (J-j)(J+j+1)\cos 2(\alpha_{j+1} - \alpha_j)]\,.&\end{aligned} \tag{2.12}$$

The equations of motion are now obtained by injecting the results (2.7)-(2.12) in

$$\frac{1}{\sqrt{g}}\,\partial_\mu(\sqrt{g}\,F_{\mu\nu}) + i\,[A_\mu, F_{\mu\nu}] = 0 \tag{2.13}$$

where $\sqrt{g} = r^2\sin\theta$.

Defining

$$u_j \equiv \cos(\alpha_{j+1} - \alpha_j) \tag{2.14}$$

one obtains finally from (2.13)

$$\begin{aligned}(\partial_t^2 + \partial_r^2)u_j + \frac{u_j}{r^2}\Bigg[1 + \frac{1}{2}(J+j)(J-j+1)u_{j-1}^2 - (J-j)(J+j+1)u_j^2&\\ + \frac{1}{2}(J-j-1)(J+j+2)u_{j+1}^2\Bigg] = 0 \quad (j = J-1, J-2, \ldots, -J)&\end{aligned} \tag{2.15}$$

which is of the general form

$$\Delta u_j + \frac{u_j}{r^2}[1 + \tilde{u}Q_j u] = 0 \qquad (\Delta \equiv (\partial_t^2 + \partial_r^2)) \tag{2.16}$$

where u is a column with elements u_k, $\tilde{u}$ is the transposed row and Q_j is a j dependent matrix, whose explicit form is given by (2.15). Let us now consider some particular cases to illustrate the content of (2.15). We note at once that assuming

$$u_{J-1} = u_{J-2} = \dots = u_{-J} = u \tag{2.17}$$

say, the system (2.15) degenerates into the single one

$$\Delta u + \frac{u}{r^2}(1 - u^2) = 0 \tag{2.18}$$

which is the known equation [I] for SU(2) gauge group. For $n = 2$ there is only one u to start with. For $n > 2$ the condition (2.17) may be assured by setting

$$\alpha_j = j\alpha \tag{2.19}$$

where $\cos(\alpha(r, t))$ satisfies the SU(2) equation (2.18). But (2.19) is not the only possibility for $n > 2$. One sees at once that *for even n*, one can satisfy both (2.2) and (2.17) by setting

$$\alpha_J = -\alpha_{J-1} = \alpha_{J-2} = -\alpha_{J-3} = \dots = -\alpha_{-J} = \frac{\alpha}{2} \tag{2.20}$$

where $\cos \alpha$ is any solution of the SU(2) equation (2.18). As will be seen later the cases (2.19) and (2.20) usually leads to different topological charges.

For SU(3), $J = 1$, and (2.15) reduces to

$$\begin{aligned} \Delta u_0 + \frac{u_0}{r^2}(1 + u_{-1}^2 - 2u_0^2) &= 0 \\ \Delta u_{-1} + \frac{u_{-1}}{r^2}(1 + u_0^2 - 2u_{-1}^2) &= 0\,. \end{aligned} \tag{2.21}$$

Setting

$$\frac{1}{2}\theta_+ = (\alpha_1 - \alpha_0) \qquad \text{and} \qquad \frac{1}{2}\theta_- = (\alpha_0 - \alpha_{-1}) \tag{2.22}$$

we get back to the SU(3) equations discussed at length in I.

For SU(4), $J = \frac{3}{2}$ and denoting

$$u_{1/2} \equiv a\,, \qquad u_{-1/2} \equiv c\,, \qquad u_{-3/2} \equiv b \tag{2.23}$$

one has from (2.15)

$$\Delta a + \frac{a}{r^2}(1 - 3a^2 + 2c^2) = 0$$

$$\Delta b + \frac{b}{r^2}(1 - 2b^2 + 2c^2) = 0 \qquad (2.24)$$

$$\Delta c + \frac{c}{r^2}\left(1 + \frac{3}{2}(a^2 + b^2) - 4c^2\right) = 0 .$$

We note the following cases.

CASE 1

For $a = b = c$, one has

$$\Delta a + \frac{a}{r^2}(1 - a^2) = 0 . \qquad (2.25)$$

Hence any solution u of (2.18) gives for

$$a = b = c = u \qquad (2.26)$$

a solution of (2.24).

(In the following cases u will always denote any solution of (2.18).)

CASE 2

For $a = b = 0$

$$\Delta c + \frac{c}{r^2}(1 - 4c^2) = 0 . \qquad (2.27)$$

Hence,

$$a = b = 0 , \qquad c = \frac{1}{2} u \qquad (2.28)$$

is a solution.

CASE 3

For $b = 0$, $c = 0$

$$\Delta a + \frac{a}{r^2}(1 - 3a^2) = 0 . \qquad (2.29)$$

Hence

$$a = \frac{1}{\sqrt{3}} u , \qquad (2.30)$$

$b = 0 = c$ is a solution.

(Due to the symmetry of (2.24) the interchange $a \rightleftarrows b$ always gives a solution. The signs of a, b, c can moreover be reversed independently. These remarks hold for all the cases.)

CASE 4

$c = 0$, $a = b$ again leads to

$$\Delta a + \frac{a}{r^2}(1 - 3a^2) = 0 . \qquad (2.31)$$

Hence

$$a = b = \frac{1}{\sqrt{3}} u , \qquad (2.32)$$

$c = 0$ is a solution.

CASE 5

For $a = 0$, $3b^2 = 4c^2$

$$\begin{aligned} \Delta b + \frac{b}{c^2}\left(1 - \frac{3}{2} b^2\right) &= 0 \\ \Delta c + \frac{c}{r^2}(1 - 2c^2) &= 0 . \end{aligned} \qquad (2.33)$$

Hence

$$a = 0 , \qquad b = \sqrt{\frac{2}{3}} u , \qquad c = \frac{1}{\sqrt{2}} u \qquad (2.24)$$

is a solution.

3. CHARGE AND ACTION

Charge.

We have utilized the neutral gauge to easily obtain the equations of motion. As discussed in detail in (I), this gauge corresponds to *zero topological charge*. We have now to calculate the charge in the « meron gauge » (2.5). But in fact we will utilize an « intermediate gauge » by transforming (2.5) by V^{-1} alone (instead of (2.6)) which does not alter the topological charge. In this gauge one has

$$\begin{aligned} A_t &= \dot{\beta}_l C_l = \mathrm{diag}\,(\dot{\alpha}_J, \ldots, \dot{\alpha}_{-J}) \\ A_r &= \beta_l' C_l = \mathrm{diag}\,(\alpha_J', \ldots, \alpha'_{-J}) \\ A_\theta &= -\frac{1}{2} e^{-i2\beta.C} J_2 e^{i2\beta.C} - \frac{1}{2} J_2 \\ A_\varphi &= \left(\frac{1}{2}\sin\theta\right)[e^{-i2\beta.C} J_1 e^{i2\beta.C} + J_1] - J_3 \cos\theta . \end{aligned} \qquad (3.1)$$

Thus,

$$\begin{aligned} (A_\theta)_{jk} &= -\frac{1}{2}(J_2)_{jk}[e^{-i2(\alpha_j - \alpha_k)} + 1] \\ (A_\varphi)_{jk} &= \frac{1}{2}\sin\theta (J_1)_{jk}[e^{-i2(\alpha_j - \alpha_k)} + 1] - (J_3)_{jk}\cos\theta . \end{aligned} \qquad (3.2)$$

The charge is defined as

$$q_p = \frac{1}{8\pi^2}\int d^4x(\partial_\mu \xi_\mu) \tag{3.3}$$

where

$$\xi_\mu = \varepsilon_{\mu\alpha\beta\gamma}\,\mathrm{Tr}\left[\mathrm{A}_\alpha\partial_\beta\mathrm{A}_\gamma + \frac{i2}{3}\mathrm{A}_\alpha\mathrm{A}_\beta\mathrm{A}_\gamma\right] \tag{3.4}$$

($\varepsilon_{0123} = 1$).

It can be shown that one finally obtains from (3.1), ..., (3.4)

$$q_p = \frac{1}{4\pi}\left[\sum_{j=-\mathrm{J}}^{\mathrm{J}-1}(\mathrm{J}-j)(\mathrm{J}+j+1)\int dtdr(\partial_t\omega_j' - \partial_r\dot{\omega}_j)\right] \tag{3.5}$$

where $\omega_j \equiv (\alpha_{j+1} - \alpha_j)$.

In particular we note that for

$$\omega_{-\mathrm{J}} = \omega_{-\mathrm{J}+1} = \ldots = \omega_{\mathrm{J}-1} = \omega \tag{3.6}$$

$$q_{\mathrm{P}} = \frac{1}{4\pi}\left(\frac{2}{3}\mathrm{J}(\mathrm{J}+1)(2\mathrm{J}+1)\right)\int dtdr(\partial_t\omega' - \partial_r\dot{\omega}), \tag{3.7}$$

where $u = \cos\omega$ is any solution of (2.18).

This permits us to immediately evaluate the charge for a class of known (SU(2)-like) solutions in the context of SU(n) by setting $\mathrm{J} = \left(\frac{n-1}{2}\right)$.

More generally, we will consider the case

$$\cos\omega_j = \frac{1}{\lambda_j}u \qquad \text{with} \quad \lambda_j \geq 1 \tag{3.8}$$

where u is a solution of (2.18).

LEMMA. — For the important case of SU(2) single meron solution $\left(u = \frac{t}{\sqrt{r^2+t^2}} \equiv \frac{t}{x}\right)$, which we will use as an example, it is useful to note the following result.

Let

$$\omega = \frac{t}{\lambda x} = \frac{1}{\lambda}\cos\alpha \tag{3.9}$$

where $\lambda \geq 1$, $t = x\cos\alpha$ and $r = x\sin\alpha$.

Then

$$\int drdt(\partial_t\omega' - \partial_r\dot{\omega}) = \left(2\int_0^\infty \delta(x)dx\right)\left(\int_{\alpha=0}^{\pi}\frac{\sin\alpha}{\sqrt{\lambda^2-\cos^2\alpha}}\,d\alpha\right)$$

$$= \int_0^\pi \frac{\sin\alpha d\alpha}{\sqrt{\lambda^2-\cos^2\alpha}} = \int_{\omega_0}^{\omega_\pi} d\omega = (\omega_\pi - \omega_0)\,. \tag{3.10}$$

Here we have used the notations

and
$$\begin{aligned} \omega &= \omega_0 \quad \text{for} \quad \alpha = 0 \\ \omega &= \omega_\pi \quad \text{for} \quad \alpha = \pi . \end{aligned} \tag{3.11}$$

(Concerning the x-integration see the discussion in (I), leading to its equation (2.11). Compare also the SU(3) case of (I) with $\lambda = 1/\sqrt{2}$ in its equation (3.23).)

Hence, *using the determination*

$$0 \leq \omega \leq \pi$$

one gets

$$\int drdt(\partial_t \omega' - \partial_r \dot{\omega}) = \left\{ \pi - 2 \cos^{-1}\left(\frac{1}{\lambda}\right) \right\}. \tag{3.12}$$

A change of sign of ω_j ($\omega_j \to -\omega_j$) simply reverses the sign of the contribution of the term in question to the total change and in fact amounts to a singular gauge transformation as discussed in (I). The result (3.12) will be used in the examples to follow.

Action.

The action is invariant under the gauge transformations considered. Using any gauge, but most easily in the neutral gauge one finds

$$\begin{aligned} S &= \frac{1}{2}\int (\mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu})r^2 \sin\theta dt dr d\theta d\varphi \\ &= 2\pi \int drdt \Bigg[\sum_{j=-J}^{J-1} (J - j)(J + j + 1)(\dot{u}_j^2 + u_j'^2) \\ &\quad + \frac{1}{r^2} \sum_{j=-J}^{J} \Big\{ j - \frac{1}{2} u_{j-1}^2 (J + j)(J - j + 1) \\ &\quad + \frac{1}{2} u_j^2 (J - j)(J + j + 1) \Big\}^2 \Bigg]. \end{aligned} \tag{3.13}$$

A class of SU(4) examples.

For $n = 2$ and 3 one easily finds back the results of (I). For $n = 4$ we consider here as examples again the class of solutions presented in section 2 (which may be compared to another class of solutions for SU(4) given in (I)). We now consider, in order, the previously noted solutions of (2.24).

In the following discussion we will always take as an illustrative example the single meron solution of (2.24)

$$u = \frac{t}{x} = \cos\alpha . \tag{3.14}$$

This will be the starting point.

CASE 1

Let

$$a = b = c = \cos\alpha \tag{3.15}$$

corresponding to the two possibilities (2.19) and (2.10) we get from (3.5)

$$q_p = \frac{5}{2} \quad \text{and} \quad q_p = \frac{1}{2} \tag{3.16}$$

respectively. Here we have the simplest case $\lambda = 1$ of (3.12), but in the second case there is negative determination for one ω. Reversing all the signs in each case one evidently obtains

$$q_p = -\frac{5}{2} \quad \text{and} \quad q_p = -\frac{1}{2} \tag{3.17}$$

respectively. We will not mention such possibilities explicitely each time.

The action is, as for SU(2), logarithmically divergent i. e.

$$S \simeq \int_0^\infty \frac{dx}{x}.$$

We may at once mention that for *all the other cases* (2, 3, 4 and 5), using (3.13) one finds that the action has the same type of divergence as for the SU(3) action in equation (3.15) of (I). That is, the leading divergence is given by a term

$$\simeq \int_0^\pi \frac{dx}{x} \int_0^\pi \frac{d\alpha}{\sin^2\alpha}.$$

(The other divergent term is of the type (3.18).)

Let us now look at the charges which are all directly given by (3.7) on using (3.12).

CASE 2

$$q_P = \frac{1}{3}. \tag{3.20}$$

Thus amusingly enough the charge is like that of a « topological quark » [2]. The other charges are however are however not even rational [3].

CASE 3

$$q_p = \frac{3}{4\pi}\left(\pi - 2\cos^{-1}\frac{1}{\sqrt{3}}\right). \tag{3.21}$$

CASE 4

$$q_P = \frac{3}{2\pi}\left(\pi - 2\cos^{-1}\frac{1}{\sqrt{3}}\right). \tag{3.22}$$

(Or $q_P = 0$ for $\omega_{1/2} = -\omega_{-3/2}$ instead of $\omega_{1/2} = \omega_{-3/2}$.)

CASE 5

$$q_P = \frac{1}{4\pi}\left(5\pi - 6\cos^{-1}\sqrt{\frac{2}{3}}\right). \tag{3.23}$$

4. GRIBOV AMBIGUITIES

In this section we generalize the treatment of [4] to study SU(n) ambiguities. For SU(2) and SU(3) there are close relations between meron solutions and Gribov vacua. The general situation can be studied from a parallel treatment of the two topics using the same type of ansatz and techniques. There will be, of course, essential differences in two contexts. We now consider static potentials satisfying

$$\vec{\nabla}\,.\,\vec{A} = 0 \tag{4.1}$$

where

$$\vec{A} = i(\vec{\nabla}u)u^{-1} \tag{4.2}$$

and

$$u = Ve^{-i\beta.C}V^{-1} \tag{4.3}$$

V and β.C are defined as in (2.4), but now the β's and hence the α's are functions of r alone. (There is now no over all factor 1/2 since (4.2) is pure gauge. In (4.3) we have taken β.C instead of 2β.C merely for convenience.) The charges corresponding to the vacua are defined through

$$Q = -\frac{i}{24\pi^2}\int d^3x\varepsilon_{ijk}\,\mathrm{Tr}\,(A_iA_jA_k) \qquad (i = 1, 2, 3)\,. \tag{4.4}$$

Passing to spherical components one finally obtains

$$Q = -\frac{i}{8\pi^2}\int drd\theta d\varphi\,\mathrm{Tr}\,(A_r[A_\theta, A_\varphi])\,. \tag{4.5}$$

(The factor $r^2 \sin\theta$ of the volume elements cancels the factor

$$|\,g\,|^{-1/2} = (r^2\sin\theta)^{-1}$$

arising in the integrand. Now $A_\theta = i(\partial_\theta u)u$ and so on.) Using such relations as

$$\begin{aligned} &-i\,\mathrm{Tr}\,(u_ru^{-1}u_\theta u^{-1}u_\varphi u^{-1}) \\ &\quad = i\,\mathrm{Tr}\,(u_ru^{-1}u_\theta u_\varphi^{-1}) \qquad (u_r \equiv \partial_r u, \text{ etc.}) \\ &\quad = \mathrm{Tr}\,\left[(\beta'.C)\left\{J_2e^{-i\beta.C} - e^{-i\beta.C}J_2\right\}\left\{e^{i\beta.C}J_1 - J_1e^{i\beta.C}\right\}\right]\sin\theta \end{aligned} \tag{4.6}$$

and the matrix elements of J_1 and J_2 one finally obtains

$$Q = -\frac{1}{2\pi}\int_0^\infty dr\left[\sum_{j=-J}^{J}\{2j\alpha'_j - (J-j)(J+j+1)\cos(\alpha_j-\alpha_{j+1})\times(\alpha'_j-\alpha'_{j+1})\}\right] \tag{4.7}$$

$$= -\frac{1}{2\pi}\sum_{j=-J}^{J}[2j\alpha_j-(J-j)(J+j+1)\sin(\alpha_{j+1}-\alpha_j)]_0^\infty\,. \tag{4.8}$$

But in this expression the α's have to be such that (4.1) is satisfied. To get the requisite constraints we resort to the technique of [4]. We consider formally a Lagrangian

$$S \simeq \int d^3x(\partial_i u \partial_i u^+)\,. \tag{4.9}$$

Varying the integrand w. r. t. u (hence w. r. t. the α's), (4.1) will be obtained as an extremum condition. One has

$$\partial_i u \partial_i u^+ = \left\{ u_r u_r^+ + \frac{1}{r^2} u_\theta u_\theta^+ + \frac{1}{r^2\sin^2\theta} u_\varphi u_\varphi^+ \right\} \tag{4.10}$$

where

$$\mathrm{Tr}\,(u_r u_r^+) = \sum_j (\alpha'_j)^2$$

and

$$\mathrm{Tr}\,(u_\theta u_\theta^+) = 2\,\mathrm{Tr}\,(J_2^2 - J_2 e^{i\beta.C} J_2 e^{-i\beta.C}) \tag{4.11}$$

$$\mathrm{Tr}\,(u_\varphi u_\varphi^+) = 2\,\mathrm{Tr}\,(J_3^2 - J_1 e^{i\beta.C} J_1 e^{-i\beta.C})\sin^2\theta\,.$$

Rejecting the α-independent terms one finally obtains

$$S \simeq \int dr \sum_j [r^2(\alpha'_j)^2 - 2(J+j+1)(J-j)\cos(\alpha_{j+1}-\alpha_j)]\,. \tag{4.12}$$

Hence the constraint equations are

$$\frac{d}{dr}\left(r^2\frac{d\alpha_j}{dr}\right) + (J+j+1)(J-j)\sin(\alpha_{j+1}-\alpha_j) - (J+j)(J-j+1)\sin(\alpha_j-\alpha_{j-1}) = 0\,. \tag{4.13}$$

Setting $r = e^\tau$, we obtain

$$\frac{d^2\alpha_j}{d^2\tau} + \frac{d\alpha_j}{d\tau} + (J+j+1)(J-j)\sin(\alpha_{j+1}-\alpha_j) - (J+j)(J-j+1)\sin(\alpha_j-\alpha_{j-1}) = 0 \qquad (j = -J, \ldots, J)\,. \tag{4.14}$$

It will be noted that in the variations we did not impose the supplementary constraint

$$\sum_j \alpha_j = 0\,.$$

But this can be introduced now without inconsistency since from (4.13) $\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\sum_j \alpha_j\right)\right) = 0$. This is now to be taken into account in considering each particular case of (4.14). This turns out to be the correct prescription.

Particular cases.

1. $n = 2\left(\mathrm{J} = \frac{1}{2}, j = -\frac{1}{2}, \frac{1}{2}\right)$.

Let

$$\alpha_{1/2} = -\alpha_{-1/2} = \alpha/2 . \tag{4.15}$$

From (4.8) and (4.13) we obtain

$$\mathrm{Q} = -\frac{1}{2\pi}[\alpha - \sin\alpha]_{r=0}^{\infty} \tag{4.16}$$

where

$$\frac{d}{dr}\left(r^2\frac{d\alpha}{dr}\right) - 2\sin\alpha = 0 . \tag{4.17}$$

This is the well-known SU(2) case. For our normalization convenient for the general case, the non-trivial solutions are $\alpha(0) = 0$, $\alpha(\infty) = \mp\pi$ giving $\mathrm{Q} = \pm\frac{1}{2}$.

2. $n = 3$ $(\mathrm{J} = 1, j = -1, 0, 1)$.

Let

$$\eta = \frac{1}{2}(\alpha_1 - \alpha_{-1})$$

and

$$\zeta = \frac{1}{2}(\alpha_1 + \alpha_{-1} - 2\alpha_0) . \tag{4.18}$$

Then

$$\mathrm{Q} = -\frac{2}{\pi}[\eta - \sin\eta\cos\zeta]_{r=0}^{\infty} \tag{4.19}$$

where

$$\frac{d}{dr}\left(r^2\frac{d\eta}{dr}\right) = 6\sin\eta\cos\zeta$$

and

$$\frac{d}{dr}\left(r^2\frac{d\zeta}{dr}\right) = 2\sin\zeta\cos\eta . \tag{4.20}$$

Thus we get back the SU(3) equations of [*4*]. Their correspondance with SU(3) merons has been noted in (I).

3. $n = 4\left(\mathrm{J} = \frac{3}{2}, j = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$.

Setting $(\alpha_{j+1} - \alpha_j) \equiv \omega_j$, the three independent equations are

$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\omega_{1/2}\right) - 6 \sin \omega_{1/2} + 4 \sin \omega_{-1/2} = 0$$
$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\omega_{-1/2}\right) + 3 \sin \omega_{1/2} - 8 \sin \omega_{-1/2} + 3 \sin \omega_{-3/2} = 0$$
$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\omega_{-3/2}\right) + 4 \sin \omega_{-1/2} - 6 \sin \omega_{-3/2} = 0 \tag{4.21}$$

The corresponding charge is

$$Q = -\frac{1}{2\pi}[3(\omega_{1/2} - \sin \omega_{1/2}) + 4(\omega_{-1/2} - \sin \omega_{-1/2}) + 3(\omega_{-3/2} - \sin \omega_{-3/2})]\Big|_{(r=0)}^{(r=\infty)}. \tag{4.22}$$

For $\alpha_j = j\alpha$, one has

$$Q = -\frac{5}{\pi}(\alpha - \sin \alpha)\Big|_{r=0}^{r=\infty} \tag{4.23}$$

where

$$\frac{d}{dr}\left(r^2 \frac{d\alpha}{dr}\right) - 2 \sin \alpha = 0. \tag{4.24}$$

For $\alpha(0) = 0$, $\alpha(\infty) = \mp \pi$ one obtains

$$Q = \pm 5. \tag{4.25}$$

For

$$\omega_{1/2} = -\omega_{-3/2} = \omega, \qquad \omega_{-1/2} = 0 \tag{4.26}$$

one has $Q = 0$ and

$$\frac{d}{dr}\left(r^2 \frac{d\omega}{dr}\right) - 6 \sin \omega = 0. \tag{4.27}$$

This equation was discussed in Ref. 4 in the SU(3) context.

For

$$\omega_{1/2} = \omega_{-3/2} = \eta, \qquad \omega_{-1/2} = \zeta \tag{4.28}$$

the equations are

$$\begin{aligned} &\frac{d}{dr}\left(r^2 \frac{d\eta}{dr}\right) - 6 \sin \eta + 4 \sin \zeta = 0 \\ &\frac{d}{dr}\left(r^2 \frac{d\zeta}{dr}\right) - 8 \sin \zeta + 6 \sin \eta = 0. \end{aligned} \tag{4.29}$$

Here we make no attempt to analyse fully the contents of the equations for such or more general cases.

5. REMARKS ON THE LIMIT $n \to \infty$

For our ansatz based on the maximal value $J = \frac{n-1}{2}$, the action diverges strongly as n and hence $J \to \infty$. However, since the equations and invariants can then be expressed in a quite simple form we display them briefly for completeness.

As $J \to \infty$, the index j $(-J \leq j \leq J)$ can be replaced by a continuous one

$$-1 \leq \xi \leq 1,$$

a limiting form of j/J. Setting

$$u_j(r, t) = u(\xi, r, t)$$

$$u_{j \mp 1} = u\left(\xi \mp \frac{1}{J}\right) = u(\xi) \mp \frac{1}{J}\frac{\partial}{\partial \xi}u(\xi) + \frac{1}{2J^2}\frac{\partial^2}{\partial \xi^2}u(\xi) + \dots \tag{5.1}$$

The equation of motion (2.15) now reduces to (the coefficients of J^2 and J cancelling)

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial r^2} + \frac{u}{r^2}\left[1 + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\{(1-\xi^2)u^2\}\right] = 0 \tag{5.2}$$

For $u(\xi, r, t) = u_0(r, t)$, independent of ξ we get back

$$\Delta u_0 + \frac{u_0}{r^2}(1 - u_0^2) = 0 \tag{5.3}$$

the SU(2) equation (2.18) as expected. As an amusing formal possibility we note that for

$$u = \left(\frac{A\xi^2 + B\xi + C}{(1-\xi^2)}\right)^{1/2} u_0(r, t) \tag{5.4}$$

A, B and C being constants, we still get a ξ-independent equation (as for (1.2))

$$\Delta u_0 + \frac{u_0}{r^2}(1 + A u_0^2) = 0. \tag{5.5}$$

But u diverges as $\xi \to \pm 1$ and u cannot be properly interpreted as the cosine of a real angle.

Using

$$\sum_{j=-J}^{J} \to J\int_{-1}^{1} d\xi$$

the action (3.13) can finally be expressed as

$$\frac{S}{J^3} = 2\pi \int drdt \int_{-1}^{1} d\xi \left[(1 - \xi^2)(\dot{u}^2 + u'^2) + \frac{1}{r^2} \left\{ \xi + \frac{1}{2} \frac{\partial}{\partial \xi} ((1 - \xi^2)u^2) \right\}^2 \right]. \quad (5.7)$$

For (5.3) one obtains

$$\frac{S}{J^3} = \left(\frac{4}{3}\right) \left\{ 2\pi \int drdt \left[\dot{u}_0^2 + u_0'^2 + \frac{1}{2r^2}(1 - u_0^2)^2 \right] \right\}. \quad (5.8)$$

The charge (3.5) can also be similarly expressed, noting that

$$\omega_j = \alpha_{j+1} - \alpha_j = \frac{1}{J} \partial_\xi \alpha(\xi). \quad (5.9)$$

In particular for (2.19),

$$\alpha_j = j\alpha \qquad \text{or} \qquad \alpha(\xi) = J\xi\alpha \quad (5.10)$$

one gets as for the action, the SU(2) expression with a factor $\left(\frac{4}{3} J^3\right)$, namely

$$\frac{q_p}{J^3} = \frac{4}{3} \int dtdr(\partial_t \alpha' - \partial_r \dot{\alpha}). \quad (5.11)$$

The equation (4.13) for Gribov ambiguity can be shown to reduce, for $\alpha = \alpha(\xi, r)$, to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \alpha}{\partial r}\right) + \frac{\partial}{\partial \xi}\left\{ (1 - \xi^2) \frac{\partial \alpha}{\partial \xi} \right\} = 0. \quad (5.12)$$

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Note added in proof. — The preprint of this paper came out in November 1979. Since then, the following related papers have appeared:

1) Z. Howath and L. Palla, O(3) symmetric merons in SU(N) gauge theory, *Phys. Rev.*, **D 21** (2953).

2) Thordur Jonsson, *Merons and elliptic equations with infinite action.* PHD Thesis, Harvard University, Cambridge (May 1980).

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[3] Irrational charges depending on an angle can arise in different contexts. See for example, E. Witten, « Dyons of charge $\frac{e\theta}{2\pi}$ ». Ref. TH. 2724, CERN (Aug. 1979).

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