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Section A :
Physique théorique.

Gauge independent formulation of dynamics of charged extended objects

by

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SUMMARY. — Dynamics of free and charged extended objects is formulated in gauge invariant phase spaces derived from the geometric interpretation of gauge fields obtained in an earlier publication [*11*].

RÉSUMÉ. — Nous formulons la dynamique des objets étendus neutres ou chargés. L'espace de phases pour cette dynamique est invariant par la jauge; sa construction est basée sur l'interprétation géométrique des champs de jauge obtenue dans notre article précédent [*11*].

1. INTRODUCTION

Extended objects: strings and membranes, were introduced in an attempt to solve certain problems in elementary particle physics. Space-time trajectories of free extended objects are minimal submanifolds. Charged extended objects interacting with gauge fields have also been considered. A geometric interpretation of gauge fields was given in [*11*].

In the present paper the dynamics of free and charged extended objects is studied as an example of a multisymplectic formulation of systems of partial differential equations describing physical systems. The present study is closely related to symplectic formulations of particle dynamics and field theory. By multisymplectic geometry we understand the geometry of k-cotangent bundles and related spaces.

Gauge independent dynamics of particles interacting with gauge fields is formulated in reduced symplectic manifolds [3], [4], [5], [7], [12]. Reduction of multisymplectic spaces has not been defined. Consequently our formulation of dynamics of extended objects is based largely on analogy with the dynamics of charged particles reviewed in Section 3. It is hoped that the example of reduction considered here will help to develop a general definition of reduction of multisymplectic spaces.

The kernel-index method of Schouten [6] is used in local coordinate representations.

2. DYNAMICS OF RELATIVISTIC PARTICLES WITHOUT CHARGE

Let X be the space-time of general relativity with a covariant metric tensor

$$\underline{g} : \mathrm{TX} \times \mathrm{TX} \to \mathbf{R} \tag{2.1}$$

and a contravariant metric tensor

$$\bar{g} : \mathrm{T^*X} \times \mathrm{T^*X} \to \mathbf{R}\,. \tag{2.2}$$

The components of the two metric tensors with respect to a coordinate system (x^κ), $\kappa = 0, 1, 2, 3$ will be denoted by $g_{\kappa\lambda}$ and $g^{\kappa\lambda}$ respectively and the associated metric connection will be represented by Christoffel symbols

$$\Gamma^\kappa_{\lambda\mu} = \frac{1}{2} g^{\kappa\nu}(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu})\,. \tag{2.3}$$

The phase space of a relativistic particle is the cotangent bundle $\mathrm{P} = \mathrm{T^*X}$. The cotangent bundle projection will be denoted by

$$\pi : \mathrm{P} \to \mathrm{X}\,. \tag{2.4}$$

The symplectic structure of P is defined by the canonical 2-form $\omega = d\theta$, where θ is the canonical 1-form on P. Coordinates (x^κ) of X induce coordinates (x^κ, p_λ) of P such that

$$\theta = p_\kappa dx^\kappa\,. \tag{2.5}$$

The velocity space of a particle is the tangent bundle TX with the projection mapping

$$\tau_\mathrm{X} : \mathrm{TX} \to \mathrm{X}\,. \tag{2.6}$$

The projection of the tangent bundle TP onto P will be denoted by

$$\tau_\mathrm{P} : \mathrm{TP} \to \mathrm{P}\,. \tag{2.7}$$

The part of the phase space P accessible to a particle of mass m is the mass shell

$$\mathrm{C}_m = \{\, p \in \mathrm{P}\,;\, \bar{g}(p, p) = m^2 \,\}\,. \tag{2.8}$$

In coordinates (x^κ, p_λ) the mass shell is described by the equation

$$g^{\kappa\lambda}p_\kappa p_\lambda = m^2 . \tag{2.9}$$

It is interesting to see that the mass shell determines completely the dynamics of the particle.

Let ω_m denote the form ω restricted (pulled back) to C_m :

$$\omega_m = \omega \mid C_m . \tag{2.10}$$

Since the codimension of C_m is 1 the characteristic distribution

$$\{ w \in TC_m ; w \lrcorner \omega_m = 0 \} \tag{2.11}$$

of ω_m is 1-dimensional. Integral manifolds of this distribution are the phase space trajectories of a particle of mass m. The characteristic distribution (2.11) can be written in the form

$$\left\{ w \in TP ; \tau_P(w) \in C_m, \exists_{\varepsilon \in \mathbf{R}} w \lrcorner \omega = -\frac{\varepsilon}{2m} dM^2 \right\}, \tag{2.12}$$

where M^2 is the function

$$M^2 : P \to \mathbf{R} : p \mapsto \bar{g}(p, p) . \tag{2.13}$$

In order to distinguish between particles and antiparticles we will orient trajectories by restricting the Lagrange multiplier ε to positive values. The infinitesimal symplectic relation [3]

$$D'_m = \left\{ w \in TP ; \tau_P(w) \in C_m, \exists_{\varepsilon \in \mathbf{R}} \varepsilon > 0, w \lrcorner \omega = -\frac{\varepsilon}{2m} dM^2 \right\} \tag{2.14}$$

represents the dynamics of a particle of mass m in infinitesimal terms. Lagrangian and Hamiltonian representations of D'_m are discussed in [10].

A curve

$$f : \mathbf{R} \to P : t \mapsto f(t) \tag{2.15}$$

is a parametrized trajectory of a particle if the image of its prolongation

$$f' : \mathbf{R} \to TP \tag{2.16}$$

is contained in D'_m. Let

$$\begin{aligned} x^\kappa &= f^\kappa(t) , \\ p_\lambda &= f_\lambda(t) \end{aligned} \tag{2.17}$$

be the coordinate expression of f in the coordinate system (x^κ, p_λ). The coordinate expression of the function M^2 is

$$M^2 = g^{\kappa\lambda}p_\kappa p_\lambda . \tag{2.18}$$

The equation

$$w \lrcorner \omega = -\frac{\varepsilon}{2m} dM^2 , \qquad \varepsilon > 0 \tag{2.19}$$

with

$$w = f'(t) = \frac{df^\kappa}{dt}\frac{\partial}{\partial x^\kappa} + \frac{df_\lambda}{dt}\frac{\partial}{\partial p_\lambda} \tag{2.20}$$

is equivalent to the system

$$\begin{aligned} \frac{df^\kappa}{dt} &= \frac{\varepsilon}{m} g^{\kappa\lambda} f_\lambda , \\ \frac{df_\lambda}{dt} &= -\frac{\varepsilon}{2m} \partial_\lambda g^{\mu\nu} f_\mu f_\nu , \\ \varepsilon &> 0 , \end{aligned} \tag{2.21}$$

or the system

$$\begin{aligned} f_\lambda &= \frac{m}{\varepsilon} g_{\lambda\kappa} \frac{df^\kappa}{dt} , \qquad \varepsilon > 0 , \\ \frac{Df_\lambda}{dt} &= \frac{df_\lambda}{dt} - \Gamma^\mu_{\lambda\nu} f_\mu \frac{df^\nu}{dt} = 0 . \end{aligned} \tag{2.22}$$

Taking into account the mass shell constraint

$$g^{\kappa\lambda} f_\kappa f_\lambda = m^2 \tag{2.23}$$

we derive the relation

$$\varepsilon^2 = g_{\kappa\lambda} \frac{df^\kappa}{dt} \frac{df^\lambda}{dt} . \tag{2.24}$$

The condition $\varepsilon = 1$ is usually imposed on the parametrization.

3. DYNAMICS OF CHARGED RELATIVISTIC PARTICLES

Let $\zeta : Z \to X$ be a principal fibre bundle. The base X of the bundle is the space-time and the structural group G_0 is the additive group of real numbers. The Lie algebra $\mathscr{G}_0$ of G_0 is the algebra of real numbers. The action of the structural group is represented by the mapping

$$\gamma : \mathbf{R} \times Z \to Z . \tag{3.1}$$

We denote by γ_s the mapping

$$\gamma_s : Z \to Z : z \mapsto \gamma(s, z) . \tag{3.2}$$

The fundamental field corresponding to $1 \in \mathscr{G}_0$ will be denoted by W.

In addition to the gravitational field represented by the metric tensor in X we have the electromagnetic field represented by the 2-form

$$\phi = -d\alpha , \tag{3.3}$$

where α is a connection form on Z satisfying

$$\langle W, \alpha \rangle = 1 \tag{3.4}$$

and

$$\mathscr{L}_W \alpha = 0 . \tag{3.5}$$

The above conditions imply the existence of a 2-form F on X such that

$$\phi = \zeta^* F . \tag{3.6}$$

The form F represents the electromagnetic field in X.

We will use in Z adapted coordinates (x^κ, y) such that

$$W = \frac{\partial}{\partial y} . \tag{3.7}$$

In terms of these coordinates we have local expressions

$$\alpha = A_\kappa(x^\mu) dx^\kappa + dy \tag{3.8}$$

and

$$\phi = \frac{1}{2} F_{\kappa\lambda}(x^\mu) dx^\kappa \wedge dx^\lambda , \tag{3.9}$$

where $A_\kappa(x^\mu)$ is the electromagnetic potential and

$$F_{\kappa\lambda}(x^\mu) = \partial_\lambda A_\kappa(x^\mu) - \partial_\kappa A_\lambda(x^\mu) \tag{3.10}$$

is the electromagnetic field.

The phase space of a charged particle is the cotangent bundle $R = T^*Z$ with the projection

$$\rho : R \to Z , \tag{3.11}$$

the canonical 1-form μ and the canonical 2-form $\nu = d\mu$. Coordinates $(x^\kappa, y, p_\lambda, q)$ such that

$$\mu = p_\kappa dx^\kappa + q dy \tag{3.12}$$

will be used in R. Each element $r \in R$ is decomposed into the horizontal part

$$\operatorname{hor}(r) = \langle W, r \rangle \alpha(\rho(r)) \tag{3.13}$$

interpreted as the charge, and the vertical part

$$\operatorname{ver}(r) = r - \operatorname{hor}(r) \tag{3.14}$$

representing the space-time momentum of the particle. We introduce mappings

$$\mathbf{h} : R \to \mathbf{R} : r \mapsto \langle W, r \rangle \tag{3.15}$$

and

$$\mathbf{v} : R \to P = T^*X \tag{3.16}$$

defined by

$$\langle u, \mathbf{v}(r) \rangle = \langle w, r \rangle = \langle w', \operatorname{ver}(r) \rangle , \tag{3.17}$$

where u is a vector at $\zeta(\rho(r))$, w is the horizontal lift of u and w' is any lift of u to $T_{\rho(r)}Z$. The number $q = \mathbf{h}(r)$ and the covector $p = \mathbf{v}(r)$ provide convenient representations of the charge and the space-time momentum respectively.

The action γ of the structural group G_0 induces an action

$$\tilde{\gamma} : \mathbf{R} \times R \to R \tag{3.18}$$

of G_0 in R. The mappings

$$\tilde{\gamma}_s : R \to R : r \mapsto \tilde{\gamma}(s, r) \tag{3.19}$$

are determined by conditions

$$\rho \circ \tilde{\gamma}_s = \gamma_s \circ \rho \tag{3.20}$$

and

$$\tilde{\gamma}_s{}^* \mu = \mu . \tag{3.21}$$

The generator of $\tilde{\gamma}$ is the vector field $\tilde{W}$ satisfying

$$T\rho \circ \tilde{W} = W \circ \rho \tag{3.22}$$

and

$$\mathscr{L}_{W}\mu = 0 . \tag{3.23}$$

Mappings $\tilde{\gamma}_s$ preserve the decomposition of elements of R into their horizontal and vertical components :

$$\begin{aligned} \text{hor}\,(\tilde{\gamma}_s(r)) &= \tilde{\gamma}_s(\text{hor}\,(r)) , \\ \text{ver}\,(\tilde{\gamma}_s(r)) &= \tilde{\gamma}_s\,(\text{ver}\,(r)) . \end{aligned} \tag{3.24}$$

Consequently

$$\mathbf{h}(\tilde{\gamma}_s(r)) = \mathbf{h}(r) \tag{3.25}$$

and

$$\mathbf{v}(\tilde{\gamma}_s(r)) = \mathbf{v}(r) . \tag{3.26}$$

We denote by

$$\tau_Z : TZ \to Z \tag{3.27}$$

and

$$\tau_R : TR \to R \tag{3.28}$$

the tangent bundle projections.

The part of the phase space R accessible to a particle of mass m and charge e is the submanifold

$$K_{m,e} = K_m \cap K_e , \tag{2.29}$$

where

$$K_m = \{ r \in R ; \bar{g}(\mathbf{v}(r), \mathbf{v}(r)) = m^2 \} \tag{3.30}$$

and

$$K_e = \{ r \in R ; \mathbf{h}(r) = e \} . \tag{3.31}$$

In terms of coordinates $(x^\kappa, y, p_\lambda, q)$ the submanifolds K_m and K_e are described by equations

$$g^{\kappa\lambda}(p_\kappa - qA_\kappa)(p_\lambda - qA_\lambda) = m^2 \tag{3.32}$$

and

$$q = e \tag{3.33}$$

respectively. Trajectories of the particle in the phase space R are integral curves of the characteristic distribution

$$\{ w \in TK_{m,e} ; w \,\lrcorner\, v_{m,e} = 0 \} \tag{3.34}$$

of the 2-form

$$v_{m,e} = v \mid \mathrm{K}_{m,e} . \tag{3.35}$$

In certain respects it is appropriate to consider integral manifolds of the characteristic distribution as trajectories of the particle. These integral manifolds are in this case 2-dimensional.

In order to prove integrability of the characteristic distribution (3.34) we must show that $\mathrm{K}_{m,e}$ is coisotropic [3]. Let M^2 and E be functions

$$\mathrm{M}^2 : \mathrm{R} \to \mathbf{R} : r \mapsto \bar{g}(\mathbf{v}(r), \mathbf{v}(r)) \tag{3.36}$$

and

$$\mathrm{E} : \mathrm{R} \to \mathbf{R} : r \mapsto \mathbf{h}(r) . \tag{3.37}$$

Since

$$\mathrm{K}_m = \{ r \in \mathrm{R} ; \mathrm{M}^2(r) = m^2 \} \tag{3.38}$$

and

$$\mathrm{K}_e = \{ r \in \mathrm{R} ; \mathrm{E}(r) = e \} \tag{3.39}$$

in order to prove that $\mathrm{K}_{m,e}$ is coisotropic it is sufficient to show that the Poisson bracket

$$\{ \mathrm{E}, \mathrm{M}^2 \} \tag{3.40}$$

vanishes [2]. From

$$\mathrm{E} = \langle \tilde{\mathrm{W}}, \mu \rangle \tag{3.41}$$

and the identity

$$\mathscr{L}_{\tilde{\mathrm{W}}}\mu = \tilde{\mathrm{W}} \lrcorner d\mu + d \langle \tilde{\mathrm{W}}, \mu \rangle \tag{3.42}$$

it follows that

$$\tilde{\mathrm{W}} \lrcorner v = - d\mathrm{E} . \tag{3.43}$$

Hence,

$$\{ \mathrm{E}, \mathrm{M}^2 \} = \mathscr{L}_{\tilde{\mathrm{W}}}\mathrm{M}^2 , \tag{3.44}$$

and

$$\mathscr{L}_{\tilde{\mathrm{W}}}\mathrm{M}^2 = 0 \tag{3.45}$$

follows from (3.26).

The characteristic distribution (3.34) can be represented in the form

$$\left\{ w \in \mathrm{TR} ; \tau_{\mathrm{R}}(w) \in \mathrm{K}_{m,e} , \exists_{\varepsilon,\delta \in \mathbf{R}} w \lrcorner v = - \frac{\varepsilon}{2m} d\mathrm{M}^2 - \delta d\mathrm{E} \right\} . \tag{3.46}$$

As in Section 2 we orient the trajectories by restricting ε to positive values. The infinitesimal symplectic relation

$$\mathrm{N}'_{m,e} = \left\{ w \in \mathrm{TR} ; \tau_{\mathrm{R}}(w) \in \mathrm{K}_{m,e}, \exists_{\varepsilon,\delta \in \mathbf{R}} \varepsilon > 0 , \; w \lrcorner v = - \frac{\varepsilon}{2m} d\mathrm{M}^2 - \delta d\mathrm{E} \right\} \tag{3.47}$$

represents the dynamics of particles in the phase space R.

Let

$$\begin{aligned} x^\kappa &= f^\kappa(t) , \; y = g(t) , \\ p_\lambda &= f_\lambda(t) , \; q = h(t) \end{aligned} \tag{3.48}$$

be the coordinate expression of a curve

$$f : \mathbf{R} \to \mathrm{R} : t \mapsto f(t). \tag{3.49}$$

We impose on the tangent vector

$$w = \frac{df^\kappa}{dt}\frac{\partial}{\partial x^\kappa} + \frac{dg}{dt}\frac{\partial}{\partial y} + \frac{df_\lambda}{dt}\frac{\partial}{\partial p_\lambda} + \frac{dh}{dt}\frac{\partial}{\partial q} \tag{3.50}$$

the equation

$$w \lrcorner v = -\frac{\varepsilon}{2m} d\mathrm{M}^2 - \delta d\mathrm{E}, \qquad \varepsilon > 0. \tag{3.51}$$

The coordinate expressions of the functions M^2 and E are

$$\begin{aligned} \mathrm{M}^2 &= g^{\kappa\lambda}(p_\kappa - q\mathrm{A}_\kappa)(p_\lambda - q\mathrm{A}_\lambda), \\ \mathrm{E} &= q \end{aligned} \tag{3.52}$$

Hence,

$$\begin{aligned} &\frac{df^\kappa}{dt} = \frac{\varepsilon}{m} g^{\kappa\lambda}(f_\lambda - e\mathrm{A}_\lambda), \\ &\frac{dg}{dt} = -\frac{\varepsilon}{m} g^{\kappa\lambda}(f_\kappa - e\mathrm{A}_\kappa)\mathrm{A}_\lambda + \delta, \\ &\frac{df_\lambda}{dt} = -\frac{\varepsilon}{2m}\partial_\lambda g^{\mu\nu}(f_\mu - e\mathrm{A}_\mu)(f_\nu - e\mathrm{A}_\nu) + \frac{\varepsilon e}{m} g^{\mu\nu}(f_\mu - e\mathrm{A}_\mu)\partial_\lambda \mathrm{A}_\nu, \\ &\varepsilon > 0, \end{aligned} \tag{3.53}$$

or

$$\begin{aligned} &f_\lambda - e\mathrm{A}_\lambda = \frac{m}{\varepsilon} g_{\lambda\kappa}\frac{df^\kappa}{dt}, \qquad \varepsilon > 0, \\ &\mathrm{A}_\kappa \frac{df^\kappa}{dt} + \frac{dg}{dt} = \delta, \\ &\frac{\mathrm{D}(f_\lambda - e\mathrm{A}_\lambda)}{dt} = e\mathrm{F}_{\mu\lambda}\frac{df^\mu}{dt}, \end{aligned} \tag{3.54}$$

where

$$\frac{\mathrm{D}(f_\lambda - e\mathrm{A}_\lambda)}{dt} = \frac{d(f_\lambda - e\mathrm{A}_\lambda)}{dt} - \Gamma^\mu_{\lambda\nu}(f_\mu - e\mathrm{A}_\mu)\frac{df^\nu}{dt}. \tag{3.55}$$

Together with

$$\begin{aligned} &g^{\kappa\lambda}(f_\kappa - e\mathrm{A}_\kappa)(f_\lambda - e\mathrm{A}_\lambda) = m^2, \\ &h = e \end{aligned} \tag{3.56}$$

equations (3.54) characterize trajectories of charged particles in the phase space R.

Dynamics of charged particles formulated in the phase space R has gauge invariant Lagrangian and Hamilton-Jacobi descriptions. If these descriptions are not used a gauge invariant formulation of dynamics in the phase space $P = T^*X$ can be obtained by reducing the symplectic manifold (R, ν) with respect to the coisotropic submanifold K_e.

The characteristic distribution

$$\{ w \in TK_e ; w \lrcorner \nu_e = 0 \} \tag{3.57}$$

of the 2-form

$$\nu_e = \nu \mid K_e \tag{3.58}$$

is the set

$$\{ w \in TR ; \tau_R(w) \in K_e , \exists_{\delta \in \mathbf{R}} w \lrcorner \nu = - \delta dE \} . \tag{3.59}$$

It follows that the integral manifolds of the characteristic distribution are orbits of the action of the structural group in K_e. The integral manifolds can also be characterized as the fibres of the mapping

$$\kappa : K_e \rightarrow P : r \mapsto \mathbf{v}(r) . \tag{3.60}$$

Consequently the quotient manifold is canonically diffeomorphic to the phase space P. The calculation

$$\begin{aligned} \langle w, \kappa^*\theta \rangle &= \langle w, \mathbf{v}^*\theta \rangle \\ &= \langle T\mathbf{v}(w), \theta \rangle \\ &= \langle T\pi(T\mathbf{v}(w)), \tau_P(T\mathbf{v}(w)) \rangle \\ &= \langle T(\pi \circ \mathbf{v})(w), \mathbf{v}(\tau_R(w)) \rangle \\ &= \langle T\zeta(T\rho(w)), \mathbf{v}(\tau_R(w)) \rangle \\ &= \langle T\rho(w), \mathrm{ver}\,(\tau_R(w)) \rangle \\ &= \langle T\rho(w), \tau_R(w) - \langle W, \tau_R(w) \rangle \alpha(\rho(\tau_R(w))) \rangle \\ &= \langle T\rho(w), \tau_R(w) - e\alpha(\tau_Z(T\rho(w))) \rangle \\ &= \langle w, \mu - e\rho^*\alpha \rangle \end{aligned} \tag{3.61}$$

shows that

$$\kappa^*\theta = (\mu - e\rho^*\alpha) \mid K_e . \tag{3.62}$$

It follows that

$$\kappa^*\omega = (\nu + e\rho^*\phi) \mid K_e \tag{3.63}$$

and

$$\nu_e = \kappa^*(\omega - e\pi^*F) , \tag{3.64}$$

since

$$\begin{aligned} \rho^*\phi \mid K_e &= \rho^*\zeta^*F \mid K_e \\ &= \mathbf{v}^*\pi^*F \mid K_e \\ &= \kappa^*\pi^*F . \end{aligned} \tag{3.65}$$

The form

$$\omega_e = \omega - e\pi^*F \tag{3.66}$$

defines the symplectic structure of the reduced phase space P. This structure is different from the canonical symplectic structure of P.

Trajectories of charged particles in the reduced phase space P are images by κ of trajectories in R. Since

$$\begin{aligned} \kappa(K_{m,e}) &= C_m \\ &= \{ p \in P ; \bar{g}(p, p) = m^2 \} \end{aligned} \tag{3.67}$$

it is clear that trajectories in P are integral curves of the infinitesimal symplectic relation

$$\begin{aligned} D'_{m,e} &= T(\kappa \mid K_{m,e})(N'_{m,e}) \\ &= \left\{ w \in TP\,;\, \tau_P(w) \in C_m,\ \exists_{\varepsilon \in \mathbf{R}} \varepsilon > 0,\ w \,\lrcorner\, \omega_e = -\frac{\varepsilon}{2m} dM^2 \right\}, \end{aligned} \tag{3.68}$$

where

$$M^2 : P \to \mathbf{R} : p \mapsto \bar{g}(p, p). \tag{3.69}$$

Let

$$\begin{aligned} x^\kappa &= f^\kappa(t), \\ p_\lambda &= f_\lambda(t) \end{aligned} \tag{3.70}$$

be the coordinate representation of a curve

$$f : \mathbf{R} \to P : t \mapsto f(t) \tag{3.71}$$

in a coordinate system (x^κ, p_λ) such that

$$\theta = p_\lambda dx^\lambda. \tag{3.72}$$

The equation

$$w \,\lrcorner\, \omega_e = -\frac{\varepsilon}{2m} dM^2, \qquad \varepsilon > 0 \tag{3.73}$$

with

$$w = \frac{df^\kappa}{dt}\frac{\partial}{\partial x^\kappa} + \frac{df_\lambda}{dt}\frac{\partial}{\partial p_\lambda} \tag{3.74}$$

leads to equations

$$\begin{aligned} &\frac{df^\kappa}{dt} = \frac{\varepsilon}{m} g^{\kappa\lambda} f_\lambda, \\ &\frac{df_\lambda}{dt} - eF_{\kappa\lambda}\frac{df^\kappa}{dt} = -\frac{\varepsilon}{2m}\partial_\lambda g^{\mu\nu} p_\mu p_\nu, \\ &\varepsilon > 0, \end{aligned}$$

or

$$\begin{aligned} f_\kappa &= \frac{m}{\varepsilon} g_{\kappa\lambda}\frac{df^\lambda}{dt}, \qquad \varepsilon > 0 \\ \frac{Df_\lambda}{dt} &= eF_{\kappa\lambda}\frac{df^\kappa}{dt}, \end{aligned} \tag{3.76}$$

where

$$\frac{Df_\lambda}{dt} = \frac{df_\lambda}{dt} - \Gamma^\mu_{\lambda\nu} f_\mu \frac{df^\nu}{dt}. \tag{3.77}$$

Equations (3.76) together with

$$g^{\kappa\lambda} f_\kappa f_\lambda = m^2 \tag{3.78}$$

characterize the trajectories of the particle in the reduced phase space P.

It is to be noted that the coordinates p_λ in P are directly components of the space-time momentum covector whereas in R only the combination $p_\lambda - qA_\lambda$ corresponds to space-time momentum.

4. DYNAMICS OF EXTENDED OBJECTS WITHOUT CHARGE

Let X denote the space-time as in Section 2. The phase space of an extended object is the k-cotangent bundle $P = \Lambda^k T^*X$ with the projection

$$\pi : P \to X \tag{4.1}$$

and the canonical k-form θ defined by

$$\langle w, \theta \rangle = \langle \Lambda^k T\pi(w), \tau_P(w) \rangle , \tag{4.2}$$

where $w \in \Lambda^k TP$ and

$$\tau_P : \Lambda^k TP \to P \tag{4.3}$$

is the k-tangent bundle projection. For $k = 1$ the extended object is the relativistic particle described in Section 2. For $k = 2$ the extended object is a string and for $k = 3$ it is a membrane. The canonical $(k + 1)$-form $\omega = d\theta$ defines in P a k-symplectic structure. Coordinates (x^κ) induce in P coordinates $(x^\kappa, p_{\lambda_1 \dots \lambda_k})$ such that

$$\theta = \frac{1}{k!} p_{\kappa_1 \dots \kappa_k} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_k}. \tag{4.4}$$

The velocity space of an extended object is the k-tangent bundle $\Lambda^k TX$ with the projection

$$\tau_X : \Lambda^k TX \to X . \tag{4.5}$$

The part of the phase space P accessible to an extended object is the submanifold

$$C_m = \{ p \in P ; \Lambda^k \bar{g}(p, p) = m^2 \} . \tag{4.6}$$

In terms of coordinates $(x^\kappa, p_{\lambda_1 \dots \lambda_k})$ this submanifold is described by the equation

$$\frac{1}{k!} g^{\kappa_1 \lambda_1} \dots g^{\kappa_k \lambda_k} p_{\kappa_1 \dots \kappa_k} p_{\lambda_1 \dots \lambda_k} = m^2 . \tag{4.7}$$

We denote by ω_m the restriction of ω to C_m :

$$\omega_m = \omega \mid C_m , \tag{4.8}$$

and introduce the characteristic distribution

$$\{ w \in \Lambda^k TC_m ; w \lrcorner \omega_m = 0 \} . \tag{4.9}$$

A submanifold of C_m of dimension k is called an integral manifold of the characteristic distribution if each k-vector tangent to the submanifold belongs to the distribution. Phase space trajectories of the extended object are integral manifolds of the characteristic distribution (4.9). The characteristic distribution (4.9) can be represented in the form

$$\left\{ w \in \Lambda^k \mathrm{TP} \,;\, \tau_{\mathrm{P}}(w) \in \mathrm{C}_m,\ \exists_{\varepsilon\in\mathbf{R}} w \lrcorner\, \omega = (-1)^k \frac{\varepsilon}{2m} d\mathrm{M}^2 \right\}, \tag{4.10}$$

where M^2 is the function

$$\mathrm{M}^2 : \mathrm{P} \to \mathbf{R} : p \mapsto \Lambda^k \bar{g}(p, p). \tag{4.11}$$

As in Section 2 we orient trajectories by restricting ε to positive values and represent dynamics of the extended object by the infinitesimal relation

$$\mathrm{D}'_m = \left\{ w \in \Lambda^k \mathrm{TP} \,;\, \tau_{\mathrm{P}}(w) \in \mathrm{C}_m,\ \exists_{\varepsilon\in\mathbf{R}} \varepsilon > 0,\ w \lrcorner\, \omega = (-1)^k \frac{\varepsilon}{2m} d\mathrm{M}^2 \right\}. \tag{4.12}$$

An embedding

$$f : \mathbf{R}^k \to \mathrm{P} : (t^1, \ldots, t^k) \mapsto f(t^1, \ldots, t^k) \tag{4.13}$$

is a parametrized trajectory of an the extended object if the image of its prolongation

$$f' : \mathbf{R}^k \to \Lambda^k \mathrm{TP} \tag{4.14}$$

is contained in D'_m. Let $k = 2$ and let

$$\begin{aligned} x^\kappa &= f^\kappa(t^1, t^2) \\ p_{\lambda_1\lambda_2} &= f_{\lambda_1\lambda_2}(t^1, t^2) \end{aligned} \tag{4.15}$$

be the coordinate expression of an embedding f. The coordinate expression of M^2 is

$$\mathrm{M}^2 = \frac{1}{2} g^{\kappa_1\lambda_1} g^{\kappa_2\lambda_2} p_{\kappa_1\kappa_2} p_{\lambda_1\lambda_2}\,. \tag{4.16}$$

If

$$w = \left(\frac{\partial f^\kappa}{\partial t^1} \frac{\partial}{\partial x^\kappa} + \frac{1}{2} \frac{\partial f_{\kappa_1\kappa_2}}{\partial t^1} \frac{\partial}{\partial p_{\kappa_1\kappa_2}} \right) \wedge \left(\frac{\partial f^\lambda}{\partial t^2} \frac{\partial}{\partial x^\lambda} + \frac{1}{2} \frac{\partial f_{\lambda_1\lambda_2}}{\partial t^2} \frac{\partial}{\partial p_{\lambda_1\lambda_2}} \right) \tag{4.17}$$

then the equation

$$w \lrcorner\, \omega = \frac{\varepsilon}{2m} d\mathrm{M}^2, \qquad \varepsilon > 0 \tag{4.18}$$

leads to the system

$$\begin{aligned} &\frac{\partial f^{\kappa_1}}{\partial t^1} \frac{\partial f^{\kappa_2}}{\partial t^2} - \frac{\partial f^{\kappa_1}}{\partial t^2} \frac{\partial f^{\kappa_2}}{\partial t^1} = \frac{\varepsilon}{m} g^{\kappa_1\lambda_1} g^{\kappa_2\lambda_2} f_{\lambda_1\lambda_2}, \\ &\frac{\partial f_{\mu\kappa}}{\partial t^1} \frac{\partial f^\mu}{\partial t^2} - \frac{\partial f_{\mu\kappa}}{\partial t^2} \frac{\partial f^\mu}{\partial t^1} = \frac{\varepsilon}{2m} \partial_\kappa g^{\lambda_1\mu_1} g^{\lambda_2\mu_2} f_{\lambda_1\lambda_2} f_{\mu_1\mu_2}, \\ &\varepsilon > 0, \end{aligned} \tag{4.19}$$

or

$$\begin{aligned}
&f_{\lambda_1\lambda_2} = \frac{m}{\varepsilon} g_{\lambda_1\kappa_1} g_{\lambda_2\kappa_2}\left(\frac{\partial f^{\kappa_1}}{\partial t^1}\frac{\partial f^{\kappa_2}}{\partial t^2} - \frac{\partial f^{\kappa_1}}{\partial t^2}\frac{\partial f^{\kappa_2}}{\partial t^1}\right), \\
&\varepsilon > 0\,, \\
&\frac{\mathrm{D}f_{\kappa\lambda}}{\partial t^1}\frac{\partial f^{\kappa}}{\partial t^2} - \frac{\mathrm{D}f_{\kappa\lambda}}{\partial t^2}\frac{\partial f^{\kappa}}{\partial t^1} = 0\,,
\end{aligned} \tag{4.20}$$

where

$$\frac{\mathrm{D}f_{\lambda\kappa}}{\partial t^i} = \frac{\partial f_{\lambda\kappa}}{\partial t^i} - \Gamma^{\mu}_{\lambda\nu} f_{\mu\kappa}\frac{\partial f^{\nu}}{\partial t^i} - \Gamma^{\mu}_{\kappa\nu} f_{\lambda\mu}\frac{\partial f^{\nu}}{\partial t^i}\,. \tag{4.21}$$

From

$$\frac{1}{2} g^{\kappa_1\lambda_1} g^{\kappa_2\lambda_2} f_{\kappa_1\kappa_2} f_{\lambda_1\lambda_2} = m^2 \tag{4.22}$$

we derive the relation

$$\varepsilon^2 = g_{\kappa_1\lambda_1} g_{\kappa_2\lambda_2}\left(\frac{\partial f^{\kappa_1}}{\partial t^1}\frac{\partial f^{\kappa_2}}{\partial t^2} - \frac{\partial f^{\kappa_1}}{\partial t^2}\frac{\partial f^{\kappa_2}}{\partial t^1}\right)\left(\frac{\partial f^{\lambda_1}}{\partial t^1}\frac{\partial f^{\lambda_2}}{\partial t^2} - \frac{\partial f^{\lambda_1}}{\partial t^2}\frac{\partial f^{\lambda_2}}{\partial t^1}\right). \tag{4.23}$$

The condition $\varepsilon = 1$ can be imposed on the parametrization.

5. DYNAMICS OF CHARGED EXTENDED OBJECTS

Let X be the space-time. We denote by

$$\eta : \Lambda^{k-1}\mathrm{T}^*\mathrm{X} \to \mathrm{X} \tag{5.1}$$

the $(k-1)$-cotangent bundle projection and by λ the canonical $(k-1)$-form on $\Lambda^{k-1}\mathrm{T}^*\mathrm{X}$. Coordinates $(x^{\kappa}, y_{\lambda_1\ldots\lambda_{k-1}})$ such that

$$\lambda = \frac{1}{(k-1)!} y_{\kappa_1\ldots\kappa_{k-1}} dx^{\kappa_1} \wedge \ldots \wedge dx^{\kappa_{k-1}} \tag{5.2}$$

will be used in $\Lambda^{k-1}\mathrm{T}^*\mathrm{X}$. We denote by G the additive group of $(k-1)$-forms on X and by G_0 the subgroup of exact forms. The Lie algebra of G, denoted $\mathcal{G}$, is the vector space of $(k-1)$-forms and the Lie algebra of G_0, denoted by $\mathcal{G}_0$, is the subspace of exact $(k-1)$-forms. The Lie bracket is trivial since G and G_0 are commutative groups.

Let

$$\zeta : \mathrm{Z} \to \mathrm{X} \tag{5.3}$$

be a differential fibration and let

$$\gamma : \Lambda^{k-1}\mathrm{T}^*\mathrm{X} \times_{\mathrm{X}} \mathrm{Z} \to \mathrm{Z} \tag{5.4}$$

be a X-morphism such that for each $x \in \mathrm{X}$ there is a neighbourhood U of x and a U-isomorphism

$$\psi : \zeta^{-1}(\mathrm{U}) \to \eta^{-1}(\mathrm{U}) \tag{5.5}$$

satisfying

$$\psi(\gamma(y, z)) = \psi(z) + y\,. \tag{5.6}$$

Isomorphisms ψ transfer coordinates $(x^\kappa, y_{\lambda_1 \ldots \lambda_{k-1}})$ from $\Lambda^{k-1}T^*X$ to Z. For each $s \in G$ we denote by γ_s the mapping

$$\gamma_s : Z \to Z : z \mapsto \gamma(s(\zeta(z)), z) \tag{5.7}$$

Each element $\sigma \in \mathscr{G}$ generates a 1-parameter subgroup of G and consequently induces a vector field on Z. This vector field will be denoted by W_σ. We will denote by W any k-vector field satisfying

$$W \lrcorner\, \zeta^*\sigma = W_\sigma \tag{5.8}$$

for each $\sigma \in \mathscr{G}$. In each coordinate neighbourhood the k-vector field

$$\frac{1}{(k-1)!}\frac{\partial}{\partial x^{\kappa_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\kappa_{k-1}}} \wedge \frac{\partial}{\partial y_{\kappa_1 \ldots \kappa_{k-1}}} \tag{5.9}$$

satisfies the condition (5.8) and a global construction of W is easily obtained with the help of a partition of unity.

A generalized connection form on Z is a k-form α satisfying

$$W_\sigma \lrcorner\, \alpha = \zeta^*\sigma \tag{5.10}$$

and

$$\mathscr{L}_{W_\sigma}\alpha = 0 \tag{5.11}$$

for each $\sigma \in \mathscr{G}_0$. The $(k + 1)$-form $\phi = -\,d\alpha$ represents a generalized gauge field. From the identity

$$\mathscr{L}_{W_\sigma}\alpha = d(W_\sigma \lrcorner\, \alpha) + W_\sigma \lrcorner\, d\alpha \tag{5.12}$$

we derive

$$W_\sigma \lrcorner\, \phi = 0 \tag{5.13}$$

and

$$\mathscr{L}_{W_\sigma}\phi = 0\,. \tag{5.14}$$

Formulae (5.10), (5.13) and (5.14) are valid for each $\sigma \in \mathscr{G}$. From (5.13) and (5.14) it follows that there is a unique $(k + 1)$-form F on X such that

$$\phi = \zeta^*F\,. \tag{5.15}$$

In terms of coordinates $(x^\kappa, y_{\lambda_1 \ldots \lambda_{k-1}})$ we have local expressions

$$\alpha = \frac{1}{k!}A_{\kappa_1 \ldots \kappa_k}dx^{\kappa_1} \wedge \ldots \wedge dx^{\kappa_k} + \frac{1}{(k-1)!}dy_{\kappa_1 \ldots \kappa_{k-1}} \wedge dx^{\kappa_1} \wedge \ldots \wedge dx^{\kappa_{k-1}} \tag{5.16}$$

and

$$\phi = \frac{1}{(k+1)!}F_{\kappa_1 \ldots \kappa_{k+1}}dx^{\kappa_1} \wedge \ldots \wedge dx^{\kappa_{k+1}}\,, \tag{5.17}$$

where

$$F_{\kappa_1 \ldots \kappa_{k+1}} = -\,(k+1)\partial_{[\kappa_1}A_{\kappa_2 \ldots \kappa_{k+1}]}\,. \tag{5.18}$$

Let

$$\rho : \mathrm{R} \to \mathrm{Z} \tag{5.19}$$

be the subbundle of $\Lambda^k \mathrm{T}^*\mathrm{Z}$ consisting of k-covectors r satisfying conditions:

$$\mathrm{W} \llcorner (v \lrcorner r) = (-1)^{k-1} \langle \mathrm{W}, r \rangle v \tag{5.20}$$

if v is a vector vertical with respect to the projection ζ, and

$$(v_1 \wedge v_2) \lrcorner r = 0 \tag{5.21}$$

if v_1 and v_2 are vertical vectors. The canonical k-form on $\Lambda^k \mathrm{T}^*\mathrm{Z}$ restricted to R will be denoted by μ, and ν will denote the $(k+1)$-form $d\mu$. We will use in R coordinates $(x^\kappa, y_{\lambda_1 \dots \lambda_{k-1}}, p_{\mu_1 \dots \mu_k}, q)$ such that

$$\mu = \frac{1}{k!} p_{\kappa_1 \dots \kappa_k} dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_k} + \frac{1}{(k-1)!} q dy_{\kappa_1 \dots \kappa_{k-1}} \wedge dx^{\kappa_1} \wedge \dots \wedge dx^{\kappa_{k-1}}. \tag{5.22}$$

The bundle R is interpreted as the phase space of a charged extended object. Each element $r \in \mathrm{R}$ is decomposed into the horizontal part

$$\mathrm{hor}(r) = (-1)^{k-1} \langle \mathrm{W}, r \rangle \alpha(\rho(r)) \tag{5.23}$$

interpreted as the charge, and the vertical part

$$\mathrm{ver}(r) = r - \mathrm{hor}(r) \tag{5.24}$$

interpreted as the space-time k-momentum of the extended object. Mappings

$$\mathbf{h} : \mathrm{R} \to \mathbf{R} \tag{5.25}$$

and

$$\mathbf{v} : \mathrm{R} \to \mathrm{P} = \Lambda^\kappa \mathrm{T}^*\mathrm{X} \tag{5.26}$$

are defined by

$$\mathbf{h}(r) = (-1)^{k-1} \langle \mathrm{W}, r \rangle \tag{5.27}$$

and

$$\langle u, \mathbf{v}(r) \rangle = \langle w, r \rangle = \langle w', \mathrm{ver}(r) \rangle, \tag{5.28}$$

where u is a k-vector at $\zeta(\rho(r))$, w is a lift of u to $\Lambda^k \mathrm{T}_{\rho(r)}\mathrm{Z}$ satisfying

$$\langle w, \alpha \rangle = 0 \tag{5.29}$$

and w' is any lift of u to $\Lambda^k \mathrm{T}_{\rho(r)}\mathrm{Z}$.

We denote by

$$\tau_\mathrm{Z} : \Lambda^k \mathrm{TZ} \to \mathrm{Z} \tag{5.30}$$

and

$$\tau_\mathrm{R} : \Lambda^k \mathrm{TR} \to \mathrm{R} \tag{5.31}$$

the k-tangent bundle projections.

The part of the phase space R accessible to the extended object is the submanifold

$$\mathrm{K}_{m,e} = \mathrm{K}_m \cap \mathrm{K}_e, \tag{5.32}$$

where

$$K_m = \{ r \in R ; \Lambda^k \bar{g}(\mathbf{v}(r), \mathbf{v}(r)) = m^2 \} \tag{5.33}$$

and

$$K_e = \{ r \in R ; \mathbf{h}(r) = e \} . \tag{5.34}$$

Equations

$$\frac{1}{k!} g^{\kappa_1 \lambda_1} \dots g^{\kappa_k \lambda_k} (p_{\kappa_1 \dots \kappa_k} - q A_{\kappa_1 \dots \kappa_k})(p_{\lambda_1 \dots \lambda_k} - q A_{\lambda_1 \dots \lambda_k}) = m^2 \tag{5.35}$$

and

$$q = e \tag{5.36}$$

describe submanifolds K_m and K_e respectively in terms of coordinates

$$(x^\kappa, y_{\lambda_1 \dots \lambda_{k-1}}, p_{\mu \dots \mu_k}, q) .$$

We denote by $v_{m,e}$ the restriction of v to $K_{m,e}$ and we consider the oriented characteristic distribution

$$N'_{m,e} = \left\{ w \in \Lambda^k TR ; \tau_{\mathbf{R}}(w) \in K_{m,e}, \exists_{\varepsilon,\delta \in \mathbf{R}} \varepsilon > 0, \right.$$
$$\left. w \lrcorner v = (-1)^k \frac{\varepsilon}{2m} dM^2 + (-1)^k \delta dE \right\}, \tag{5.37}$$

where M^2 and E are the functions

$$M^2 : R \to \mathbf{R} : r \mapsto \Lambda^k \bar{g}(\mathbf{v}(r), \mathbf{v}(r)) \tag{5.38}$$

and

$$E : R \to \mathbf{R} : r \mapsto \mathbf{h}(r) . \tag{5.39}$$

An embedding

$$f : \mathbf{R}^k \to R : (t^1, \dots, t^k) \mapsto f(t^1, \dots, t^k) \tag{5.40}$$

is a parametrized trajectory of the extended object if the image of the prolongation

$$f' : \mathbf{R}^k \to \Lambda^k TR \tag{5.41}$$

is contained in $N'_{m,e}$.

Let $k = 2$ and let

$$\begin{aligned} x^{\kappa_1 \kappa_2} &= f^{\kappa_1 \kappa_2}(t^1, t^2) , \\ y_\lambda &= g_\lambda(t^1, t^2) , \\ p_{\mu_1 \mu_2} &= f_{\mu_1 \mu_2}(t^1, t^2) , \\ q &= h(t^1, t^2) \end{aligned} \tag{5.42}$$

be the coordinate expression of an ambedding f. The tangent 2-vector

$$w = \left(\frac{\partial f^\kappa}{\partial t^1} \frac{\partial}{\partial x^\kappa} + \frac{\partial g_\kappa}{\partial t^1} \frac{\partial}{\partial y_\kappa} + \frac{1}{2} \frac{\partial f_{\kappa\mu}}{\partial t^1} \frac{\partial}{\partial p_{\kappa\mu}} + \frac{\partial h}{\partial t^1} \frac{\partial}{\partial q} \right)$$
$$\wedge \left(\frac{\partial f^\lambda}{\partial t^2} \frac{\partial}{\partial x^\lambda} + \frac{\partial g_\lambda}{\partial t^2} \frac{\partial}{\partial y_\lambda} + \frac{1}{2} \frac{\partial f_{\lambda\nu}}{\partial t^2} \frac{\partial}{\partial p_{\lambda\nu}} + \frac{\partial h}{\partial t^2} \frac{\partial}{\partial q} \right) \tag{5.43}$$

belongs to $N'_{m,e}$ if equations

$$
\begin{aligned}
&\frac{\partial f^{\kappa_1}}{\partial t^1}\frac{\partial f^{\kappa_2}}{\partial t^2}-\frac{\partial f^{\kappa_1}}{\partial t^2}\frac{\partial f^{\kappa_2}}{\partial t^1}=\frac{\varepsilon}{m}g^{\kappa_1\lambda_1}g^{\kappa_2\lambda_2}(f_{\lambda_1\lambda_2}-eA_{\lambda_1\lambda_2}),\\
&\frac{\partial f_{\kappa\mu}}{\partial t^1}\frac{\partial f^{\kappa}}{\partial t^2}-\frac{\partial f_{\kappa\mu}}{\partial t^2}\frac{\partial f^{\kappa}}{\partial t^1}=\frac{\varepsilon}{2m}\partial_\mu g^{\kappa_1\lambda_1}g^{\kappa_2\lambda_2}(f_{\kappa_1\kappa_2}-eA_{\kappa_1\kappa_2})(f_{\lambda_1\lambda_2}-eA_{\lambda_1\lambda_2})\\
&\qquad\qquad-\frac{\varepsilon e}{2m}g^{\kappa_1\lambda_1}g^{\kappa_2\lambda_2}(f_{\kappa_1\kappa_2}-eA_{\kappa_1\kappa_2})\partial_\mu A_{\lambda_1\lambda_2},\\
&\frac{\partial g_{\kappa}}{\partial t^1}\frac{\partial f^{\kappa}}{\partial t^2}-\frac{\partial g_{\kappa}}{\partial t^2}\frac{\partial f^{\kappa}}{\partial t^1}=-\frac{\varepsilon}{2m}g^{\kappa_1\lambda_1}g^{\kappa_2\lambda_2}(f_{\kappa_1\kappa_2}-eA_{\kappa_1\kappa_2})A_{\lambda_1\lambda_2}+\delta,\\
&\varepsilon>0
\end{aligned}
\tag{5.44}
$$

are satisfied together with

$$
\begin{aligned}
&\frac{1}{k!}g^{\kappa_1\lambda_1}g^{\kappa_2\lambda_2}(f_{\kappa_1\kappa_2}-eA_{\kappa_1\kappa_2})(f_{\lambda_1\lambda_2}-eA_{\lambda_1\lambda_2})=m^2,\\
&h=e.
\end{aligned}
\tag{5.45}
$$

Equations (5.44) are equivalent to the system

$$
\begin{aligned}
&f_{\lambda_1\lambda_2}-eA_{\lambda_1\lambda_2}=\frac{m}{\varepsilon}g_{\lambda_1\kappa_1}g_{\lambda_2\kappa_2}\left(\frac{\partial f^{\kappa_1}}{\partial t^1}\frac{\partial f^{\kappa_2}}{\partial t^2}-\frac{\partial f^{\kappa_1}}{\partial t^2}\frac{\partial f^{\kappa_2}}{\partial t^1}\right),\\
&\varepsilon>0,\\
&\frac{1}{2}A_{\kappa_1\kappa_2}\left(\frac{\partial f^{\kappa_1}}{\partial t^1}\frac{\partial f^{\kappa_2}}{\partial t^2}-\frac{\partial f^{\kappa_1}}{\partial t^2}\frac{\partial f^{\kappa_2}}{\partial t^1}\right)+\frac{\partial g_{\kappa}}{\partial t^1}\frac{\partial f^{\kappa}}{\partial t^2}-\frac{\partial g_{\kappa}}{\partial t^2}\frac{\partial f^{\kappa}}{\partial t^1}=\delta,\\
&\frac{D(f_{\kappa\lambda}-eA_{\kappa\lambda})}{\partial t^1}\frac{\partial f^{\kappa}}{\partial t^2}-\frac{D(f_{\kappa\lambda}-eA_{\kappa\lambda})}{\partial t^2}\frac{\partial f^{\kappa}}{\partial t^1}\\
&\qquad=\frac{1}{2}eF_{\kappa\mu\lambda}\left(\frac{f^{\kappa}}{\partial t^1}\frac{f^{\mu}}{\partial t^2}-\frac{f^{\kappa}}{\partial t^2}\frac{f^{\mu}}{\partial t^1}\right),
\end{aligned}
\tag{5.46}
$$

where

$$
\frac{D(f_{\kappa\lambda}-eA_{\kappa\lambda})}{\partial t^i}=\frac{\partial(f_{\kappa\lambda}-eA_{\kappa\lambda})}{\partial t^i}-\Gamma^{\mu}_{\kappa\nu}(f_{\mu\lambda}-eA_{\mu\lambda})\frac{\partial f^{\nu}}{\partial t^i}-\Gamma^{\mu}_{\lambda\nu}(f_{\kappa\mu}-eA_{\kappa\mu})\frac{\partial f^{\nu}}{\partial t^i}.
\tag{5.47}
$$

A formulation of dynamics in the phase space $P=\Lambda^k T^*X$ can be obtained by a reduction analogous to that used in Section 3. Let

$$\kappa: K_e\to P \tag{5.48}$$

be the restriction of the mapping $\mathbf{v}$ to K_e. Then

$$\kappa*\omega_e=\nu_e \tag{5.49}$$

and

$$\kappa(K_{m,e}) = C_m = \{ p \in P ; M^2(p) = m^2 \} , \tag{5.50}$$

where

$$\omega_e = \omega - e\pi^*F , \tag{5.51}$$

$$\nu_e = \nu \mid K_e \tag{5.52}$$

and M^2 is the function

$$M^2 : P \to \mathbf{R} : p \mapsto \Lambda^k \bar{g}(p, p) . \tag{5.53}$$

Trajectories of the extended object in the phase space P are integral manifolds of the oriented characteristic distribution

$$D'_{m,e} = \left\{ w \in \Lambda^k TP ; \tau_P(w) \in C_m, \exists_{\varepsilon \in \mathbf{R}} \varepsilon > 0, \; w \;\lrcorner\; \omega_e = (-1)^k \frac{\varepsilon}{2m} dM^2 \right\} \tag{5.54}$$

of the form

$$\omega_{m,e} = \omega_e \mid C_m . \tag{5.55}$$

Let

$$\begin{aligned} x^\kappa &= f^\kappa(t^1, t^2) , \\ p_{\lambda_1\lambda_2} &= f_{\lambda_1\lambda_2}(t^1, t^2) \end{aligned} \tag{5.56}$$

be the coordinate expression of an embedding

$$f : \mathbf{R}^2 \to P \tag{5.57}$$

in a coordinate system $(x^\kappa, p_{\lambda_1\lambda_2})$. The tangent 2-vector

$$w = \left(\frac{\partial f^\kappa}{\partial t^1} \frac{\partial}{\partial x^\kappa} + \frac{\partial f_{\lambda_1\lambda_2}}{\partial t^1} \frac{\partial}{\partial p_{\lambda_1\lambda_2}} \right) \wedge \left(\frac{\partial f^\mu}{\partial t^2} \frac{\partial}{\partial x^\mu} + \frac{\partial f_{\nu_1\nu_2}}{\partial t^2} \frac{\partial}{\partial p_{\nu_1\nu_2}} \right) \tag{5.58}$$

belongs to $D'_{m,e}$ if equations

$$\begin{aligned} &\frac{\partial f^{\kappa_1}}{\partial t^1} \frac{\partial f^{\kappa_2}}{\partial t^2} - \frac{\partial f^{\kappa_1}}{\partial t^2} \frac{\partial f^{\kappa_2}}{\partial t^1} = \frac{\varepsilon}{m} g^{\kappa_1\lambda_1} g^{\kappa_2\lambda_2} f_{\lambda_1\lambda_2} , \\ &\frac{\partial f_{\kappa\lambda}}{\partial t^1} \frac{\partial f^\kappa}{\partial t^2} - \frac{\partial f_{\kappa\lambda}}{\partial t^2} \frac{\partial f^\kappa}{\partial t^1} - \frac{1}{2} eF_{\kappa\mu\lambda} \left(\frac{\partial f^\kappa}{\partial t^1} \frac{\partial f^\mu}{\partial t^2} - \frac{\partial f^\kappa}{\partial t^2} \frac{\partial f^\mu}{\partial t^1} \right) \\ &\qquad = \frac{\varepsilon}{2m} \partial_\lambda g^{\kappa_1\mu_1} g^{\kappa_2\mu_2} f_{\kappa_1\kappa_2} f_{\mu_1\mu_2} , \\ &\varepsilon > 0 \end{aligned} \tag{5.59}$$

are satisfied together with

$$\frac{1}{2} g^{\kappa_1\lambda_1} g^{\kappa_2\lambda_2} f_{\kappa_1\kappa_2} f_{\lambda_1\lambda_2} = m^2 . \tag{5.60}$$

Equations (5.59) are equivalent to

$$\begin{aligned} & f_{\kappa_1\kappa_2} = \frac{m}{\varepsilon} g_{\kappa_1\lambda_1} g_{\kappa_2\lambda_2} \left(\frac{\partial f^{\lambda_1}}{\partial t^1} \frac{\partial f^{\lambda_2}}{\partial t^2} - \frac{\partial f^{\lambda_1}}{\partial t^2} \frac{\partial f^{\lambda_2}}{\partial t^1} \right), \\ & \varepsilon > 0 , \\ & \frac{\mathrm{D} f_{\kappa\lambda}}{\partial t^1} \frac{f^{\kappa}}{\partial t^2} - \frac{\mathrm{D} f_{\kappa\lambda}}{\partial t^2} \frac{f^{\kappa}}{\partial t^1} = \frac{1}{2} e \mathrm{F}_{\kappa\mu\lambda} \left(\frac{\partial f^{\kappa}}{\partial t^1} \frac{\partial f^{\mu}}{\partial t^2} - \frac{\partial f^{\kappa}}{\partial t^2} \frac{\partial f^{\mu}}{\partial t^1} \right). \end{aligned} \tag{5.61}$$

Although both formulations of dynamics, in R and in P, are gauge invariant, the formulation of dynamics in R is distinguished by having a gauge invariant Lagrangian description.

REFERENCES

[1] P. DEDECKER, *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, Heidelberg, New York, t. **570**, 1977, p. 395-456.
[2] A. LICHNEROWICZ, *C. R. Acad. Sci. Paris*, Sér. A, t. **280**, 1975, p. 37-40, 523-527.
[3] M. R. MENZIO and W. M. TULCZYJEW, *Ann. Inst. Henri Poincaré*, t. **28**, 1978. p. 349-367.
[4] J. SNIATYCKI and W. M. TULCZYJEW, *Ann. Inst. Henri Poincaré*, t. **14**, 1971, p. 177-187.
[5] S. STERNBERG, *Proc. Nat. Acad. Sci. USA*, t. **74**, 1977, p. 5253-5254.
[6] J. A. SCHOUTEN, *Ricci-Calculus*, *Second edition*, Springer-Verlag, Berlin, Götingen, Heidelberg, 1954.
[7] R. J. TORRENCE and W. M. TULCZYJEW, *J. Math. Phys.*, t. **14**, 1973, p. 1725-1732.
[8] A. TRAUTMAN, *Rep. Math. Phys.*, t. **1**, 1970, p. 29.
[9] W. M. TULCZYJEW, *Ann. Inst. Henri Poincaré*, t. **27**, 1977, p. 101-114.
[10] W. M. TULCZYJEW, *Acta Phys. Polon.*, Sér. B, t. **8**, 1977, p. 431-447.
[11] W. M. TULCZYJEW, *Rep. Math. Phys.*, t. **16**, 1978, p. 227-244.
[12] A. WEINSTEIN, *Lett. Math. Phys.*, t. **2**, 1978, p. 417-420.

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