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An exactly solvable model in Predictive Relativistic Mechanics. I

by

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ABSTRACT. — We present a family of hamiltonian models for N-particle systems in Predictive Relativistic Mechanics. The relation between position and velocity variables and the non-physical canonical ones is derived. The case of a harmonic oscillator potential is studied in detail. We compare our model with some singular lagrangian systems already appeared in the literature.

1. INTRODUCTION

This is the first of a series of papers about a class of systems of particles interacting at a distance and their quantization. The interest of relativistic action-at-a-distance theories has recently grown [*1*] [*2*] [*3*] [*4*] [*5*] [*6*]. From the theoretical viewpoint this is due to the fact that the simultaneous character of the interaction has been made compatible with Poincaré invariance by Predictive Relativistic Mechanics. Also, hamiltonian systems with constraints have been better understood. These systems are only defined on a submanifold of the phase space and this enables one to circumvent the no-interaction theorem. The connexion between both

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formalisms, Predictive Relativistic Mechanics and Singular Lagrangian Theory has been recently established by the authors [7] [8].

From a phenomenological viewpoint the use of covariant harmonic oscillator models has allowed to obtain the mass spectrum of hadrons and has provided a theoretical way to explain the quark-confinement [9] [10].

Before the work of Droz-Vincent [5] only perturbative solutions of physically interesting problems were known [1] [19] [20] within the framework of Predictive Relativistic Mechanics.

We present here a family of exactly solvable models which includes as a particular case for two particles the Droz-Vincent system.

Although we do not believe that this kind of simple models can describe the interaction between quarks, this will be only possible in the framework of a relativistic second quantization. Nevertheless, the clearness, the simplicity and the exact solvability of these models permit a better understanding of the main features of quark interaction.

In sec. 2 we present a short reviews of Predictive Relativistic Mechanics and the general characteristics of our model. It is inspired on Dirac's idea [11] of distinguishing the kinematic generators from the dynamic ones among those of the Poincaré Group. In our case we make the above mentioned distinction among the generators of the Complete Symmetry Group which is an abelian extension of the Poincaré Group.

In sec. 3 we present the model in detail. In order to circumvent the No-Interaction Theorems [12] [13] we deal with a non-physical canonical coordinates. The relation between the position and velocity variables and the canonical ones is derived in sec. 4. In sec. 5 we present some applications of the model: harmonic oscillator potential, free particles system and comparison with some singular lagrangian models.

2. PREDICTIVE HAMILTONIAN SYSTEMS

A Predictive Poincaré invariant system is a second order differential system

$$\frac{\partial x_a^\mu}{\partial \sigma_b} = \delta_{ab}\pi_a^\mu \qquad \frac{\partial \pi_a^\mu}{\partial \sigma_b} = \delta_{ab}\theta_a^\mu(x, \pi) \tag{2.1}$$

$$a, b, c = 1, \ldots, \mathrm{N}; \qquad \mu, \nu, \ldots = 0, 1, 2, 3$$

which describes the dynamics of N particles.

The acceleration θ_a^μ must satisfy [14] [15]:

$$\theta_a^\mu(x, \pi)\pi_{a\mu} = 0 \tag{2.2}$$

$$\pi_{a'}^\nu \frac{\partial \theta_a^\mu}{\partial x^{a'\nu}} + \theta_{a'}^\nu \frac{\partial \theta_a^\mu}{\partial \pi^{a'\nu}} = 0 \tag{2.3}$$

where a'/a, the summation convention holds for the $a, b, c, \ldots$ and $\mu, \nu, \lambda, \ldots$ indices, and the metric is taken $(-+++)$.

Equation (2.3) exhibits that σ_a is the proper time of the particle « a » appart from a multiplicative constant which we shall take equal to $\frac{1}{m_a}$. If σ_a were the proper time, it would yield

$$\pi_a^\mu \pi_{a\mu} = -1$$

and the coordinates (x_a^μ, π_b^ν) would no longer be independent. This would obstruct the construction of a halmitonian formalism. On the contrary, the choice of σ_a mentioned above allows π_a^2 to take on any value which, according to eq. (2.2), will be an integral of motion.

Equations (2.3) show that the solution x_a of system (2.1) only depends on the parameter σ_a. Furthermore, they guarantee the integrability of the system. Equations (2.3) can also be written as:

$$[\vec{H}_a, \vec{H}_{a'}] = 0, \qquad \vec{H}_a = \pi_a^\mu \frac{\partial}{\partial x^{a\mu}} + \theta_a^\mu(x, \pi) \frac{\partial}{\partial \pi^{a\mu}} \tag{2.4}$$

The invariance of (2.1) under the Poincaré group implies that the functions $\theta_a^\mu(x, \pi)$ behave as four-vectors invariant under translations. This is equivalent to

$$[\vec{P}_\mu, \vec{H}_a] = 0 \tag{2.5a}$$

$$[\vec{J}_{\mu\nu}, \vec{H}_a] = 0 \tag{2.5b}$$

where:

$$\vec{P}_\mu = -\varepsilon_a \frac{\partial}{\partial x_a^\mu} \tag{2.6a}$$

$$\vec{J}_{\mu\nu} = (\eta_\mu^\lambda \eta_{\nu\rho} - \eta_\nu^\lambda \eta_{\mu\rho})\left(x_a^\rho \frac{\partial}{\partial x_a^\lambda} + \pi_a^\rho \frac{\partial}{\partial \pi_a^\lambda}\right) \tag{2.6b}$$

are the infinitesimal generators of the Poincaré Group.

Equations (2.4) and (2.5) exhibit that $\vec{H}_a$, $\vec{P}_\mu$, $\vec{J}_{\mu\nu}$ span a realization of an abelian extension of the Poincaré algebra [*16*]. The associated transformation group $\mathscr{P} \otimes \mathscr{A}_N$ is the direct product of the Poincaré group (purely kinematic) and the dynamical group $\mathscr{A}_N \approx \mathbb{R}^N$.

A hamiltonian formalism for the system (2.1) is determined by a symplectic form Ω on $(TM^4)^N$ invariant under $\mathscr{P} \otimes \mathscr{A}_N$.

As it is well known, a function f can be associated to each field $\vec{\Lambda}$ that leaves Ω invariant. This function is defined by [*16*].

$$df = -i_{\vec{\Lambda}}\Omega \tag{2.7}$$

This function is defined at least locally on $(TM^4)^N$ and it is unique except for an arbitrary additive constant.

In particular, the functions associated to $\vec{P}_\mu$, $\vec{J}_{\mu\nu}$ and $\vec{H}_a$ are the linear momentum, the angular momentum and the hamiltonians, respectively.

This is the way to follow in order to quantize a classical system like (2.1) [17]. We must point out that this procedure involves perturbative solutions as in the cases of electromagnetic interaction and of short range scalar and vector interactions [18].

We will follow, however, Droz-Vincent's approach [5]. Instead of constructing a hamiltonian formalism for a given interaction (e. g. electromagnetism...), we display some exactly solvable models which serve us to know better the structure of these theories. We also want some of these models to describe the interaction between quarks.

We work in an adapted canonical coordinates system (q_a^μ, p_b^ν) of Ω [16], i. e.: these coordinates satisfy the following conditions:

i) q_a^μ and p_b^ν $(a, b = 1, \ldots, N)$ are four-vectors and q_a^μ behaves like x_a^μ under translations.

ii) the sympletic form Ω can be expressed as:

$$\Omega = dq_a^\mu \wedge dp_\mu^a \tag{2.8}$$

or, equivalently, the Poisson bracket associated to Ω is such that:

$$\{ q_a^\mu, p_b^\nu \} = \delta_{ab}\eta^{\mu\nu}, \qquad \{ q_a^\mu, q_b^\nu \} = \{ p_a^\mu, p_b^\nu \} = 0 \tag{2.8a}$$

Then, if P_μ and $J_{\mu\nu}$ are the functions associated by eq. (2.7) to the generators $\vec{P}_\mu$ and $\vec{J}_{\mu\nu}$ of $\mathscr{P}$, we immediately obtain from the condition (*i*) that:

$$P_\mu = \varepsilon_a p_\mu^a, \qquad J_{\mu\nu} = q_{a\mu}p_\nu^a - q_{a\nu}p_\mu^a \tag{2.9}$$

Therefore, the dynamical aspects of the system will be present in the functions $H_a(q, p)$ (associated to $\vec{H}_a$) and in the functions $x_a^\mu(q, p)$, $\pi_a^\mu(q, p)$ which relate the position and velocity variables to the canonical coordinates and momenta.

Hence, a model will consist in giving N function $H_a(q, p)$ satisfying the following conditions:

i) each $H_a(q, p)$ is invariant under the Poincaré group

ii)
$$\{ H_a, H_{a'} \} = 0, \qquad a \neq a' \tag{2.10}$$

Both conditions guarantee that eq. (2.4) and (2.5) will be fulfilled by the generators $\vec{H}_a$ associated to $H_a(q, p)$ by (2.7).

After that several problems still remain. First we have to derive the relationship between the adapted canonical coordinates (q, p)—which have not any physical significance—and the position and velocity variables (x_a^μ, π_b^ν). As far as we do not know these relations our hamiltonian model will neither be able to predict anything nor to be compared with other physical models.

Taking into account eq. (2.1), the functions x_a^μ must satisfy:

$$\{ H'_a, x_a^\mu \} = 0 \tag{2.11}$$

This means that the evolution of x_a^μ depends only on the parameter σ_a, unlike the canonical coordinates q_a^μ which depend on the whole set of parameters $(\sigma_1, \ldots, \sigma_N)$.

If we can solve eq. (2.11) we will know the relation $x_a^\mu(q, p)$. As we shall see later a good set of initial data will permit us to collect one among the infinity of solutions of (2.11).

Secondly, according to (2.2) $\{ H_a, x_a^\mu \} \{ H_a, x_{a\mu} \}$ ought to be an integral of motion. It is extremely complicated to impose this condition. However we can circumvent this difficulty by assuming that the field associated trough Ω to $H_a(q, p)$ is not equal to $\vec{H}_a$ but only proportional. That is to say, if we write:

$$\{ H_a, f(q, p) \} = \frac{\partial f}{\partial \tau_a} \tag{2.12}$$

the parameter τ_a is no longer equal to the proper time σ_a.

Now, given a solution $x_a^\mu = f_a^\mu(q, p)\,(a = 1, \ldots, N)$ of eq. (2.10), we need only to require the change of variables

$$x_a^\mu = f_a^\mu(q, p) \qquad \dot{x}_a^\mu = \{ H_a, f_a^\mu \} \tag{2.13}$$

to be invertible.

There is at last one problem left: what is the relation between $(x, \dot{x})$ and (x, π)? As $\dot{x}_a^\mu$ and π_a^μ are parallel we only need the ratio between their lengths in order to define precisely the change of variables. This can be done by giving a fixed value to each integral of motion $H_1, \ldots, H_N$. Similarly to the case of free particles or of separable interaction [*16*] we shall take

$$H_a(q, p) = F_a(x, \dot{x}) = -\frac{1}{2}\pi_a^2 = \frac{1}{2}m_a^2 \tag{2.14}$$

The relation between the parameters σ_a and τ_a can be easily obtained:

$$\frac{d\sigma_a}{d\tau_a} = \sqrt{\frac{\dot{x}_a^2}{\pi_a^2}} = \frac{1}{m_a}\sqrt{-\dot{x}_a^2(\tau_a\,;\,\text{C. I.})} \tag{2.15}$$

where C. I. means the set of initial conditions which determine the trajectories. The change of parameter $\sigma_a(\tau_a)$ can be now obtained from (2.15) by a quadrature.

3. A FAMILY OF MODELS

We shall choose the hamiltonians H_a with the following form:

$$H_a = T_a + V, \qquad a = 1, \ldots, N \tag{3.1}$$

where: T_a is a quadratic function of p_b^μ, V is a Poincaré invariant function and it is the same for all a.

For the sake of simplicity we shall also require that both T_a and V are well behaved under particle interchange. That is to say, if $\sigma \in S_N$ (the symmetric group of N elements), then:

$$T_a(q_{\sigma(b)}, p_{\sigma(c)}) = T_{\sigma(a)}(q_b, p_c)$$
$$V(q_{\sigma(b)}, p_{\sigma(c)}) = V(q_b, p_c)$$

This condition implies that:

$$T_a = \alpha_1 p_a^2 + \beta_1(Pp_a) + \gamma_1 \varepsilon^b p_b^2 + \delta_1 \sum_{b<b'} (p_b p_{b'}) \tag{3.2}$$

where α_1, β_1, γ_1 and δ_1 are arbitrary constants.

Instead of using the variables (q_a^μ, p_b^ν) it is more suitable to perform the canonical transformation:

$$X^\mu = \frac{1}{N}\varepsilon^a q_a^\mu \qquad z_A^\mu = q_1^\mu - q_A^\mu$$
$$P^\mu = \varepsilon^a p_a^\mu \qquad y_A^\mu = \frac{1}{N}P^\mu - p_A^\mu \tag{3.3}$$

$A = 2, 3, \ldots, N$

The new variables P^μ, y_A^ν, z_B^λ are translation invariant. Hence, $V(q, p)$ must be some function of the following scalar variables:

$$P^2, (P, y_A), (P, z_A), (\tilde{y}_A, \tilde{y}_B), (\tilde{y}_A, \tilde{z}_B), (\tilde{z}_A, \tilde{z}_B) \tag{3.4}$$

where $\tilde{y}_A^\mu$, $\tilde{z}_B^\nu$ are the projections of y_A^μ, z_B^ν perpendicular to P^μ:

$$\tilde{y}_A^\mu = \pi^\mu{}_\nu y_A^\nu \qquad \pi^\mu{}_\nu = \eta^\mu{}_\nu - \frac{P^\mu P_\nu}{P^2} \tag{3.5}$$

In terms of the new variables, eq. (3.2) yields:

$$T_a = \alpha(P, y_a) - \beta P^2 - \gamma\left[\varepsilon^A y_A^2 + \sum_{A<A'} (y_A, y_{A'})\right] + \delta y_a^2 \tag{3.6}$$

where, in orden to simplify the notation, we have introduced:

$$y_1^\mu = \frac{1}{N}P^\mu - p_1^\mu = -\varepsilon^A y_A^\mu$$

and α, β, γ, δ are arbitrary constants.

The functions H_a that we have chosen must satisfy the predictivity equations (2.10). As the hamiltonians H_a are well behaved under particle interchange it suffices to require: $\{H_1, H_A\} = 0$ $A = 2, 3, \ldots, N$. Using (3.6) these conditions yield:

$$\alpha\{Py_A, V\} + \alpha\varepsilon^B\{Py_B, V\} - \delta\frac{\partial(y_1^2 - y_A^2)}{\partial(y_B, y_C)}\{(y_B, y_C), V\} = 0 \tag{3.7}$$

Taking into account that V depends on the variables (3.4) we obtain:

$$\{ Py_A, V \} = - P^2 \frac{\partial V}{\partial (Pz_A)} \tag{3.8}$$

The simplest models are those with $\delta = 0$. Otherwise the differential system (3.7) becomes extremely complicated. For these simplified models eq. (3.7) yields:

$$\frac{\partial V}{\partial (Pz_A)} = 0 \tag{3.9}$$

Hence, if we choose $\delta = 0$ and $V(P^2, Py_A, \tilde{y}_A\tilde{y}_B, \tilde{y}_A\tilde{z}_B, \tilde{z}_A\tilde{z}_B)$ the predictivity condition (2.9) is automatically fulfilled.

Since V does not depen on (Pz_A), the functions Py_A are integrals of motion. And, as P^2 is also an integral of motion the products (Pp_a) are conserved quantities, too.

Other integrals of motion are those associated to the generators of the Lorentz group:

$$J_{\mu\nu} = q^a_\mu p_{a\nu} - q^a_\nu p_{a\mu}$$

which can be expressed as:

$$J_{\mu\nu} = \frac{1}{M}(K_\mu P_\nu - K_\nu P_\mu) + \frac{1}{M}\eta_{\mu\nu\lambda\rho}W^\lambda P^\rho \tag{3.10}$$

with:

$$\begin{aligned} K^\mu &= \frac{1}{M}J^{\nu\mu}P_\nu = MX^\mu + \frac{(PX)}{M}P^\mu + \frac{1}{M}\varepsilon^A[(Pz_A)y^\mu_A - (Py_A)z^\mu_A] \\ W^\mu &= \frac{1}{2M}\eta^\mu{}_{\nu\lambda\rho}P^\nu J^{\lambda\rho} = \frac{1}{M}\varepsilon^A \eta^\mu{}_{\nu\lambda\rho}P^\nu z^\lambda_A y^\rho_A \end{aligned} \tag{3.11}$$

where: $M = (-P^2)^{1/2}$ and $\eta^{\mu\nu\lambda\rho}$ is the four-dimensional Levi-Civita tensor.

Analogously to the canonical transformation (3.3) we perform a change of parameters:

$$\lambda = \frac{1}{N}\varepsilon^a\tau_a, \qquad \frac{\partial}{\partial\lambda} = \varepsilon_a\frac{\partial}{\partial\tau_a} = \varepsilon^a\{H_a, -\} \tag{3.12a}$$

$$\lambda_A = \tau_1 - \tau_A, \qquad \frac{\partial}{\partial\lambda_A} = \frac{1}{N}\frac{\partial}{\partial\lambda} - \frac{\partial}{\partial\tau_A} = -\alpha\{(Py_A), -\} \tag{3.12b}$$

$$a = 1, \ldots, N; \qquad A = 2, 3, \ldots, N$$

It immediately follows from eq. (3.12b) that $\tilde{y}^\mu_A$ and $\tilde{z}^\mu_A$ do not depend on λ_B and they are solely functions of λ. Therefore, the transverse internal motion of the system (i. e.: the evolution of the relative coordinates and momenta in the orthogonal hiperplane to P^μ) is governed by the ordinary second order differential system:

$$\frac{d\tilde{y}_A^\mu}{d\lambda} = N\{V, \tilde{y}_A^\mu\} \tag{3.13a}$$

$$\frac{d\tilde{z}_A^\mu}{d\lambda} = N\{V, \tilde{z}_A^\mu\} + N\gamma(\tilde{y}_A^\mu + \varepsilon^B \tilde{y}_B^\mu) \tag{3.13b}$$

The potential V can depend on $\tilde{y}_A^\mu$, $\tilde{z}_B^\nu$ but also on P^2 and (Py_A). However, as P^2 and Py_A are integrals of motion they can be regarded as parameters in order to integrate the system (3.13). Therefore, eqs. (3.13) allows us to obtain $\tilde{y}_A^\mu(\lambda;$ C. I.) and $\tilde{z}_A^\mu(\lambda;$ C. I.) independently of the evolution of the other variables.

Finally, the evolution equations of (PX) and (Pz_A) yield:

$$\frac{\partial(PX)}{\partial\delta_B} = \alpha(Py_B)\,;\; \frac{\partial(Pz_A)}{\partial\lambda_B} = \alpha\delta_{AB}P^2$$

$$\frac{\partial(PX)}{\partial\lambda} = 2N\beta P^2 - 2NP^2\frac{\partial V}{\partial P^2} - N(Py_A)\frac{\partial V}{\partial(Py_A)} \tag{3.14}$$

$$\frac{\partial(Pz_A)}{\partial\lambda} = N\gamma[(Py_A) + \varepsilon^B(Py_B)] - NP^2\frac{\partial V}{\partial(Py_A)}$$

If V does not depends on P^2 and Py_A, the system (3.13) would be immediately integrable, because P^2 and Py_A are conserved quantities. As we shall see later, if we know (PX) and (Pz_A) in terms of λ, λ_A and the initial conditions then we can easily integrate the position equations (2.11). Therefore we shall consider hereafter only potentials of the form: $V(\tilde{y}_B, \tilde{z}_C)$.

In this case, by integration of eq. (3.14) we obtain:

$$\begin{aligned}(PX) &= \alpha\varepsilon^A(Py_A)\lambda_A + 2N\beta P^2\lambda + \text{cte}\\ (Pz_A) &= \alpha P^2\lambda_A + N\gamma[2(Py_A) + \varepsilon^{A'}(Py_{A'})]\lambda + \text{cte}\end{aligned} \tag{3.15}$$

The hamiltonians to be considered hereafter are therefore:

$$H_a = \alpha(Py_a) - \beta P^2 - \gamma\left[\varepsilon^A y_A^2 + \sum_{A<A'} (y_A, y_{A'})\right] + V(\tilde{y}_B, \tilde{z}_C) \tag{3.16}$$

4. INTEGRATION OF THE POSITION EQUATIONS

In order to complete the model proposed in the last section we must integrate the position equation (2.11).

By analogy with the two-particle model of Droz-Vincent [5] and with some singular lagrangians [2] [3] [4] we shall take the following initial conditions:

$$x^{\mu}_{a|\Sigma} = q^{\mu}_{a} \, ; \qquad \Sigma = \{ (q, p) \in (TM^4)^N \mid P z_A = 0, \, A = 2, \dots, N \} \quad (4.1)$$

The Frobenius theorem guarantees us that there exists a unique solution of (2.11) satisfying the initial conditions (4.1). Given one point $Q = (q_{(0)}, p_{(0)}) \in (TM^4)^N$ there exists a unique integral $\varphi^{\mu}_{a}(\tau_1, \dots, \tau_N)$, $\psi^{\nu}_{b}(\tau_1, \dots, \tau_N)$ of the generalized Hamilton equations depending on N parameters:

$$\frac{\partial q^{\mu}_{a}}{\partial \tau_b} = \{ H_b, q^{\mu}_{a} \} \, ; \qquad \frac{\partial p^{\mu}_{a}}{\partial \tau_b} = \{ H_b, p^{\mu}_{a} \} \quad (4.2)$$

Satisfying $\varphi^{\mu}_{a}(0, \dots, 0) = q^{\mu}_{a}$, $\psi^{\mu}_{a}(0, \dots, 0) = p^{\mu}_{a}$.

By variation of all the parameters but one τ_a, this integral submanifold permits us to go from Q to $Q_a = (\underset{a}{q^{\mu}_{b}}, \underset{a}{p^{\mu}_{b}}) \in \Sigma$:

$$\underset{a}{q^{\mu}_{b}} = \varphi^{\mu}_{b}(\underset{a}{\delta}\tau_c) \, , \qquad \underset{a}{p^{\mu}_{b}} = \psi^{\mu}_{b}(\underset{a}{\delta}\tau_c)$$

where: $\underset{a}{\delta}\tau_a = 0$ and $\underset{a}{\delta}\tau_{a'}$ are suitable variation of the parameters.

From eq. (2.11) $x^{\mu}_{a}(q, p)$ is not altered by a change of $\tau_{a'}(a' \neq a)$. Hence $x^{\mu}_{a}(Q)$ is equal to $x^{\mu}_{a}(Q_a)$ and using eq. (4.1) we have:

$$x^{\mu}_{a}(Q) = q^{\mu}_{a}(Q_a) \quad (4.3)$$

in terms of $Q = (q^{\mu}_{a(0)}, p^{\nu}_{b(0)})$.

Therefore, if we can express $q^{\mu}_{a}(Q_a)$ in terms of $Q = (q^{\mu}_{a}, p^{\nu}_{b})$ the integration of (2.11) with the initial conditions (4.1) will be concluded.

From eq. (3.3), (3.11) and (4.3) we obtain

$$x^{\mu}_{a}(Q) = \frac{K^{\mu}}{M} + (PX)_{Q_a} \frac{P^{\mu}}{P^2} + \varepsilon^{a'} \frac{(P p_{a'})}{P^2} (\tilde{q}^{\mu}_{a} - \tilde{q}^{\mu}_{a'})_{Q_a} \quad (4.4)$$

where the subindex has been omitted from K^{μ}, M, P^{ν} and $(P p_b)$ because they are integrals of motion.

Writing eq. (3.15) at Q and Q_a, substracting them and taking into account eq. (4.1), we have:

$$\begin{aligned} (P z_A)_Q &= N\gamma(P, y_1 - y_A)\underset{a}{\delta}\lambda - \alpha P^2 \underset{a}{\delta}\lambda_A \\ (PX)_Q - (PX)_{Q_a} &= -2N\beta P^2 \underset{a}{\delta}\lambda - \alpha\varepsilon^A (P y_A) \underset{a}{\delta}\lambda_A \end{aligned} \quad (4.5)$$

where $\underset{a}{\delta}\lambda_A$, $\underset{a}{\delta}\lambda$ are expressed in terms of the (N − 1) parameters $\delta_a \tau_{a'}$ by (3.12).

We can find the unknowns $(\mathrm{PX})_{\mathrm{Q}_a}$ and $\underset{a}{\delta\lambda}$ from eq. (4.5) and we obtain:

$$(\mathrm{PX})_{\mathrm{Q}_a} = (\mathrm{PX}) - \frac{1}{\mathrm{P}^2}\varepsilon^{\mathrm{A}}(\mathrm{P}y_{\mathrm{A}})(\mathrm{P}z_{\mathrm{A}}) + \frac{(\mathrm{P}, \mathrm{N}q_a - \varepsilon^b q_b)}{\mathrm{P}^2[\alpha\mathrm{P}^2 + \mathrm{N}\gamma(\mathrm{P}y_a)]}[2\beta\mathrm{P}^4 + \gamma\varepsilon^{\mathrm{A}}(\mathrm{P}y_{\mathrm{A}})(\mathrm{P}, y_1 - y_{\mathrm{A}})] \quad (4.6)$$

$$\underset{a}{\delta\lambda} = \frac{(\mathrm{P}, \mathrm{N}q_a - \varepsilon^b q_b)}{\mathrm{N}[\alpha\mathrm{P}^2 + \mathrm{N}\gamma(\mathrm{P}y_a)]} \quad (4.7)$$

where the subindex Q is understood in the quantities on the right hand side. Now the quantities $(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{\mathrm{Q}_a}$ are still to be found. Let us write (3.13) as:

$$\begin{aligned}\frac{d(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)}{d\lambda} &= \mathrm{N}\{\mathrm{V}, \tilde{q}_a^\mu - \tilde{q}_{a'}^\mu\} + \mathrm{N}\gamma(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)\\ \frac{d(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)}{d\lambda} &= \mathrm{N}\{\mathrm{V}, \tilde{p}_a^\mu - \tilde{p}_{a'}^\mu\}\end{aligned} \quad (4.8)$$

Let $\{f_{a'}^\mu(\lambda; \text{I. C.}), g_{a'}^\mu(\lambda; \text{I. C.})\}$ be the general integral of (4.8). Then, it is quite clear that:

$$(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{\mathrm{Q}_a} = f_{a'}^\mu(\underset{a}{\delta\lambda}; (\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu, \tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)_{(0)}) \quad (4.9)$$

where $\underset{a}{\delta\lambda}$ is given by eq. (4.7).

Substituting eq. (4.9) and (4.6) in eq. (4.4) the integration of the position equations (2.11) is complete. We have to point out that only the term $(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{\mathrm{Q}_a}$ depends on the potential. Since Q is arbitrary, the result is general, so the indices Q and (0) will be dropped from now on.

The change of variables $(q, p) \to (x, \dot{x})$ can be finally written as:

$$x_a^\mu = q_a^\mu - \varepsilon^{\mathrm{A}}\frac{(\mathrm{P}z_{\mathrm{A}})}{\mathrm{P}^2}y_{\mathrm{A}}^\mu + \varepsilon^{a'}\frac{(\mathrm{P}p_{a'})}{\mathrm{P}^2}[(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{\mathrm{Q}a} - (\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)] + \frac{(\mathrm{P}, \mathrm{N}q_a - \varepsilon^b q_b)}{\alpha\mathrm{P}^2 + \mathrm{N}\gamma(\mathrm{P}y_a)}\left\{2\beta - \frac{\alpha}{\mathrm{N}} + \frac{\gamma}{\mathrm{P}^\mu}\varepsilon^{a'}(\mathrm{P}p_{a'})(\mathrm{P}, p_a - p_{a'})\right\}\mathrm{P}^\mu \quad (4.10)$$

$$\dot{x}_a^\mu = \alpha\mathrm{N}\frac{2\beta\mathrm{P}^4 - \gamma\varepsilon^b(\mathrm{P}y_b)^2}{\alpha\mathrm{P}^2 + \mathrm{N}\gamma(\mathrm{P}y_a)}\frac{\mathrm{P}^\mu}{\mathrm{P}^2} + \varepsilon^{a'}\frac{(\mathrm{P}p_{a'})}{\mathrm{P}^2}\{\mathrm{H}_a, (\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{\mathrm{Q}a}\} \quad (4.11)$$

were the subindex (0) is dropped from (4.9).

If this change of variables is to be invertible and $\dot{x}_b^\mu$ and P^μ well behaved, the following conditions must be also satisfied:

$$\begin{gathered}\dot{x}_a^2 < 0, \qquad \dot{x}_a^c > 0, \qquad \mathrm{P}^2 < 0, \qquad \mathrm{P}^0 > 0\\ \det\left(\frac{\partial(x, \dot{x})}{\partial(q, p)}\right) \neq 0\end{gathered} \quad (4.12)$$

In any case we can choose α, β, γ and V such that the conditions above hold in $(\mathrm{TM}^4)^{\mathrm{N}}$ except, perhaps, in a set of measure zero.

5. APPLICATIONS

a) Harmonic oscillator.

In order to explain the interaction between quarks, different models of relativistic harmonic oscillators have appeared in the literature [*21*] [*22*] [*5*] [*6*]. We shall now present a model of N-particle harmonic oscillator, which for N = 2 includes the model of Droz-Vincent [*5*] mentioned above as a particular case. We shall use the potential:

$$V = -\frac{1}{2}K\sum_{a<a'}(\tilde{q}_a - \tilde{q}_{a'})^2 = -K\left[\frac{N-1}{2}\varepsilon^A \tilde{z}_A^2 - \sum_{B<B'}(\tilde{z}_B, \tilde{z}_{B'})\right] \quad (5.1)$$

Then, the equations of motion for the relative transverse coordinates and momenta can be written as:

$$\begin{aligned} \frac{d(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)}{d\lambda} &= N\gamma(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu) \\ \frac{d(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)}{d\lambda} &= -N^2K(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu) \end{aligned} \quad (5.2)$$

Then, we obtain the following equation for $\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu$:

$$\frac{d^2(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)}{d\lambda^2} + \omega^2(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu) = 0, \qquad \omega^2 = N^3\gamma K \quad (5.3)$$

In other words, all the relative transverse coordinates oscillate with the same frequency $\omega = N\sqrt{N\gamma K}$. On the other hand, eq. (3.15) shows that the component parallel to P^μ of these coordinates depends linearly on τ_a.

To completely solve this model we must derive the change of variables $(q, p) \to (x, \dot{x})$. From (4.9) and (5.3) we obtain easily:

$$\begin{aligned} &(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{Q_a} \\ &= (\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)\cos(\omega\underset{a}{\delta\lambda}) + \frac{N\gamma}{\omega}(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)\sin(\omega\underset{a}{\delta\lambda})\{H_a, (\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)_{Q_a}\} \\ &= \frac{\alpha NP^2}{\alpha P^2 + N\gamma(Py_a)}\left[\gamma(\tilde{p}_a^\mu - \tilde{p}_{a'}^\mu)\cos(\omega\underset{a}{\delta\lambda}) + \frac{\omega}{N}(\tilde{q}_a^\mu - \tilde{q}_{a'}^\mu)\sin(\omega\underset{a}{\delta\lambda})\right] \end{aligned} \quad (5.4)$$

where $\underset{a}{\delta\lambda}$ is given by (4.6). The introduction of the results (5.4) in eq. (4.10) and (4.11) determines the above mentioned change of variables. For the particular case $N = 2$, $\alpha = +\frac{1}{2}$, $\beta = +\frac{1}{8}$, $\gamma = +\frac{1}{2}$, our hamiltonians yield:

$$H_a = -\frac{1}{2}p_a^2 - \frac{1}{2}K\tilde{z}^2 \quad (5.5)$$

which coincides with the harmonic oscillator hamiltonian presented by Droz-Vincent [5], if we take: $K_{DV} = \frac{1}{2} K$.

In this case the position and velocity variables can be expressed in a particularly simple form:

$$x_a^\mu = q_a^\mu - \frac{(Pz)}{P^2}\tilde{y}^\mu - \eta_a \frac{(Pp_{a'})}{P^2}\left[\tilde{z}^\mu\left(1 - \cos\frac{\omega(Pz)}{2(Pp_{a'})}\right) - \frac{2\eta_a}{\omega}\tilde{y}^\mu \sin\frac{\omega(Pz)}{2(Pp_{a'})}\right]$$
$$\dot{x}_a^\mu = \frac{(Pp_a)}{P^2}P^\mu + \eta_a \tilde{y}^\mu \cos\frac{\omega(Pz)}{2(Pp_{a'})} - \frac{1}{2}\omega\tilde{z}^\mu \sin\frac{\omega(Pz)}{2(Pp_{a'})} \tag{5.6}$$

where: $y = y_2$, $z = z_2$, $\eta_a = (-1)^{a+1}$, $a = 1, 2$.

Equations (5.6) coincide with Droz-Vincent's expressions. We have to point out that, when restricted to $\Sigma = \{ (TM^4)^2 \mid P \cdot (q_1 - q_2) = 0 \}$, not only $x_a^\mu(q, p) = q_a^\mu$, but also: $\dot{x}_a^\mu(q, p) = p_a^\mu$.

On the other hand, if $N > 2$ it is no longer possible to find suitable values of α, β, γ such that $T_a = -\frac{1}{2}p_a^2$. Because of this it was necessary to generalize the expression of T_a in order to describe (in a solvable way) a model involving three or more particles.

b) Singular lagrangian systems.

Most of the singular lagrangian systems existing in the literature present the constraints:

$$\Sigma^* = \{ (Pz_A) = 0, \quad (Py_A) = 0, \quad A = 2, \ldots, N \} \tag{5.7}$$

We are going to show that, when $N = 3$ and for suitable values of α, β, γ and V, our hamiltonian model is a predictive extension of the Takabayasi lagrangian system [4]. In other words, we are going to prove that, when restricted to the submanifold Σ^*, the equations of motion derived from our model yield the equations of motion of Takabayasi (*).

Each integral of our generalized Hamilton equations (4.2) is a N-submanifold parametrized by $\tau_1, \ldots, \tau_N$ which intersects Σ^* in a curve. This is the world line of the system on Σ^*. It is immediate to show that the derivative $\partial/\partial\lambda$ is tangent to Σ^*, while $\partial/\partial\lambda_A$ ($A = 2, \ldots, N$) are not. Therefore, λ is a good parameter to describe the evolution of the system on Σ^*. Hereafter we shall write $d/d\lambda$ instead of $\partial/\partial\lambda \mid \Sigma^*$ and we shall consider a potential of the form $V(z_A)$—we have to realize that $\dot{z_A} = \tilde{z}_A$ on Σ^*.

(*) The same can be done easily for the Kalb and Van Alstine Lagrangian [2] which has the same structure as (5.10] with $N = 2$.

The equations of motion (4.2) restricted to Σ^* yield:

$$\begin{aligned} \frac{dX^\mu}{d\lambda} &= 2N\beta P^\mu & \frac{dP^\mu}{d\lambda} &= 0 \\ \frac{dz_A^\mu}{d\lambda} &= N\gamma(2y_A^\mu + \varepsilon^{A'}y_{A'}^\mu) & \frac{dy_A^\mu}{d\lambda} &= N\frac{\partial V}{\partial z_{A\mu}} \end{aligned} \tag{5.8}$$

from which we obtain the following second order differential equations:

$$\begin{aligned} \frac{d^2X^\mu}{d\lambda^2} &= 0 \\ \frac{d^2z_A^\mu}{d\lambda^2} &= N^2\gamma\left(2\frac{\partial V}{\partial z_{A\mu}} + \varepsilon^{A'}\frac{\partial V}{\partial z_{A'\mu}}\right) \end{aligned} \tag{5.9}$$

Let us now consider the Takabayasi Lagrangian [*4*]:

$$\mathscr{L} = -\sqrt{-\mathscr{U}(z_A)[\underset{\sim}{X}'^2 + z_2'^2 + z_3'^2 - (z_2', z_3')]} \tag{5.10}$$

where (′) means the derivative referred to any Poincaré invariant parameter τ and $\underset{\sim}{X}'^\mu$ is the projection of X'^μ perpendicular to the relative coordinates z_2 and z_3.

In fact, the model presented in ref. [*4*] does not exhibit the cross term (z_2', z_3'). We have added it in order to maintain the invariance under particle interchange. It is immediate to prove that there is a linear transformation which changes the Takabayasi lagrangian into the form (5.10).

From the lagrangian (5.10) we derive the equations of motion:

$$\begin{aligned} \frac{d^2X^\mu}{d\tau^2} &= lX'^\mu \\ \frac{d^2z_A^\mu}{d\tau^2} &= lz_A'^\mu - \frac{X'^2}{3P^2}\left(2\frac{\partial\mathscr{U}}{\partial z_{A\mu}} + \frac{\partial\mathscr{U}}{\partial z_{A'\mu}}\right) \end{aligned} \tag{5.11}$$

which hold on the constrained submanifold $\Sigma^* = \{ Pz_A = 0, Py_A = 0, A = 2, 3 \}$. Here P^μ and y_A^μ are the conjugated momenta of X^μ and z_A^μ:

$$\begin{aligned} P^\mu &= \sqrt{\frac{P^2}{X'^2}}\,X'^\mu \\ y_A^\mu &= \sqrt{\frac{P^2}{X'^2}}\left(z_A'^\mu - \frac{1}{2}z_{A'}'^\mu\right) \end{aligned} \tag{5.12}$$

From eq. [*5*] [*8*] [*9*] [*11*] [*12*] we see easily that both models coincide if we take:

$$\lambda = C\tau, \qquad \beta = 1, \qquad \gamma = \frac{4}{3}, \qquad C = \frac{1}{6}\sqrt{\frac{X'^2}{P^2}} \tag{5.13}$$

and
$$\mathscr{U}(z_A) = b - V(z_A) \tag{5.14}$$

where b and C are numerical constants and it is clear that $\sqrt{X'^2/P^2}$ is an integral of motion. We have to point out that eq. (5.14) is the expression of $U(z_A)$ given by Takabayasi in order to obtain a free particle system when $V = 0$.

c) Free particle system.

When $V = 0$ the equations of motion (4.2) read:

$$\begin{aligned} \frac{\partial q_a^\mu}{\partial \tau_b} &= -(\delta_1 + \delta_{ab}\beta_1)P^\mu - (2\gamma_1 - \delta_1)p_a^\mu - \beta_1 p_b^\mu \\ \frac{\partial p_a^\mu}{\partial \tau_b} &= 0 \end{aligned} \tag{5.15}$$

Hence, the coordinates q_a^μ depend linearly on τ_b and the momenta p_a^μ are integrals of motion.

We have to point out that the relation between p_a^μ and $\dot{q}_a^\mu$ is not the usual simple one: $\dot{q}_a^\mu = p_a^\mu$.

This is due to the fact that we have had to give up the usual free particle form of $T_a = -\frac{1}{2}p_a^2$, $a = 1, \ldots, N$.

Nevertheless, in terms of the position and velocity variables x_a^μ, $\dot{x}_a^\mu$ we can see immediately from eq. (2.10) (4.10) (4.11) and (5.15) that:

$$\begin{aligned} x_a^\mu &= \dot{\underset{0}{x}}{}_a^\mu \tau_a + \underset{0}{x}{}_a^\mu \\ \dot{x}_a^\mu &= \dot{\underset{0}{x}}{}_a^\mu \end{aligned} \tag{5.16}$$

where $\underset{0}{x}{}_a^\mu$ and $\dot{\underset{0}{x}}{}_a^\mu$ are the positions and velocities at $\tau_1 = \ldots = \tau_N = 0$.

6. CONCLUSION AND OUTLOOK

In earlier models with interaction-at-a-distance [*2*] [*3*] [*4*] [*5*] two or three particle systems have been treated separately. They neither can be generalized to N-particle systems nor permit to deal with quark interaction for mesons and baryons in a unified way. Here we have presented a family of N-particle systems which includes the earlier quoted ones when $N = 2$ or $N = 3$—it is a predictive extension of some of them [*2*] [*3*] [*4*] and coincides with the predictive Droz-Vincent's model [*5*].

Many of those models are incomplete because the relation between the position and velocity variables and the canonical ones is not specified. Hence, the physical meaning of the coordinates involved remains unknown.