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Cylindrically symmetric Einstein-Maxwell and scalar fields and stiff fluid distributions

by

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ABSTRACT. — The problem of interacting charged perfect fluid distribution and massive scalar field is investigated when the space-time is described by cylindrically symmetric Einstein-Rosen metric. It is shown that the meson rest mass M and the cosmological constant Λ vanish, the electromagnetic field is source free and the perfect fluid reduces to « stiff fluid » ($p = \rho$). Sets of exact solutions have been obtained and the physics exemplified by some of the solutions is discussed.

RÉSUMÉ. — On étudie le problème d'une distribution de fluide parfait chargé en interaction avec un champ scalaire massif lorsque l'espace temps est décrit par une métrique d'Einstein-Rosen à symétrie cylindrique. On montre que la masse au repos M du méson et la constante cosmologique Λ sont nulles, que le champ électromagnétique est sans source, et que le fluide parfait se réduit à un « fluide rigide » ($p = \rho$). On obtient des solutions exactes et on discute la physique décrite par quelques-unes d'entre elles.

I. INTRODUCTION

Cylindrically symmetric physical distributions arise in many realistic problems of interest in general theory of relativity. The work of Weyl [*1*] [*2*] and Levi-civita [*3*] regarding the static space-times and that of Einstein-

Rosen [*4*] and Rosen [*5*] [*6*] with respect to dynamical space-times are well-known. We may also mention the work of Marder [*7*] [*8*], Witten [*9*], Bonnor [*10*] [*11*] and Misra and Radhakrishna [*12*] in this regard.

Rao *et al.* [*13*] [*14*] [*15*] have obtained a class of solutions for cylindrically symmetric coupled zero-mass and source free electromagnetic fields described by Einstein-Rosen metric and have interpreted these solutions mainly from the view point of their singular behaviour. In a separate investigation they (Rao *et al.* [*16*] [*17*] [*18*] [*19*] have extended the study to the case of Brans-Dicke theory.

In the present investigation we have considered the problem of interacting charged perfect fluid and massive scalar fields with the presence of cosmological constant in the field equations. We have taken Λ to be positive as it allows the particle physics to be considered as an analogue of general theory of relativity (Wesson [*20*]). It turns out that the mass parameter M of the scalar field and the cosmological constant Λ vanish. Further, electromagnetic field becomes source free and the perfect fluid reduces to « stiff fluid ». Sets of exact solutions of this physically realistic situation have been obtained. The singular structure of some of the solutions has been investigated. The nature of the electromagnetic fields is also analysed.

II. FIELD EQUATIONS

The relativistic cosmological field equations for the region of space-time containing charged perfect fluid and scalar field are

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij}R + \Lambda g_{ij} = - K(T_{ij} + E_{ij} + M_{ij}) \tag{2.1}$$

where

$$T_{ij} = \frac{1}{4\pi} [V_{,i}V_{,j} - (1/2)g_{ij}(V_{,s}V^{,s} - M^2V^2)], \tag{2.1a}$$

$$E_{ij} = \frac{1}{4\pi} [- F_{is}F_j^{\ s} + (1/4)g_{ij}F_{ab}F^{ab}] \tag{2.1b}$$

and

$$M_{ij} = \frac{1}{4\pi} [(p + \rho)U_iU_j - pg_{ij}], \qquad (g^{ij}U_iU_j = 1) \tag{2.1c}$$

are stress tensors of massive scalar field, electromagnetic field and perfect fluid distribution. Units are chosen so that the velocity of light C = 1 and ρ, p and U_i are respectively proper mass density, pressure and four-velocity vector of the distribution. In addition to these the electromagnetic field tensor F_{ij} and massive scalar field V also satisfy the following equations

$$F_{ij} = A_{i.j} - A_{j,i}, \tag{2.2}$$

$$F^{ij}_{\ \ ;j} = - 4\pi\sigma U^i. \tag{2.3}$$

and

$$g^{ij}V_{;ij} + M^2V = 0 \tag{2.4}$$

where A_i is the electromagnetic four-potential and σ is the charge density. Comma and semicolon denote partial and covariant differentiations respectively.

We now form the field equations (2.1) to (2.4) in terms of the non-static cylindrically symmetric Einstein-Rosen metric

$$ds^2 = e^{2\alpha-2\beta}(dt^2 - dr^2) - r^2e^{-2\beta}d\theta^2 - e^{2\beta}dz^2\,, \tag{2.5}$$

where $\alpha = \alpha(r, t)$ and $\beta = \beta(r, t)$, and r, θ, z and t correspond to the coordinates x^1, x^2, x^3 and x^4 respectively. Due to the symmetry of the space-time imposed by the metric (2.5), the field variables are independent of the coordinates θ and z. The coordinates are chosen to be comoving so that

$$U_1 = U_2 = U_3 = 0 \qquad \text{and} \qquad U_4 = (g^{44})^{-1/2} \tag{2.6}$$

The consistency of two of the field equations (2.1), viz.,

$$G_{11} = -K(T_{11}+M_{11}+E_{11}) \qquad \text{and} \qquad G_{44} = -K(T_{44}+M_{44}+E_{44})$$

demands that

$$\Lambda = 0, \quad p = \rho, \quad M = 0, \quad F_{14} = 0 \qquad \text{and} \qquad F_{23} = 0\,. \tag{2.7}$$

As a consequence of (2.7), we get from (2.3)

$$\sigma = 0 \qquad (\text{since } U_4 \neq 0)\,. \tag{2.8}$$

Thus all the surviving components of F_{ij} can be obtained from A_2 and A_3 of the electromagnetic four potential A_i. For the sake of simplicity we take

$$A_2 = \phi \qquad \text{and} \qquad A_3 = \psi\,. \tag{2.9}$$

Using (2.5) to (2.9), the field equations (2.1) to (2.4) finally take the form:

$$2\alpha_1 = 2(\Delta\alpha + 2\beta_1^2) + \frac{kr}{4\pi}V_1^2 + \frac{kr}{4\pi}\left[\frac{e^{2\beta}}{r^2}\phi_4^2 + e^{-2\beta}\psi_1^2\right], \tag{2.10}$$

$$\alpha_4 = 2r\beta_1\beta_4 + \frac{kr}{4\pi}V_1V_4 + \frac{kr}{4\pi}\left[\frac{e^{2\beta}}{r^2}\phi_1\phi_4 + e^{-2\beta}\psi_1\psi_4\right], \tag{2.11}$$

$$p(=\rho) = \frac{2\pi}{k}e^{-2\alpha+2\beta}\left[\Delta\alpha - 2\beta_4^2 - \frac{k}{4\pi}V_4^2 - \frac{k}{4\pi}\left\{\frac{e^{2\beta}}{r^2}\phi_1^2 + e^{-2\beta}\psi_4^2\right\}\right], \tag{2.12}$$

$$\Delta\beta = \frac{k}{8\pi}\left[\frac{e^{2\beta}}{r^2}(\phi_1^2 - \phi_4^2) - e^{-2\beta}(\psi_1^2 - \psi_4^2)\right], \tag{2.13}$$

$$\phi_1\psi_1 - \phi_4\psi_4 = 0\,, \tag{2.14}$$

$$\phi_{11} - \phi_{44} - \frac{\phi_1}{r} = 2\beta_4\phi_4 - 2\beta_1\phi_1\,, \tag{2.15}$$

$$\Delta\psi = 2\beta_1\psi_1 - 2\beta_4\psi_4\,, \tag{2.16}$$

$$\Delta V = 0 \tag{2.17}$$

where $\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}$, is the Laplace operator in cylindrical coordinate systems.

To determine the unknowns ϕ, ψ, α, β, V and $p(= \rho)$, we need to solve the field equations (2.10)-(2.17). The question of overdeterminacy has been settled in each case by the satisfaction of all the field equations verified by actual substitution of the solutions in the field equations.

III. SOLUTIONS

It is difficult to find the exact solution in general due to the nonlinearity of the field equations. Therefore, we solve them by imposing certain restrictions on electromagnetic fields. Equation (2.4) is identically satisfied in the following cases:

i) $\phi_1 = \phi_4 = \psi_1 = 0$,
ii) $\phi_1 = \phi_4 = \psi_4 = 0$,
iii) $\psi_1 = \psi_4 = \phi_1 = 0$,
iv) $\psi_1 = \psi_4 = \phi_4 = 0$,
v) $\phi_1 = \psi_4 = 0$,
vi) $\phi_4 = \psi_1 = 0$.
vii) both ϕ and ψ exist and are functions of $(r - t)$ or $(r + t)$.

CASE (*i*) $\underline{\phi_1 = \phi_4 = \psi_1 = 0}$.

This case implies that ψ is a function of time only and the equation (2.15) is satisfied identically. Then from (2.16), we get

$$\beta = \frac{1}{2} [\log \psi_4 - \mathrm{M}(r)]$$

(where M is an arbitrary function of r only) which when substituted in (2.13) yields

$$\frac{d^2\mathrm{M}}{dr^2} + \frac{1}{r}\frac{d^2}{dt^2} \log \psi_4 = \frac{k}{4\pi} \psi_4 e^{\mathrm{M}}. \tag{3.2}$$

Since ψ is a function of time only and M is a function r only, the equation (3.2) will hold if and only if either ψ_4 or L is a constant.

SUB CASE (*i*)*a*

We first assume ψ_4 to be a constant, say f. Then

$$\psi = ft + d \tag{3.3}$$

With this value of ψ, equation (3.2) yields a solution as

$$M = -\log\left[\frac{2u}{q^2}r^2 \cos h^2\left(\frac{q}{2}\log r + s\right)\right] \tag{3.4}$$

where we take $u = \frac{kf}{4\pi}$. Here and in what follows small latin letters except r, z and t denote arbitrary constants unless otherwise stated.

With the help of (3.1), (3.3) and (3.4), we obtain

$$\beta = \frac{1}{2}\log\left[\frac{2uf}{q^2}r^2 \cos h^2\left(\frac{q}{2}\log r + s\right)\right]. \tag{3.5}$$

We consider the particular solution of (2.17) as

$$V = k \log r + at + e. \tag{3.6}$$

Now substituting ψ, β and V from (3.3), (3.5) and (3.6) in (2.11) and integrating, we get

$$\alpha = \frac{Kka}{4\pi}t + S(r)$$

where S is an arbitrary function of r only.

Using (3.3), (3.5), (3.6) and (3.7) in (2.10) ,we get

$$r^2 S_{11} - rS_1 = -\left(2 + \frac{Kk^2}{4\pi}\right) - \frac{q^2}{2}\tan h\left(\frac{q}{2}\log r + s\right) - 2q \tan h\left(\frac{q}{2}\log r + s\right).$$

As it is difficult to get the particular integral of the above differential equation for general value of q, we solve it for certain specific values of q, say $q = 2$. Thus the solution is

$$\alpha = \frac{Kka}{4\pi}t + \log\left[r^{\left(2+\frac{Kk^2}{8\pi}\right)}\cos h^2(\log r + s)\right] + hr^2 + g \tag{3.8}$$

The pressure can be obtained from (2.12) as

$$p(=\rho) = \left(f^2 h - \frac{ufa^2}{4}\right) r^{-\left(2+\frac{Kk^2}{4\pi}\right)} \sec h^2(\log r + s) e^{-2\frac{Kka}{4\pi}thr^2+g}. \tag{3.9}$$

In this sub case the solution is given by (3.3), (3.5), (3.6), (3.8) and (3.9) when $q = 2$.

SUB CASE *(i)b*

In the alternative case when M is a constant, say

$$M = \log f \tag{3.10}$$

the general solution of (3.2) is

$$\psi = b - \frac{q}{u} \tan h\left(- \frac{q}{2} t + s\right). \tag{3.11}$$

We get from (3.1)

$$\beta = \frac{1}{2} \log \left[\frac{q^2}{2uf} \sec h^2\left(- \frac{q}{2} t + s\right)\right]. \tag{3.12}$$

As before considering the solution (2.17), viz.,

$$\mathrm{V} = k \log r + at + e, \tag{3.13}$$

we calculate α and $p(= \rho)$ from (2.10) to (2.12) as

$$\alpha = \frac{\mathrm{K}ka}{4\pi} t + \frac{\mathrm{K}k^2}{8\pi} \log r + hr^2 + g \tag{3.14}$$

and

$$p(=\rho) = \frac{q^2}{u^2}\left(h - \frac{q^2}{8} - \frac{a^2u}{4f}\right) r^{-\frac{\mathrm{K}k^2}{4\pi}} \sec h^2\left(- \frac{q}{2} t + s\right) e^{-2\left(\frac{\mathrm{K}ka}{4\pi} t + hr^2 + g\right)}. \tag{3.15}$$

Thus (3.11) to (3.14) constitute the solution of the field equations.

CASE (*ii*) $\underline{\phi_1 = \phi_4 = \psi_4 = 0}$

This case implies that ψ is a function of r only. Proceeding as before we get the following two sets of solutions:

CASE (*ii*)*a*

$$\psi = h - \frac{q}{u} \tan h\left(- \frac{q}{2} \log r + s\right), \quad \beta = \frac{1}{2} \log\left[\frac{q^2}{2uf} \sec h^2\left(- \frac{q}{2} \log r + s\right)\right],$$

$$\mathrm{V} = k \log r + at + e, \quad \alpha = \frac{\mathrm{K}ka}{4\pi} t + \left(\frac{q^2}{4} + \frac{\mathrm{K}k^2}{8\pi}\right) \log r + hr^2 + g,$$

$$p(=\rho) = \frac{q^2}{u}\left(\frac{h}{u} - \frac{a^2}{4f}\right) r^{-\left(\frac{q^2}{2} + \frac{\mathrm{K}k^2}{4\pi}\right)} \sec h^2\left(- \frac{q}{2} \log r + s\right) e^{-2\left(\frac{\mathrm{K}ka}{4\pi} t + hr^2 + g\right)}. \tag{3.16}$$

SUB CASE (*ii*)*b*

$$\psi = \frac{1}{2} fr^2 + d, \quad \beta = \frac{1}{2} \log\left[\frac{2uf}{q^2} r^2 \cos h^2 \left(\frac{q}{2} t + s\right)\right],$$

$$\mathrm{V} = k \log r + at + e, \quad \alpha = \log\left[\cos h^2\left(\frac{q}{2} t + s\right)\right] + \frac{\mathrm{K}ka}{4\pi} t + hr^2 + g,$$

$$p(=\rho) = \frac{f^2}{q^2}\left(4h - \frac{q^2}{2} - \frac{ua^2}{f}\right) r^{-\frac{\mathrm{K}k^2}{4\pi}} \sec h^2 \left(\frac{q}{2} t + s\right) e^{-2\left(\frac{\mathrm{K}ka}{4\pi} t + hr^2 + g\right)}. \tag{3.17}$$

CASE (*iii*) $\underline{\psi_1 = \psi_4 = \phi_1 = 0}$

This case suggests us that ϕ depends on time only. Following the method of Case (*i*), we get the following two sets of solutions:

SUB CASE $(iii)a$

$$\phi=b-\frac{q}{u}\tan h\left(-\frac{q}{2}t+s\right),\quad \beta=\frac{1}{2}\log\left[\frac{2uf}{q^2}r^2\cos h^2\left(-\frac{q}{2}t+s\right)\right],$$

$$\mathrm{V}=k\log r+at+e,\ \alpha=\frac{\mathrm{K}ka}{4\pi}t+\log\left[r^{\left(1+\frac{\mathrm{K}k^2}{8\pi}\right)}\cos h^2\left(-\frac{q}{2}t+s\right)\right]+hr^2+g,$$

$$p(=\rho)=\frac{f^2}{q^2}\left(4h-\frac{q^2}{2}-\frac{ua^2}{f}\right)r^{-\frac{\mathrm{K}k^2}{4\pi}}\sec h^2\left(-\frac{q}{2}t+s\right)e^{-2\left(\frac{\mathrm{K}k^2}{4\pi}t+hr^2+g\right)}. \tag{3.18}$$

SUB CASE $(iii)b$

$$\phi=ft+d,\quad \beta=\frac{1}{2}\log\left[\frac{q^2}{2uf}\sec h^2\left(-\frac{q}{2}\log r+s\right)\right],$$

$$\mathrm{V}=k\log r+at+e,\quad \alpha=\frac{\mathrm{K}ka}{4\pi}t+\left(\frac{q^2}{4}+\frac{\mathrm{K}k^2}{8\pi}\right)\log r+hr^2+g,$$

$$p(=\rho)=\frac{q^2}{4u^2}\left(4h-\frac{ha^2}{f}\right)r^{-\left(\frac{q^2}{2}+\frac{\mathrm{K}k^2}{4\pi}\right)}\sec h^2\left(-\frac{q}{2}\log r+s\right)e^{-2\left(\frac{\mathrm{K}k^2}{4\pi}t+hr^2+g\right)}. \tag{3.19}$$

CASE (iv) $\underline{\psi_1=\psi_4=\phi_4=0}$

In this case ϕ depends on r only. The following two sets of solutions are obtained:

$$\phi=b+\frac{2}{u}\tan h(\log r+s),\quad \beta=\frac{1}{2}\log\left[\frac{uf}{2}r^2\cos h^2(\log r+s)\right]$$

$$\mathrm{V}=k\log r+at+e,\quad \alpha=\frac{\mathrm{K}ka}{4\pi}t+\log\left[r^{\left(2+\frac{\mathrm{K}k^2}{8\pi}\right)}\cos h^2(\log r+s)\right]+hr^2+g,$$

$$p(=\rho)=\left(hf^2-\frac{fua^2}{4}\right)r^{\left(2+\frac{\mathrm{K}k^2}{4\pi}\right)}\sec h^2(\log r+s)e^{-2\left(\frac{\mathrm{K}ka}{4\pi}thr^2+g\right)} \tag{3.20}$$

SUB CASE $(iv)b$

$$\phi=fr^2+d,\quad \beta=\frac{1}{2}\log\left[\frac{q^2}{2uf}\sec h^2\left(-\frac{q}{2}t+s\right)\right],$$

$$\mathrm{V}=k\log r+at+e,\quad \alpha=\frac{\mathrm{K}ka}{4\pi}t+\frac{\mathrm{K}k^2}{8\pi}\log r+hr^2+g,$$

$$p(=\rho)=\frac{q^2}{4u^2}\left(4h-\frac{q}{2}-\frac{ua^2}{f}\right)r^{-\frac{\mathrm{K}k^2}{4\pi}}\sec h^2\left(-\frac{q}{2}t+s\right)e^{-2\left(\frac{\mathrm{K}k^2}{4\pi}t+hr^2+g\right)}. \tag{3.21}$$

CASE (v) $\underline{\phi_1=\psi_4=0}$

Proceeding as in the case (i), we obtain the following two sets of solutions:

SUB CASE $(v)a$

$$\psi = fr^2 + d, \quad \phi = b - \frac{q}{2u}\tan h\left(-\frac{q}{2}t + s\right), \quad V = k\log r + at + e,$$

$$\beta = \frac{1}{2}\log\left[\frac{4uf}{q^2}r^2\cos h^2\left(-\frac{q}{2}t + s\right)\right],$$

$$\alpha = \frac{Kka}{4\pi}t + \log\left[\cos h^2\left(-\frac{q}{2}t + s\right)\right] + \left(1 + \frac{Kk^2}{8\pi}\right)\log r + hr^2 + g,$$

$$p(=\rho) = \frac{2f^2}{q^2}\left(4h - \frac{q^2}{2} - \frac{ua^2}{f}\right)r^{-\frac{Kk^2}{4\pi}}\sec h^2\left(-\frac{q}{2}t + s\right)e^{-2\left(\frac{Kka}{4\pi}t + hr^2 + g\right)}. \tag{3.22}$$

SUB CASE $(v)b$

$$\phi = ft + d, \quad \psi = b - \frac{q}{2u}\tan h\left(-\frac{q}{2}\log r + s\right), \quad V = k\log r + at + e,$$

$$\beta = \frac{1}{2}\log\left[\frac{q^2}{4uf}\sec h^2\left(-\frac{q}{2}\log r + s\right)\right],$$

$$\alpha = \frac{Kka}{4\pi}t + \left(\frac{q^2}{4} + \frac{Kk^2}{8\pi}\right)\log r + hr^2 + g,$$

$$p(=\rho) = \frac{q^2}{8u^2}\left(4h - \frac{ua^2}{f}\right)r^{-\left(\frac{q^2}{2} + \frac{Kk^2}{4\pi}\right)}\sec h^2\left(-\frac{q}{2}\log r + s\right)e^{-2\left(\frac{Kk^2}{4\pi}t + hr^2 + g\right)}. \tag{3.23}$$

CASE (vi) $\underline{\phi_4 = \psi_1 = 0}$

As before, in this case we get the following two sets of solutions:

CASE $(vi)a$

$$\phi = fr^2 + d, \quad \psi = b - \frac{q}{2u}\tan h\left(-\frac{q}{2}t + s\right), \quad V = k\log r + at + e,$$

$$\beta = \frac{1}{2}\log\left[\frac{q^2}{ruf}\sec h^2\left(-\frac{q}{2}t + s\right)\right], \quad \alpha = \frac{Kka}{4\pi}t + \frac{Kk^2}{8\pi}\log r + hr^2 + g,$$

$$p(=\rho) = \frac{q^2}{8u^2}\left(4h - \frac{q^2}{2} - \frac{ua^2}{f}\right)2^{-\frac{Kk^2}{4\pi}}\sec h^2\left(-\frac{q}{2}t + s\right)e^{-2\left(\frac{Kka}{4\pi}t + hr^2 + g\right)}. \tag{3.24}$$

CASE $(vi)b$

$$\psi = ft + d, \quad \phi = b + \frac{1}{u}\tan h(\log r + s), \quad V = K\log r + at + e,$$

$$\beta = \frac{1}{2}\log ufr^2\cos h^2(\log r + s), \quad \alpha = \frac{Kka}{4\pi}t$$

$$+ \log\left[r^{\left(2 + \frac{Kk^2}{8\pi}\right)}\cos h^2(\log r + s)\right] + hr^2 + g,$$

$$p(=\rho) = \frac{f^2}{2}\left(rh - \frac{ua^2}{f}\right)r^{-\left(2 + \frac{Kk^2}{4\pi}\right)}\sec h^2(\log r + s)e^{-2\left(\frac{Kka}{4\pi}t + hr^2 + g\right)}. \tag{3.25}$$

CASE (vii) *Both ϕ and ψ exist and are functions of $(r - t)$ or $(r + t)$*

Eventhough ϕ and ψ can be arbitrary functions of $(r-t)$ or $(r+t)$, for an explicit form of solution, we choose

$$\phi = \psi = (r - t)^n , \tag{3.26}$$

where $n\ (\neq 0,\ 1/2)$ is any real number. The values 0 and 1/2 for n are excluded as these values remove electromagnetic as well as matter fields. In view of (3.25), equations (2.15) and (2.16) lead to a single equation

$$\beta_1 + \beta_4 = \frac{1}{2r} \tag{3.27}$$

and (2.13) reduces to

$$\Delta\beta = 0 . \tag{3.28}$$

A common solution of (3.27) and (3.28) is

$$\beta = \frac{1}{2}\log\left(\frac{r}{m}\right). \tag{3.29}$$

Considering the particular solution of (2.17) as

$$\mathrm{V} = k \log r + at + e , \tag{3.30}$$

we get α and $p(=\rho)$ as

$$\alpha = \frac{\mathrm{K}ka}{4\pi} t + \frac{\mathrm{K}n^2(1+m^2)}{4\pi m(2n-1)}(r-t)^{2n-1} + \left(\frac{1}{4} + \frac{\mathrm{K}k^2}{8\pi}\right)\log r + hr^2 + g , \tag{3.31}$$

and

$$p(=\rho) = \frac{2\pi}{\mathrm{K}m}\left(4h - \frac{\mathrm{K}a^2}{4\pi}\right) r^{\left(\frac{1}{2}-\frac{\mathrm{K}k^2}{4\pi}\right)} e^{-2\frac{\mathrm{K}ka}{4\pi}t + \frac{un^2(1+m^2)}{mf(2n-1)}(r-t)^{2n-1} + hr^2 + g} . \tag{3.32}$$

In this case the solution of the field equations is given by (3.26) and (3.29) to (3.32).

IV. PHYSICAL DISTRIBUTIONS REPRESENTED BY SOME OF THE SOLUTIONS

1) Energy conditions.

All the solutions obtained under the cases (*i*) to (*vii*) satisfy the weak energy condition $\mathrm{T}^{ab}\mathrm{U}_a\mathrm{U}_b \geq 0$ and the Hawking and Penrose [*21*] energy condition $(\mathrm{T}_{ab} - 1/2g_{ab}\mathrm{T})\mathrm{U}^a\mathrm{U}^b \geq 0$ as well.

2) Curvature scalar.

The curvature scalar R is obtained by contracting field equations (2.1) as

$$\mathrm{R} = \frac{\mathrm{K}}{4\pi}(2p + \mathrm{V}_s\mathrm{V}^s)$$

Since $p(=\rho) \geq 0$ and $\mathrm{V}_s\mathrm{V}^s \geq 0$, $\mathrm{R} = 0$ iff $p = \rho = \mathrm{V}_s\mathrm{V}^s = 0$.

The condition $V_sV^s = 0$ implies that V = constant hypersurfaces are characteristic surfaces.

For the solutions under the cases (*i*), (*ii*) and (*vii*) on t = constant hypersurface, we have

$$R \to \infty \quad \text{as} \quad r \to 0$$

and

$$R \to 0 \quad \text{as} \quad r \to \infty .$$

Similarly keeping r = constant, we get R $\to$ *a* finite value as $t \to 0$, and as $t \to \infty$, R $\to$ 0 or finite value or ∞, depending on the nature of the constants involved in the solutions.

3) **Kretschmann curvature invariant.**

The Kretschmann curvature invariant $L = R_{ejkl}R^{ejkl}$ for the solutions under cases (*i*), (*ii*) and (*vii*) behaves as follows: For t = constant hypersurface, we have

$$\text{as} \quad r \to 0, \quad L \to \infty \quad \text{and} \quad \text{as} \quad r \to \infty, \quad L \to 0 .$$

Similarly, keeping r = const., we obtain, as $t \to 0$, L $\to$ a finite value and as $t \to \infty$, L $\to$ 0 or a finite value or ∞ depending on the different conditions on the arbitrary constants involved on the solutions. Thus the behaviour of L suggests that the solution is singular on the axis of symmetry.

4) **Bonnor's criteria concerning the singularities.**

We use the criteria suggested by Bonor [*11*] in studying the singularities of our solutions. In order to avoid the coordinate singularities we adopt pseudo-cartesian coordinates. It may be verified that the solutions of case (*i*), (*ii*) and (*vii*) are singular on the axis of symmetry and at spatial and temporal infinities. This study supports the behaviour of curvature scalar R and the Kretschmann curvature invariant studied before.

5) **Scalar pressure.**

In each of the above cases the pressure is physically realistic with possible suitable relations between the arbitrary constants involved. We may mention that in a few cases the axis of symmetry is singular eventhough there is no matter field on it.

6) **Electromagnetic fields.**

The solutions obtained under the cases (*i*) and (*ii*) are dual solutions of the cases (*iii*) and (*iv*) as per Bonnor's [*10*] duality theorem.

The condition for the existence of null electromagnetic fields, viz.,

$$w(\equiv [(F_{ij}F^{ij})^2 + (F_{ij}F^{*ij})^2]^{1/2}) = 0$$

is examined for all the solutions. It turns out that the solution (VII) above only is a null field and the others are nonnull. We have also verified whether the electromagnetic fields are uniform, i. e. $F_{ij;k} = 0$, and found that all the electromagnetic fields in all the solutions are non-uniform.

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