L. Mangiarotti
M. Modugno

Some results on the calculus of variations on jet spaces


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Some results on the calculus
of variations on jet spaces (*)

by

L. MANGIAROTTI (**) and M. MODUGNO (***)

ABSTRACT. — The basic object is a fibered manifold $p : E \to M$ and
the framework is that of jet spaces. Given a Lagrangian form $\Lambda$ on $JE$,
we work with the space $\{ \Lambda \}$ of variational forms associated to $\Lambda$. It is
this space which is important in the calculus of variations. We study a
new operator $\delta_{\Lambda}$ (defined only on $\text{Ker} \ \mathfrak{g} \subset TJE$ where $\mathfrak{g}$ is the fundamen-
tal 1-form) canonically associated to $\{ \Lambda \}$. This operator is well suited
for studying critical sections and functorial properties. The so called
Euler-Lagrange operator $E_{\Lambda}$ appears as an extension of $\delta_{\Lambda}$. Variational
symmetries are introduced as morphisms of a category whose objects
are the variational forms $\{ \Lambda \}$. The uniqueness of the Poincaré-Cartan
form $\Theta_{\Lambda}$ is proved under certain circumstances. Various interesting rela-
tions between $\Lambda, E_{\Lambda}$ and $\Theta_{\Lambda}$ are investigated.

RÉSUMÉ. — L'objet fondamental est une variété avec un espace fibré
$p : E \to M$ et le cadre est celui des espaces de jets. Étant donnée une
forme Lagrangienne $\Lambda$ sur $JE$, on considère l'espace $\{ \Lambda \}$ des formes
variationnelles associées à $\Lambda$, qui est l'espace important pour le calcul
des variations. On étudie un nouvel opérateur $\delta_{\Lambda}$ (défini seulement sur
$\text{Ker} \ \mathfrak{g} \subset TJE$, où $\mathfrak{g}$ est la 1-forme fondamentale), canoniquement associé
à $\{ \Lambda \}$. Cet opérateur est bien adapté à l'étude des sections critiques et
des propriétés fonctorielles. L'opérateur d'Euler-Lagrange $E_{\Lambda}$ apparaît
comme extension de $\delta_{\Lambda}$. Les symétries variationnelles sont introduites

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Matematica).

(**) Permanent address: Istituto di Matematica, Università, Camerino (MC).

(***) Permanent address: Istituto di Matematica Applicata, Università, Firenze.
comme morphismes d'une catégorie dont les objets sont les formes variationnelles de \{\Lambda\}. On montre l'unicité de la forme \(\Theta_\Lambda\) de Poincaré-Cartan dans certaines circonstances. On étudie quelques relations intéressantes entre \(\Lambda\), \(E_\Lambda\) et \(\Theta_\Lambda\).

INTRODUCTION

In recent years much work has been done on the calculus of variations on jet spaces \([2]\) \([3]\) \([4]\) \([6]\) \([7]\) \([9]\) \([10]\) \([11]\) and \([12]\). Following this stream, the present paper emphasizes both the geometrical basis of the theory as well as its functorial nature. Moreover, it contains novelties in the treatment as well as new results.

The basic geometric object is a fibered manifold \(p : E \to M\) and the basic spaces are the first two jet spaces, i.e. \(J^1E\) and \(J^2E\). In Section 1 we recall the two main facts about the geometry of jet spaces, namely the affine structures and the fundamental 1-form \(\theta\). Then the subject of infinitesimal contact transformations is developed in a new way. These transformations are important because a « variational problem » is « infinitesimally functorial » with respect to them. A deeper analysis at all orders of jet spaces will appear in a related paper \([8]\).

In Section 2 we introduce Lagrangian forms \(\Lambda\) on \(JE\), i.e. horizontal forms with respect to the canonical projection \(JE \to M\) (in physical terms, these are Lagrangian densities). Affine Lagrangian forms are studied in detail. Then we define the space \(\{\Lambda\}\) of variational forms associated with a Lagrangian form \(\Lambda\) (already used in \([10]\) in the context of affine bundles). It is \(\{\Lambda\}\) which is important in the Calculus of Variations. Various properties of \(\{\Lambda\}\) in relation with automorphisms \(\varphi : E \to E\) (for brevity we consider only automorphisms of \(E\) though many properties hold in a more general context) as well as infinitesimal contact transformations are demonstrated. Moreover, the uniqueness of the Poincaré-Cartan form is proved under certain circumstances.

A main result in Section 3 is the introduction of a new operator \(\delta_\Lambda\) (defined only on \(\text{Ker} \ \delta \subset TJE\)) canonically associated to the set \(\{\Lambda\}\). This operator is well suited for studying critical sections and functorial properties. The classical Euler-Lagrange operator \(E_\Lambda\) is an extension of \(\delta_\Lambda\). Various interesting relations between \(\Lambda\), \(E_\Lambda\) and \(\Theta_\Lambda\) are investigated. Variational symmetries are introduced as morphisms of a category whose objects are the variational forms \(\{\Lambda\}\). However, this is only a first insight into such an intricate question \([2]\) \([4]\) \([6]\) and \([11]\). The analysis of the subject, also with respect to Noether's theorem, will be pursued in a future work.
In Section 4 we treat of critical sections. The functorial nature of a variational problem is exhibited in a very direct way. A new characterization of critical sections in terms of $\delta_{\alpha}$ is given.

1. NOTATION AND PRELIMINARY RESULTS

a) All manifolds and maps will be $C^\infty$. The notation used is that of modern differential geometry. The basic object is a fibered manifold $p : E \to M$ (i.e. $p$ is a surjective submersion), dim $M = m$, dim $E = m + l$. Fibered charts are denoted by $(x^a, y^i)$, $1 \leq a \leq m$, $1 \leq i \leq l$. Induced fibered charts on the 1-jet space $JE$ are denoted by $(x^a, y^i, y^i_a)$. The canonical projection $JE \to E$ is denoted by $p_E$. Then $p_M = p \circ p_E$ is the canonical projection $JE \to M$. If $s$ is a (local) section of $E$, then $js : M \to JE$ is the 1-jet prolongation of $s$. If $\varphi : E \to E$ is an automorphism of $E$ (over the diffeomorphism $\varphi : M \to M$), then $J\varphi : JE \to JE$ is the 1-jet prolongation of $\varphi$. Recall that $J\varphi$ is characterized by $J\varphi \circ js = J(\varphi \ast s) \circ \varphi_M$ for any section $s$ of $E$, $\varphi \ast s = \varphi \circ s \circ \varphi_M^{-1}$.

There are two basic properties concerning $JE$. First, $JE$ is an affine bundle over $E$ with vector bundle the tensor product bundle $T^*M \otimes VE$ [3]. Here $T$ and $V$ denote the tangent and vertical functors, respectively ($VE = \text{Ker } Tp \subset TE$). As a rule, obvious pull-backs will be omitted.

Second, there is a (first order) canonical inclusion over $JE \to E$, namely

$$c : JE \times_M TM \to TE$$

characterized by $c \circ (js, u) = Tjs \circ u$ for any section $s : M \to E$ and any vector field $u : M \to TM$. It induces the canonical projection

$$\theta : JE \times_M TE \to VE,$$

the so called structure 1-form (see [2] and [3]), whose local expression is

$$(x^a, y^i, y^i_a ; \dot{y}^i) \circ \theta = (x^a, y^i, y^i_a ; \dot{y}^i - y^i_a \dot{x}^a).$$

By means of the tangent map $Tp_E : TJE \to TE$, $\theta$ can also be considered as a linear morphism $\theta : TJE \to VE$ over $p_E : JE \to E$. Hence $\theta$ is a 1-form on $JE$ valued over $VE$ whose local expression is

$$\theta^i = dy^i - y^i_a dx^a.$$
associated to the fibered charts \((x^a, y^i, y^i_a)\). Let us note that \(\mathcal{J}\) (and hence \(\Delta\)) is invariant with respect the automorphisms \(\varphi : E \to E\), i.e.

\[
\mathcal{J} \circ T\varphi = V\varphi \circ \mathcal{J}.
\]

In the usual pull-back notation, \((1.5)\) is written as \(\varphi^*\mathcal{J} = \mathcal{J}\).

All that we have said holds for higher order jet spaces \([3]\) \([4]\) and \([8]\). Nevertheless we need only to consider the 2-jet space \(J^2E\) (because the Euler-Lagrange equation is of the second order). Induced fibered charts on \(J^2E\) are denoted by \((x^a, y^i, y^i_a, y^i_{ab})\). Note that there is a canonical injection \(J^2E \subset (J(E))\).

The 2-jet space \(J^2E\) is an affine bundle over \(JE\) with vector bundle \((\bigvee_{JE} T^*M) \otimes_{JE} V\), where \(\bigvee\) denotes the second order symmetric tensor product.

From the second order canonical inclusion

\[
c : J^2E \times TM \to \Delta \subset TJE,
\]

we get the (second order) structure 1-form on \(J^2E\) which can be considered as a linear morphism \(TJ^2E \to VJE\) over the canonical projection \(J^2E \to E\) (here \(VJE = \text{Ker} Tp_M \subset TJE\)). Its local expression is

\[
\mathcal{J}^i = dy^i - y^i_a dx^a, \quad \mathcal{J}^i_a = dy^i_a - y^i_{ab} dx^b.
\]

b) There are distinguished vector fields on \(JE\), namely the so called infinitesimal contact transformations (in short: i. c. t.) \([2]\) \([4]\). They are important in the calculus of variations, essentially because a « variational problem » is « infinitesimally functorial » with respect to them. We give here a new treatment of these vector fields (compare with \([2]\) and \([4]\)). A comprehensive analysis of the subject, involving jet spaces of any order, will appear in \([8]\).

**Definition.** — A vector field \(u\) on \(JE\) is an i. c. t. (of first order) iff \(L_u\Delta \subset \Delta\) (here \(L_u\) is the Lie derivative with respect to \(u\)).

Let \(u = u^a \partial_a + u^i \partial_i + u^i_a \partial^i_a\) be the local expression of \(u\). Then it is easily proved that \(u\) is an i. c. t. iff

\[
u^a \partial_a u^i + y^i_a \partial_a \mu^i - y^i_p (\partial_a \mu^a + y^i_a \partial_a \mu^a), \quad \partial^a_a u^i + y^i_p \partial^a_a \mu^a = 0.
\]

There is an interesting way of characterizing the i. c. t. which is based on the existence of a canonical projection \(r : JTE \to TJE\) \([8]\). Let us remark preliminarily that \(JTE\) is an affine bundle over the fibered product \(JE \times TM\) with vector bundle (the pull-back over \(JE \times TM\) of) \(VJE\). Also \(TJE\) is an affine bundle over \(JE \times TM\) with vector bundle (the pull-back over \(JE \times TM\) of) \(VJE\).
**Proposition.** There is a unique affine morphism \( r: \text{JTE} \to \text{TJE} \) over the canonical projection \( \text{JE} \times \text{JTM} \to \text{JE} \times \text{TM} \) such that

1) \( r \circ JTs = Tjs \) for any section \( s: \text{M} \to \text{E} \),

2) its fiber derivative \( Dr \) restricts to the identity over the fibers.

The local expression of \( r \) is

\[
(x^a, y^i, y_a^i, \dot{x}^a, \dot{y}^i) \circ r = (x^a, y^i, y_a^i, \dot{x}^a, \dot{y}^i - y_a^i \dot{x}^a).
\]

**Proposition.** A vector field \( u \) on \( \text{JE} \) is an i. c. t. iff we have \( u = r \circ J(Tp_E \circ u) \). If \( u_E \) is a vector field on \( E \), then \( u = r \circ Ju_E \) is the unique i. c. t. projectable over \( u_E \) (1-jet prolongation of \( u_E \)). Moreover, in the particular case when \( u_E \) is projectable over \( M \), \( u \) is the infinitesimal generator of \( JF_t \), where \( F_t \) is the flux of \( u_E \).

If \( u_E = u^a \partial_a + u^i \partial_i \) is the local expression of \( u_E \), then the local expression of the 1-jet prolongation of \( u_E \) is

\[
u = u^a \partial_a + u^i \partial_i + u^a_1 \partial_i^a,
\]

where \( u^a_1 \) is given by (1.8).

It is obvious how the previous definition works on \( J^2E \). If

\[
u = u^a \partial_a + u^i \partial_i + u^a_2 \partial_i^a + u^a_3 \partial_i^a^2
\]

is the local expression of a vector field on \( J^2E \), it follows from (1.7) that \( u \) is a second order i. c. t.

\[
u = u^a \partial_a + y^i \partial_i + y^a_1 \partial_i^a + y^a_2 \partial_i^a^2,
\]

(1.11)

It follows easily from (1.8) and (1.11) that if \( u \) is projectable over a vector field \( u_E \) on \( E \) and also over a vector field on \( JE \), then this projection is the 1-jet prolongation of \( u_E \). As in the last proposition, it is clear that (1.11) gives us the local expression of the 2-jet prolongation of a vector field \( u_E \) on \( E \).

**2. VARIATIONAL FORMS**

\( a \) Let us recall that \( p_M \) denotes the canonical projection \( JE \to M \). A \( p_M \)-horizontal \( m \)-form on \( JE \) is called a Lagrangian form on \( JE \). An \( m \)-form \( \Lambda \) on \( JE \) is a Lagrangian form iff its local expression is

\[
\Lambda = \mathcal{L} \omega, \quad \omega = dx^1 \wedge \ldots \wedge dx^m,
\]

where \( \mathcal{L} \) is a local function on \( JE \).

If the manifold \( M \) is orientable, then the choice of a volume form on \( M \) induces a bijection between Lagrangian forms and functions on \( J \mathcal{E} \). In physical terms, \( \Lambda \) is a Lagrangian density while \( \mathcal{L} \) is a Lagrangian.

Let us make some remarks about Lagrangian forms of the affine type. Clearly, \( \Lambda \) is an affine map (over \( E \)) iff it can be written as

\[
\mathcal{L} = \pi^*_i y^i_a + \lambda,
\]

where \( \pi^*_i \) and \( \lambda \) are local functions on \( E \). To go further, we need the fact that the affine bundle \( p_E : J \mathcal{E} \to E \) has a canonical « linearizing bundle » \( L \to E \).

To see this, note that the product bundle \( E \times \mathbb{R} \to E \) is a vector subbundle of \( T^*M \otimes TM \) in a canonical way. Hence \( L \) is the vector subbundle of \( T^*M \otimes TE \) which is projected on \( E \times \mathbb{R} \) by \( id_{T^*M} \otimes Tp \).

Moreover, from the canonical injection \( J \mathcal{E} \subset T^*M \otimes TE \) (cf. (1.1)) which takes its values in \( L \), we get a canonical affine injection \( j : J \mathcal{E} \to L \).

Then the local expression of the unique \( \tilde{\Lambda} \in L^* \otimes \bigwedge^m T^*M \) such that \( \Lambda = \tilde{\Lambda} \circ j \) is

\[
\tilde{\Lambda} = (\pi^*_i \partial_a \otimes dy^i + \lambda \partial_a \otimes dx^a) \otimes \omega.
\]

Since there is a canonical injection of

\[
L^* \otimes \bigwedge^m T^*M \quad \text{into} \quad T^*E \otimes \bigwedge^{m-1} T^*M,
\]

from \( \Lambda \), by using the projection \( Tp \), we get an \( m \)-form \( \Lambda_E \) on \( E \) whose local expression is

\[
\Lambda_E = \pi^*_i dy^i \wedge \omega_a + \lambda \omega, \quad \omega_a = \partial_a \mathcal{J} \omega,
\]

where \( \mathcal{J} \) denotes the inner product.

Now let \( \rho \) be a \( p \)-horizontal \((m - 1)\)-form on \( E \) and let

\[
\rho = \rho^a \omega_a,
\]

be its local expression (\( \rho^a \) are local functions on \( E \)). Then by using the functor \( J \) we get an affine Lagrangian form (of the so called divergence type: cf. [1] [5] and [6]), namely \( \Lambda = \text{div} \rho \), whose local expression is

\[
\mathcal{L} = (\partial_i \rho^a) y^i_a + \partial_a \rho^a.
\]

In terms of the form \( \Lambda_E \) we have \( \Lambda_E = d\rho \). Hence \( \Lambda_E \) is closed. Conversely, if \( \Lambda \) is an affine Lagrangian form such that \( d\Lambda_E = 0 \), \( \Lambda \) is locally a Lagrangian form of the divergence type (as can easily be seen).
Let $\Lambda$ be a Lagrangian form on $\mathcal{J}E$. By taking the fiber derivative $D\Lambda$ (with respect to $E$), followed by an obvious canonical contraction, we get an $(m - 1)$-form $\Omega_\Lambda$ on $\mathcal{J}E$ valued on $V^*E$, that is

\[ (2.7) \quad \Omega_\Lambda : \mathcal{J}E \to V^*E \bigotimes\limits_E^{m-1} T^*M, \]

whose local expression is

\[ (2.8) \quad \Omega_\Lambda = \pi^*_i dy^i \otimes \omega_a, \quad \pi^*_i = \partial_i \mathcal{L}. \]

The map $\Omega_\Lambda$ is the so-called Legendre transformation associated to $\Lambda$ [2]. Note that $\Lambda$ is affine iff $D\Omega_\Lambda = 0$.

The map $\Omega_\Lambda$ can also be viewed, in a canonical way, as an $m$-form on $\mathcal{J}E$. For this, it is sufficient to observe that the projection $\partial$, as defined in (1.2), allows us to construct the canonical injection

\[ (2.9) \quad V^*E \bigotimes\limits_{\mathcal{J}E}^{m-1} T^*M \subset T^*E \bigotimes\limits_{\mathcal{J}E}^{m-1} T^*E. \]

It is clear that $\Omega_\Lambda$ is a $p_{\mathcal{J}E}$-horizontal $m$-form on $\mathcal{J}E$ whose local expression is

\[ (2.10) \quad \Omega_\Lambda = \pi^*_i \mathcal{G}^i \wedge \omega_a, \]

as follows from (2.8) and (2.9).

Let $\varphi : E \to E$ be an automorphism. Then $(J\varphi)^*\Lambda$ is again a Lagrangian form and it is easily proved that

\[ (2.11) \quad (J\varphi)^*\Omega_\Lambda = \Omega_{(J\varphi)^*\Lambda}. \]

Moreover, let $u_E$ be a vector field on $E$, projectable on $M$, and let $u$ be the 1-jet prolongation of $u_E$. Then the infinitesimal version of (2.11) is

\[ (2.12) \quad L_u\Omega_\Lambda = \Omega_{L_u\Lambda}. \]

b) Now we consider variations of Lagrangian forms. The basic tool is the subbundle $\Delta \subset \mathcal{T}JE$ [10].

**Definition.** — Let $\Lambda$ be a Lagrangian form on $\mathcal{J}E$. A variation of $\Lambda$ is a $p_{\mathcal{J}E}$-horizontal $m$-form $\Theta$ on $\mathcal{J}E$ such that

\[ i) \quad \Theta | \Delta = \Lambda | \Delta, \]
\[ ii) \quad d\Theta | \Delta = 0. \]

Let $\{ \Lambda \}$ be the set of variations of $\Lambda$. Clearly $\{ \Lambda \}$ is an affine space (over $\mathbb{R}$) with vector space the space $[\Theta]$ of the $p_{\mathcal{J}E}$-horizontal $m$-form $\Theta$ on $\mathcal{J}E$ such that $\Theta | \Delta = 0$ and $d\Theta | \Delta = 0$. Note that $\Lambda \in \{ \Lambda \}$ iff $\Omega_\Lambda = 0$.

There is a canonical form $\Theta_\Lambda \in \{ \Lambda \}$, namely the so-called Poincaré-Cartan form

\[ (2.13) \quad \Theta_\Lambda = \Lambda + \Omega_\Lambda. \]
The assignment \( \Lambda \mapsto \{ \Lambda \} \) is functorial, i.e. if \( \varphi : E \to E \) is an automorphism, we find that

\[
(\Lambda^\varphi)^* \{ \Lambda \} = \{ (\Lambda^\varphi)^* \Lambda \}.
\]

In particular, from (2.11) and (2.13) it follows that

\[
(\Lambda^\varphi)^* \Theta_\Lambda = \Theta_{(\Lambda^\varphi)^* \Lambda}.
\]

If \( u \) is an i.c.t. on \( JE \), we have

\[
L_u \{ \Lambda \} \subset \{ L_u \Lambda \}.
\]

More particularly, if \( u \) is the 1-jet prolongation of a vector field \( u_E \) on \( E \), projectable on \( M \), from (2.12) and (2.13) we get the infinitesimal version of (2.15), namely

\[
L_u \Theta_\Lambda = \Theta_{L_u \Lambda}.
\]

Note that, since \( 0A = 0 \) iff \( \Lambda = 0 \) (as is easily seen), from (2.17) it follows that

\[
L_u \Theta_\Lambda = 0 \iff L_u \Lambda = 0.
\]

c) At least in some particular cases, we can say more about the set of variational forms \( \{ \Lambda \} \). For example, in Classical Analytical Mechanics one is interested to the case in which \( M = \mathbb{R} \), the absolute time. Another case which occurs often in practice is that in which the fibers of \( E \) are 1-dimensional. In both these cases \( [\Theta] \) is trivial.

**Proposition.** — Let \( \Lambda \) be a Lagrangian form on \( JE \). Then if \( \dim M = 1 \) or also if \( \dim E = m + 1 \), the Poincaré-Cartan form is the unique variational form of \( \Lambda \).

**Proof.** — Let \( \dim M = 1 \) and let \( \overline{\Theta} \) be a \( p_E \)-horizontal 1-form on \( JE \) such that \( \overline{\Theta} | \Lambda = 0 \). Then locally we must have

\[
\overline{\Theta} = f_i \delta^i,
\]

where \( f_i \) are local functions on \( JE \). The result follows now from

\[
\partial_i \overline{\Theta} = - f_i dx^1 + (\partial_i f_j) g^j.
\]

The case \( \dim E = m + 1 \) is treated in the same way.

Apart from these two cases, in general \( [\Theta] \) is not trivial. For example, let \( M = \mathbb{R}^2 \) and let \( E = \mathbb{R}^2 \times \mathbb{R}^2 \). Then the 2-form on \( JE \)

\[
\overline{\Theta} = (y_1^2 - y_2 y_1^1) dx^1 \wedge dx^2 + g^1 \wedge dy^2 + dy^1 \wedge g^2 - dy^1 \wedge dy^2,
\]

belongs to \( [\Theta] \).
3. VARIATIONAL STRUCTURES

a) The concept of variational form leads us to introduce, in a natural way, \(m\)-forms defined on \(\Delta\). This is done in the following proposition.

**Proposition.** — Let \(\Lambda\) be a Lagrangian form on \(J E\). Then there exists a unique \(m\)-form \(\Theta_{\Lambda}\) defined on \(\Delta\) and valued on \(V^*E\) such that

\[
(js)^*(u_0 \perp d \Theta) = - \left< (js)^* \Theta_{\Lambda}, \vartheta \circ u_0 \circ js \right>,
\]

where \(\Theta \in \{ \Lambda \}, s : M \to E\) is any section and \(u_0\) is any vector field on \(J E\). The local expression of \(\Theta_{\Lambda}\) is

\[
(\Theta_{\Lambda})_i = [d\pi^*_i \wedge \omega_x - (\partial_i \vartheta)\omega] \mid \Delta.
\]

**Proof.** — The uniqueness of \(\Theta_{\Lambda}\) follows easily from (3.1). To prove the existence, we use the well known formula

\[
d\Theta(u_0, u_1, \ldots, u_m) = \sum_{i=0}^{m} (-1)^i u_i \cdot \Theta(u_0, \ldots, \hat{u}_i, \ldots, u_m)
\]

\[
+ \sum_{0 \leq i < j \leq m} (-1)^{i+j} \Theta([u_i, u_j], u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_m),
\]

where \(u_1, \ldots, u_m\) are vector fields on \(J E\) belonging to \(\Delta\). Let us note preliminarily that

\[
d\Theta(\partial_x, u_1, \ldots, u_m) = - y_x^i d\Theta(\partial_i, u_1, \ldots, u_m),
\]

since \(d\Theta \mid \Delta = 0\) and the vector fields \(w_x = \partial_x + y_x^i \partial_i \in \Delta\).

By putting \(u_0 = \partial_x^*\), \(u_x = w_x\) into (3.9) and by using the properties of \(\Theta\), it follows that

\[
\pi^*_x = (-1)^{x-1} \Theta(\partial_i, w_1, \ldots, \hat{w}_x, \ldots, w_m),
\]

because \([\partial_x^*, w_p] = \partial_p^* \partial_i\).

Let now \(s : M \to E\) be a section and let

\[
u_x = w_x + (\partial^2_x s^i \partial_i^p) \in \Delta, \quad s^i = y^i \circ s.
\]

By putting again these vector fields in (3.9), since \([u_x, u_p] = 0\), we get

\[
(js)^*(u_0 \perp d \Theta)(\partial_1, \ldots, \partial_m) = d\Theta(u_0, u_1, \ldots, u_m) \circ js
\]

\[
= (js)^* \left[ u_0 \vartheta + \sum_{x=1}^{m} (-1)^x u_x \cdot \Theta(u_0, w_1, \ldots, \hat{w}_x, \ldots, w_m) \right.
\]

\[
+ \sum_{x=1}^{m} (-1)^x \Theta([u_0, u_x], w_1, \ldots, \hat{w}_x, \ldots, w_m) \Big] .
\]
where we have used the properties that $\Theta$ is $p_E$-horizontal and that $\Theta | \Delta = \Lambda | \Delta$. By putting $u_0 = \partial_i$ into (3.7) and by taking into account (3.5), it follows that

$$\text{(3.8)} \quad (js)^* (\partial_i \downarrow d\Theta)(\partial_1, \ldots, \partial_m) = - (js)^* [d\pi_i^* \wedge \omega_x - (\partial_i \mathcal{L})\omega](\partial_1, \ldots, \partial_m).$$

Finally, if the local expression of $u_0$ is $u_0 = u^\alpha \partial_x + u^i \partial_i + u_0^\alpha \partial_x$, since $d\Theta | \Delta = 0$, we get from (3.4) and (3.8)

$$\text{(3.9)} \quad (js)^* (u_0 \downarrow d\Theta) = - (js)^* [d\pi_i^* \wedge \omega_x - (\partial_i \mathcal{L})\omega](u^i - y_0^\alpha u^\alpha) \circ js.$$

Hence the proof is complete.

The functoriality of the assignment $\Lambda \to \mathcal{E}_\Lambda$ can easily be proved. Let $\varphi : E \to E$ be an automorphism (over $\varphi_M : M \to M$). Then by taking the pull-back of both sides of (3.1) by means of $\varphi^*$ and by using (1.5), we have

$$\text{(3.10)} \quad (js)^* \{ [(J\varphi)^* u_0] \downarrow d(J\varphi)^* \Theta \} = - \langle (js)^* \varphi^* \mathcal{E}_\Lambda, \varphi (J\varphi)^* u_0 \circ js^* \rangle,$$

where $s^* = \varphi^* s$. By recalling (2.14) and the uniqueness property of $\mathcal{E}(J\varphi)^\Lambda$, as stated before, (3.10) implies that

$$\text{(3.11)} \quad \varphi^* \mathcal{E}_\Lambda = \mathcal{E}(J\varphi)^\Lambda.$$

b) Let $\Lambda$ be a Lagrangian form on $je$. Let us recall that $\mathcal{E}_\Lambda$ is an $m$-form defined only on $\Delta \subset T\mathcal{E}$ and valued in $V^*E$. Then the second order canonical inclusion (1.6) induces the operator

$$\text{(3.12)} \quad E_\Lambda : J^2E \to V^*E \otimes E \wedge ^m T^*M.$$

It is characterized by

$$\text{(3.13)} \quad E_\Lambda \circ j^2 s = (js)^* \mathcal{E}_\Lambda,$$

for any section $s : M \to E$. This second order operator is the Euler-Lagrange operator. Its local expression, as follows from (3.2) and (3.13), is

$$\text{(3.14)} \quad E_\Lambda = [(\partial_j^\beta \pi_i^\gamma) y_j^\beta + (\partial_j \pi_i^\gamma) y_j^i + \partial_i \pi_\gamma - \partial_i \mathcal{L}] dy^i \otimes \omega.$$

Note that $E_\Lambda$ is an affine map over $je$.

Following the procedure as in (2.9), it is clear that $E_\Lambda$ can be viewed as an $(m + 1)$-form on $J^2E$ whose local expression is

$$\text{(3.15)} \quad E_\Lambda = [(\partial_j^\beta \pi_i^\gamma) y_j^\beta + (\partial_j \pi_i^\gamma) y_j^i + \partial_i \pi_\gamma - \partial_i \mathcal{L}] dy^i \wedge \omega.$$

The functoriality of $\Lambda \mapsto E_\Lambda$ can be seen in this way. If $\varphi : E \to E$ is an automorphism, by taking the pull-back of both sides of (3.13) by means of $\varphi^*$, we get

$$\text{(3.16)} \quad (\varphi^* E_\Lambda) \circ j^2 s^* = (js)^* \mathcal{E}(J\varphi)^\Lambda, \quad s^* = \varphi^* s,$$

where we have used (3.11). Hence the uniqueness property of $E(J\varphi)^\Lambda$ implies that

$$\text{(3.17)} \quad \varphi^* E_\Lambda = E(J\varphi)^\Lambda.$$
The following proposition gives the infinitesimal version of (3.17) in terms of i. c. t.

**Proposition.** — Let \( u_E \) be a vector field on \( E \), projectable on \( M \). Then we have

\[
L_u E_\Lambda = E_{L_\omega L_\Lambda}.
\]

where with \( u \) on the right and the left side of (3.18) we denote the 1 and 2-jet prolongation of \( u_E \), respectively.

**Proof.** — We use the formula

\[
d\Theta_\Lambda = - E_\Lambda + d_\nu \Omega_\Lambda,
\]

where \( d_\nu \) is the vertical derivation on jet spaces [12]. Formula (3.19) can be proved by a straightforward calculation. The local expression of \( d_\nu \Omega_\Lambda \) is

\[
d_\nu \Omega_\Lambda = (\partial \pi^i_\nu) \partial^j \wedge \partial^j \wedge \omega_x + (\partial^j \pi^i_\nu) \partial^j \wedge \partial^j \wedge \omega_x.
\]

Since the following commutation relation holds

\[
L_\omega d_\nu \Omega_\Lambda = d_\nu L_\omega \Omega_\Lambda,
\]

from (3.19) we get

\[
d\Theta_{L_\omega L_\Lambda} = - L_\omega E_\Lambda + d_\nu \Omega_{L_\omega L_\Lambda},
\]

where we have used (2.12) and (2.17). Hence the result follows by comparing (3.22) with (3.23) with

\[
d\Theta_{L_\omega L_\Lambda} = - E_{L_\omega L_\Lambda} + d_\nu \Omega_{L_\omega L_\Lambda}.
\]

c) There are some useful relations between \( \Lambda, E_\Lambda \) and \( \Theta_\Lambda \). Note that \( \delta_\Lambda = 0 \) is equivalent to \( E_\Lambda = 0 \), as follows from (3.13). From (3.1) it follows that \( d \Theta_\Lambda = 0 \) implies \( E_\Lambda = 0 \) (and also that \( \Lambda \) is affine). Also the converse is true. Hence we have

\[
d \Theta_\Lambda = 0 \iff \Lambda \text{ is affine and } E_\Lambda = 0.
\]

Another statement equivalent to (3.24) is: \( \Lambda \) is affine and \( E_\Lambda \) is closed (cfr. (2.4)).

Let us remark that if \( \dim M = 1 \) or also if \( \dim E = m + 1 \), then \( E_\Lambda = 0 \) implies that \( \Lambda \) is affine. It follows that also \( d \Theta_\Lambda = 0 \). However, apart from these special cases, in general \( E_\Lambda = 0 \) does not implies that \( \Lambda \) is affine [6].

d) Given a Lagrangian form \( \Lambda \), intuitively, the « variational symmetries of \( \Lambda \) » are the morphisms of a category whose objects are the variational forms \( \{ \Lambda \} \). This is made precise by the following definition.

**Definition.** — Let \( \Lambda \) be a Lagrangian form on \( JE \). Then an automor-
phism \( \varphi : E \to E \) is said to be a variational symmetry (in short: a v. s.) of \( \Lambda \) iff there is some variational form \( \Theta \in \{ \Lambda \} \) such that
\[
(J\varphi)^*d\Theta \in d\{ \Lambda \}.
\]

(3.25)

Note that by taking the automorphisms \( \varphi : E \to E \) such that
\[
(J\varphi)^*\Theta \in \{ \Lambda \},
\]
for some \( \Theta \in \{ \Lambda \} \), we get a subcategory of the previous one. It is clear that if \( \varphi \) is a v. s. of \( \Lambda \), then \( \varphi \) is a symmetry of the Euler-Lagrange operator \( E_\Lambda \), i.e. \( \varphi^*E_\Lambda = E_\Lambda \). This follows from (3.1) in the same way used to prove (3.10).

The existence of a canonical variational form of \( \Lambda \), namely the Poincaré-Cartan \( \Theta_\Lambda \), allows us to consider some distinguished v. s. By recalling (2.15) and the equivalence between \( \Theta_\Lambda = 0 \) and \( \Lambda = 0 \), it follows that
\[
\varphi \in \text{Aut} \{ \Theta_\Lambda \} \iff \varphi^*\Lambda = \Lambda.
\]

(3.27)

Hence these v. s. are just the symmetries of \( \Lambda \). Moreover, by recalling (3.24), we get
\[
\varphi \in \text{Aut} \{ d\Theta_\Lambda \} \iff (J\varphi)^*\Lambda - \Lambda \text{ is affine and } \varphi^*E_\Lambda = E_\Lambda.
\]

(3.28)

Note that if \( \dim M = 1 \) or also if \( \dim E = m + 1 \), then a symmetry of \( E_\Lambda \) is also an automorphism of \( d\Theta_\Lambda \).

We finish by considering briefly the infinitesimal symmetries. Let \( u_E \) be a vector field on \( E \) projectable on \( M_1 \) and let \( u \) be its 1-jet prolongation. Then \( u_E \) is said to be an infinitesimal variational symmetry (in short: i. v. s.) of a Lagrangian form \( \Lambda \) iff
\[
L_u d\Theta = 0,
\]
for some \( \Theta \in \{ \Lambda \} \). It is clear that if \( u_E \) is an i. v. s. of \( \Lambda \), then \( u \) (the 2-jet prolongation of \( u_E \)) is an infinitesimal symmetry of the Euler-Lagrange operator \( E_\Lambda \), i.e. \( L_u E_\Lambda = 0 \), as follows from (3.1) and (3.18).

Clearly, we get i. v. s. of \( \Lambda \) by taking those vector fields \( u_E \) on \( E \), projectable on \( M \), such that
\[
L_u \Theta = 0,
\]
for some \( \Theta \in \{ \Lambda \} \) (as before \( u \) is the 1-jet prolongation of \( u_E \)).

The infinitesimal versions of (3.27) and (3.28) are
\[
L_u \Theta_\Lambda = 0 \iff L_u \Lambda = 0,
\]
and
\[
L_u d\Theta_\Lambda = 0 \iff L_u \Lambda \text{ is affine and } L_u E_\Lambda = 0,
\]
respectively.
4. CRITICAL SECTIONS

a) In order to exhibit the functorial nature of a variational problem in the most direct way, we start with the following definition.

**Definition.** — Let $s : M \to E$ be a section. A variation of $s$ is a pair $(P, (\psi_t)_*s)$ where

i) $P \subset M$ is an orientable $m$-dimensional compact submanifold of $M$ with boundary (denoted by $\partial P$),

ii) $\psi_t$ is a one-parameter family of automorphisms $\psi_t : E \to E$ (over $\chi_t : M \to M$) such that

\begin{equation}
\psi_t|_{\partial P} = id_{\partial P}, \quad (\psi_t)_*s|_{\partial P} = s|_{\partial P} \quad \text{for any } t.
\end{equation}

The existence of $\psi_t$ follows easily by considering, for example, vertical fields on $E$ with compact support. Note that (4.1) implies that variations of $s$ behave in the right way under automorphisms of $E$. In fact, let $\phi : E \to E$ be an automorphism (over $\phi_M : M \to M$) and let $(P, (\psi_t)_*s)$ be a variation of $s$. Then

\begin{equation}
\phi^*(P, (\psi_t)_*s) = (N, \phi^*(\psi_t)_*s),
\end{equation}

where $N = \phi_M^{-1}(P)$, is a variation of $\phi^*s$ since

\begin{equation}
\phi^*(\psi_t)_*s = (\phi_t)_*\phi^*s, \quad \phi_t = \phi^*\psi_t = \phi_M^{-1} \circ \psi_t \circ \phi_M.
\end{equation}

**Definition.** — Let $\Lambda$ be a Lagrangian form on $JE$. A section $s : M \to E$ is said to be critical (with respect to $\Lambda$) iff the function

\begin{equation}
t \mapsto \int_{\chi_t(P)} j[(\psi_t)_*s]^*\Lambda
\end{equation}

is stationary at $t = 0$ for any variation $(P, (\psi_t)_*s)$ of $s$.

Let $\phi : E \to E$ be an automorphism (over $\phi_M : M \to M$). The $\phi^*s$ is a critical section of the Lagrangian form $(J\phi)^*\Lambda$ iff $s$ is a critical section of $\Lambda$, as follows from (4.4)

\begin{equation}
\int_{\chi_t(P)} j[(\psi_t)_*s]^*\Lambda = \int_{\phi_M^{-1}(\chi_t(P))} \phi_M^*j[(\psi_t)_*s]^*\Lambda
\end{equation}

\begin{equation}
= \int_{\lambda_t(N)} (J\phi \circ j[(\phi_t)_*s])^*\Lambda = \int_{\lambda_t(N)} j[(\phi_t)^*\phi^*s]^*\Lambda,
\end{equation}

where $\lambda_t = \phi_M^{-1} \chi_t$ is the diffeomorphism induced by $\phi_t$.

b) Given a Lagrangian form $\Lambda$, critical sections of $\Lambda$ can be characterized by using the form $\partial_\Lambda$ as follows.

PROPOSITION. — Let $\Lambda$ be a Lagrangian form on $J\xi E$. Then a section $s : M \to E$ is critical (with respect to $\Lambda$) iff $(js)^*\delta_\Lambda = 0$.

Proof. — The derivative of the function $(4.5)$ at $t = 0$ is

$$D\left(\int_{\xi(t)} j[(\psi_t)_s]^{^*}\Lambda\right)_{t=0} = \int_p (js)^*L_u\Lambda,$$

where $u$ is the 1-jet prolongation of $u_E$, the vector field on $E$ determined by $\psi_t$. This result follows by applying the change of variables formula to $(4.5)$, i.e.

$$\int_{\xi(t)} j[(\psi_t)_s]^{^*}\Lambda = \int_p (js)^*(J\psi_t)^*\Lambda.$$

Note that $u$ is just the vector field on $J\xi E$ determined by the one parameter family $J\psi_t$.

By using successively $(2.16)$, the well known identity

$$L_u\Theta = u \perp d\Theta + d(u \perp \Theta)$$

and Stokes' theorem, $(4.7)$ can be written as

$$\int_p (js)^*L_u\Lambda = \int_p (js)^*(u \perp d\Theta) + \int_{\partial p} (js)^*(u \perp \Theta),$$

where $\Theta \in \{\Lambda\}$. The last integral vanishes since $\Theta$ is $p_E$-horizontal and $u_E|p^{-1}(\partial P) = 0$. The result follows now from $(3.1)$.

By choosing $\Theta \in \{\Lambda\}$, from $(3.1)$ we get another characterization of critical sections, namely a section $s$ is critical iff

$$(4.10) \quad (js)^*(u \perp d\Theta) = 0,$$

for any vector field $u$ on $J\xi E$ (or also for any i. c. t. $u$ projectable over $E$). In terms of the Euler-Lagrange operator $E_{\Lambda}$, the condition for $s$ to be critical is the well known one, namely $E_{\Lambda} \circ j^2s = 0$, as follows from $(3.13)$.

REFERENCES

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