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## On the structure of relative identification operators for quantum fields and their connection with the Haag-Ruelle scattering theory

by

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**ABSTRACT.** — The paper recalls the notion of a relative identification operator  $K$  of a Wightman field  $A(\cdot)$  with respect to a corresponding free field  $A^0(\cdot)$ , useful for the definition of wave operators with respect to the field  $A(\cdot)$ . The corresponding pre-wave operator  $e^{iH}Ke^{-iH_0}$  can be linked directly with the Haag-Ruelle approximants of the field  $A(\cdot)$ . Thus the Haag-Ruelle scattering theory can be embedded formally into the framework of the abstract scattering theory. Some structural properties of  $K$  are presented. It is pointed out that  $K$  is uniquely determined by a single field operator  $A(h(p)\gamma_0(p))$ , where  $h(p)$  is a smooth function with support in a sufficiently small neighbourhood of the discrete mass hyperboloid characterized by the mass  $m_0 > 0$  belonging to  $A^0(\cdot)$  (as is usually introduced within the Haag-Ruelle framework) and where  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  is multiplicative-generating.

**RÉSUMÉ.** — On rappelle la notion d'opérateur d'identification relatif  $K$  d'un champ de Wightman  $A(\cdot)$  par rapport au champ libre correspondant  $A^0(\cdot)$ , utile pour la définition des opérateurs d'onde du champ  $A(\cdot)$ . Le pré-opérateur d'onde  $e^{iH}Ke^{-iH_0}$  correspondant peut être relié directement aux approximants de Haag-Ruelle pour le champ  $A(\cdot)$ . Ainsi la théorie de la diffusion de Haag-Ruelle peut être formellement incorporée dans le cadre de la théorie abstraite de la diffusion. On donne quelques propriétés structurelles de  $K$ . On remarque que  $K$  est déterminé de façon unique par un seul opérateur de champ  $A(h(p)\gamma_0(p))$ , où  $h(p)$  est une fonction lisse à support contenu dans un voisinage assez petit de l'hyperboloïde de

masse caractérisé par la masse  $m_0 > 0$  du champ  $A^0(\cdot)$  (comme on l'introduit de façon usuelle dans la théorie de Haag-Ruelle), et où  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  est génératrice par multiplication (voir Définition 3).

§ 1. INTRODUCTION

Let  $A(\cdot)$  be a Wightman field on a (separable) Hilbert space  $\mathcal{H}$ . For convenience we collect the properties of such a field. The tensor algebra over the Schwartz space  $\mathcal{S}(\mathbb{R}^4)$  is denoted by  $\mathcal{F}$ . Its elements are finite sequences  $f := \{ f_0, f_1, \dots, f_N, 0, \dots \}$ , where  $N$  depends on  $f$  and where  $f_n \in \mathcal{S}(\mathbb{R}^{4n})$ .  $\mathcal{F}$  is equipped with the usual topology  $\tau$  (locally convex direct sum of the Schwartz space topologies of the  $\mathcal{S}(\mathbb{R}^{4n})$ ). The Wightman functional  $W(\cdot) : \mathcal{F} \mapsto \mathbb{C}$  of the field  $A(\cdot)$  is assumed to be linear, normed, positive, continuous and Poincaré invariant. There is a unique vacuum  $\omega \in \mathcal{H}$  and the field is assumed to be spectral and local. The continuous linear functionals on  $\mathcal{F}$  have the form  $W = \{ W_0, W_1, W_2, \dots \}$ , where  $W_n$  is a continuous linear functional on  $\mathcal{S}(\mathbb{R}^{4n})$ , a so-called  $n$ -point functional, and  $W(f) = \sum W_n(f_n)$ . Recall the special form of the functionals  $W_0, W_1, W_2$  :

- (1)  $W_0(f_0) = f_0,$
- (2)  $W_1(f_1) = \gamma f_1(0), \quad \gamma \in \mathbb{R},$
- (3)  $W_2(f_2) = \gamma f_2(0, 0) + \int_0^\infty \int_{H_m} f_2(-p, p) \mu_m(dp) \rho(dm).$

Note that we prefer to work with momentum coordinates, that is with functions  $f_n(p_1, p_2, \dots, p_n) \in \mathcal{S}(\mathbb{R}^{4n})$  which are Fourier transforms

$$f_n(p_1, \dots, p_n) = (2\pi)^{-2n} \int e^{-i \sum_{j=1}^n (p_j, x_j)} \check{f}_n(x_1, \dots, x_n) dx_1 \dots dx_n$$

of functions  $\check{f}_n \in \mathcal{S}(\mathbb{R}^{4n})$  depending on position coordinates.  $(\cdot, \cdot)$  denotes the Cartesian scalar product in  $\mathbb{R}^4$ .  $H_m$  denotes the mass hyperboloid  $H_m := \{ p : p_0^2 - |p|^2 = m^2, p_0 > 0 \}$ ,  $\mu_m(\cdot)$  denotes the Lorentz invariant measure on  $H_m$ , given by  $\mu_m(dp) = dp / (m^2 + |p|^2)^{1/2}$  and  $\rho(\cdot)$  is a characteristic polynomially bounded Borel measure on  $m \geq 0$ . Formula (3) is called the Källen-Lehmann representation of the 2-point functional.

The set of all  $f \in \mathcal{F}$  satisfying  $W(f^*f) = 0$  is denoted by lker  $W$  (left kernel). Note that  $f^*$  denotes the usual conjugation in  $\mathcal{F}$ ,  $f_n(p_1, \dots, p_n) = \overline{f_n(-p_n, -p_{n-1}, \dots, -p_1)}$ .

The field  $A(\cdot)$  is defined on  $\mathcal{F}$ , i. e.  $A(f), f \in \mathcal{F}$ , is a generalized field

operator, the usual field operator is given by  $A(\tilde{f}_1)$  where  $\tilde{f}_1 = (0, f_1, 0, \dots) \in \mathcal{F}$ . For brevity we write also  $A(f_1)$  in this case.

Furthermore, an upper and lower mass gap is assumed, the discrete mass is denoted by  $m_0$ , the corresponding one-particle subspace of  $\mathcal{H}$  is denoted by  $\mathcal{H}_1$ , it is assumed to be irreducible with respect to the Poincaré group, the corresponding representation is labeled by  $m_0$  and  $s = 0$ . Recall the representation (SNAG-theorem)

$$(4) \quad U_a = \int_{\mathbb{R}^4} e^{-i(a,p)} E(dp)$$

for the unitary representation  $U_a$  of the translation group  $a \in \mathbb{R}^4$  associated with the field. In terms of  $E(\cdot)$  the mass gap is expressed by

$$(5) \quad \text{supp}^m E = \{0\} \cup \{m_0\} \cup \Lambda, \Lambda \subseteq [m_0 + \varepsilon, \infty), \varepsilon > 0, m_0 > 0,$$

where  $\text{supp}^m E$  denotes the mass spectrum (note that  $\text{supp} E \subseteq \text{clo } V^+$ ,  $V^+$  the forward cone, and that  $\text{supp} E$  is Lorentz invariant, i. e. it contains only full mass hyperboloids  $H_m$ , then the mass spectrum is the closure of all  $m$  such that  $H_m \subset \text{supp} E$ ).

Finally, the condition of « coupling of the vacuum to the one-particle states » is assumed to be satisfied (see for example M. Reed and B. Simon [1, p. 319]. Note that this condition is satisfied if and only if

$$(6) \quad m_0 \in \text{supp } \rho$$

is valid. That is, in this case one obtains

$$(7) \quad \{m_0\} \subseteq \text{supp } \rho \subseteq \text{supp}^m E,$$

(the latter inclusion is obvious).

The free (scalar) field, corresponding to  $m_0 > 0$  and  $s = 0$ , is denoted by  $A^0(\cdot)$ , acting on the Hilbert space  $\mathcal{H}^0$ . Its measure «  $\rho$  » (mass distribution of the Källén-Lehmann representation) is given by the Dirac measure  $\rho(m) = \delta(m - m_0)$ .

In H. Baumgärtel *et al.* [2] a so-called relative identification operator  $K$  is introduced, useful for the definition of wave operators with respect to the field  $A(\cdot)$ . For convenience, we recall the definition and simple properties of  $K: \mathcal{H}^0 \mapsto \mathcal{H}$ . First, by  $h \in C^\infty(\mathbb{R}^4)$  we denote a fixed real-valued function,  $0 \leq h \leq 1$ , with the following properties:

$$i) \text{supp } h \subseteq \bigcup_{m \in [m_0 - \delta, m_0 + \delta]} H_m \quad \delta > 0,$$

$$ii) h(\Lambda p) = h(p) \text{ for all } \Lambda \in \mathcal{L}_+^\uparrow \text{ (proper Lorentz group),}$$

$$iii) h \upharpoonright H_{m_0} = 1.$$

Second, we define a certain linear manifold  $\mathcal{L} \subset \mathcal{F}$  by:  $g \in \mathcal{L}$  if and only if  $g_0 \in \mathbb{C}$  arbitrary,  $g_n(p_1, p_2, \dots, p_n) = h(p_1)h(p_2) \dots h(p_n)\gamma_n(p_1, p_2, \dots, p_n)$ ,  $\gamma_n \in \mathcal{S}(\mathbb{R}^{3n})$ ,  $\gamma_n$  symmetric with respect to  $p_1, p_2, \dots, p_n$ .

Then, if  $(W^0(\cdot))$  denotes the Wightman functional and  $J^0$  denotes the (absolute) identification operator of the free field, it turns out that  $\mathcal{L}$  contains exactly one element from each equivalence class mod  $\text{lker } W^0$ , that is,  $\text{ima } (J^0 \upharpoonright \mathcal{L}) = \text{ima } J^0$  and  $f \in \mathcal{L}$  and  $J^0 f = 0$  imply  $f = 0$ , in other words,  $\mathcal{L} \cap \text{lker } W^0 = \{0\}$  and  $\mathcal{L} \oplus \text{lker } W^0 = \mathcal{F}$ . Now an operator  $K: \mathcal{H}^0 \mapsto \mathcal{H}$  can be defined by

$$(8) \quad K \{ A^0(f)\omega^0 \} := A(f)\omega, \quad f \in \mathcal{L},$$

or

$$(9) \quad K \{ J^0 f \} := Jf, \quad f \in \mathcal{L},$$

where  $J$  denotes the (absolute) identification operator of the field  $A(\cdot)$ . (9) means that  $K$  is a certain factorization of  $J$ ,  $J = KJ^0$ , on  $\mathcal{L}$ .  $K$  is called the *relative identification operator* between  $A^0(\cdot)$  and  $A(\cdot)$  with respect to  $\mathcal{L}$ . Recall the following simple properties of  $K$ :

- I)  $K$  is densely defined,  $\text{dom } K = \text{ima } J^0$ , which is dense in  $\mathcal{H}^0$ .
- II)  $\text{dom } K$  is invariant with respect to  $U_g^0$  (the unitary representation of the Poincaré group belonging to the free field).
- III)  $K$  is continuous with respect to the Schwartz space topology  $\tau$  of  $\mathcal{F}$  (more precisely :  $\text{dom } K$  may be equipped with this topology by the bijection  $\mathcal{L} \ni f \leftrightarrow J^0 f \in \text{dom } K$ , then  $K$  is continuous with respect to this topology).
- IV)  $K\omega^0 = \omega$ .
- V) The intertwining relation

$$(10) \quad U_g K = K U_g^0$$

is valid if  $g = \{ \Lambda_0, (0, \alpha) \}$ , where  $\alpha \in \mathbb{R}^3$  is a pure spatial translation and where  $\Lambda_0$  is a pure rotation in the  $\alpha$ -space (the intertwining relation (10) is not valid in general for time translations).

Using  $K$ , the standard two-space pre-wave operator is given by

$$(11) \quad e^{itH} K e^{-itH^0} u, \quad u \in \text{dom } K,$$

where  $e^{-itH} = U_{(t,0)}$ ,  $e^{-itH^0} = U_{(t,0)}^0$  denote the unitary representations of the time translations in  $\mathcal{H}$ ,  $\mathcal{H}^0$ , respectively.

In this paper some further structural properties of  $K$  are presented. In fact, it is shown that the expression (11) is intimately connected with the Haag-Ruelle approximants with respect to the field  $A(\cdot)$ .

If  $\mathcal{M}$  is a subset of  $\mathcal{F}$ , for brevity we denote by  $\mathcal{M}^{(n)}$  the set of all  $f \in \mathcal{M}$  with  $f = \{ 0, \dots, 0, f_m, 0, \dots \}$ , i. e. the intersection of  $\mathcal{M}$  with  $\mathcal{S}(\mathbb{R}^{4n})$ .

Finally recall the assignment between one-particle states and field operators.

If  $f \in \mathcal{L}^{(1)}$  then  $A(f)\omega \in \mathcal{H}_1$ . Moreover, the assignment  $\mathcal{L}^{(1)} \ni f \mapsto A(f)\omega \in \mathcal{H}_1$  is an injection, the image  $\{ A(f)\omega, f \in \mathcal{L}^{(1)} \}$  coincides with

$$\{ A(f)\omega, f \in \mathcal{F}^{(1)}, \text{supp } f \cap \text{supp } E \subseteq H_{m_0} \}$$

and this linear manifold is dense in  $\mathcal{H}_1$ .

That is, the vector  $u = A(f)\omega \in \mathcal{H}_1$  with  $f \in \mathcal{L}^{(1)}$  is in one-to-one correspondence with  $f$ . Therefore, one has an assignment of one-particle states  $u \in \mathcal{H}_1$  to certain field operators  $B_u = A(f)$ . This assignment satisfies the property  $B_u\omega = u$ .

On the other hand, the assignment can be considered as the assignment of vectors of the free one-particle space to vectors  $u \in \mathcal{H}_1$  via  $K$ . Namely, if  $f \in \mathcal{L}^{(1)}$ , then  $A^0(f)\omega^0 = \{0, \gamma(p), 0, \dots\}$ , where  $f_1(p) = h(p)\gamma(p)$  and where  $\gamma(p)$  is to be considered as an element of  $L^2(\mathbb{R}^3, dp/\mu(p))$ . That is, we have

$$\mathcal{H}_1^0 = L^2(\mathbb{R}^3, dp/\mu(p)) \ni \gamma(p) \xrightarrow{K} A(f)\omega = u \in \mathcal{H}_1,$$

where  $\mu(p) := (m_0^2 + |p|^2)^{1/2}$ .

### § 2. CALCULATION OF $K$

In this paragraph we calculate  $K$  on a dense subdomain of  $\text{dom } K = \text{ima } J^0 = \text{ima } (J^0 \upharpoonright \mathcal{L})$ . For this purpose we use the structure of  $J^0$ , which is given in [2]. According to this paper (Corollary 1) we have

$$J^0 f = \{ f_0, (Tf)_1 \upharpoonright H_{m_0}, S_2(Tf)_2 \upharpoonright H_{m_0} \times H_{m_0}, \dots \}, \quad f \in \mathcal{F},$$

where  $T$  denotes a certain continuous linear operator, acting in  $\mathcal{F}$  (see [2, Theorem 2]) and where  $S_n$  denotes the symmetrization operator

$$(S_n f_n)(p_1, \dots, p_n) = (n!)^{-1} \sum_{\pi} f_n(p_{\pi(1)}, p_{\pi(2)}, \dots, p_{\pi(n)}).$$

Note that  $Tf = f$  for  $f \in \mathcal{L}$  (cf. [2, Lemma 5]), that is, one obtains

$$J^0 f = \{ f_0, f_1 \upharpoonright H_{m_0}, S_2 f_2 \upharpoonright H_{m_0} \times H_{m_0}, \dots \}, \quad f \in \mathcal{L}.$$

Recall that  $f \in \mathcal{L}$  means  $f_n(p_1, \dots, p_n) = \prod_{j=1}^n h(p_j)\gamma_n(p_1, p_2, \dots, p_n)$ , where  $\gamma_n$  is symmetric. Therefore

$$J^0 f = \{ f_0, \gamma_1(p_1), \gamma_2(p_1, p_2), \dots \}, \quad f \in \mathcal{L},$$

where  $\gamma_n(p_1, \dots, p_n)$  is to be considered as an element of

$$L^2\left(\mathbb{R}^{3n}, \bigotimes_{j=1}^n dp_j/\mu(p_j)\right),$$

where  $\mu(p) := (m_0^2 + |p|^2)^{1/2}$ .

Now let  $K_n$  be the  $n$ -particle component of  $K$ , that is

$$K_n := K \upharpoonright S_n(\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1),$$

such that

$$Ku = \sum_{n=0}^{\infty} K_n u_n, \quad u = J^0 f, \quad f \in \mathcal{L}, \quad u_n = J^0 f_n.$$

Furthermore, let

$$(12) \quad \gamma_n(\mathfrak{p}_1, \dots, \mathfrak{p}_n) = S_n(\alpha_1(\mathfrak{p}_1) \otimes \alpha_2(\mathfrak{p}_2) \otimes \dots \otimes \alpha_n(\mathfrak{p}_n)), \quad \alpha_j \in \mathcal{S}(\mathbb{R}^3).$$

According to (8) we obtain

$$(13) \quad K_n \{ S_n(\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n) \} \\ = S_n \left\{ \prod_{j=1}^n A(h(p)\alpha_j(\mathfrak{p})) \right\} \omega, \quad n = 1, 2, \dots,$$

where we write for brevity  $A(h(p)\alpha_j(\mathfrak{p}))$  instead of  $A(\{0, h(p)\alpha_j(\mathfrak{p}), 0, \dots\})$ . We denote the linear submanifold of  $\mathcal{L}$  defined by (12) by  $\mathcal{L}_0 \subset \mathcal{L}$ .  $\mathcal{L}_0$  is dense in  $\mathcal{L}$  with respect to  $\tau$ . Then we have

**PROPOSITION 1.** — *The relative identification operator  $K: \mathcal{H}^0 \mapsto \mathcal{H}$  is given on the (dense) submanifold  $\text{ima}(J^0 \upharpoonright \mathcal{L}_0)$  of  $\text{dom } K$  by formula (13).*

*Proof.* — Obvious by the preceding arguments of this paragraph. ■

According to (13),  $K \upharpoonright \text{ima}(J^0 \upharpoonright \mathcal{L}_0)$ , hence  $K$  itself, is already uniquely determined by the field operators  $A(h(p)\alpha(\mathfrak{p}))$ ,  $\alpha \in \mathcal{S}(\mathbb{R}^3)$ . But the Poincaré covariance property of the field  $A(\cdot)$  implies a strong connection between these field operators. Namely, we have

**LEMMA 2.** — *Let  $\alpha \in \mathcal{S}(\mathbb{R}^3)$ ,  $f \in \mathcal{S}(\mathbb{R}^4)$ . Then*

$$(14) \quad A(\hat{\alpha}(\mathfrak{p})\hat{f}(p)) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha(\mathfrak{a}) U_{\mathfrak{a}} A(f) U_{-\mathfrak{a}} d\mathfrak{a},$$

where  $\hat{\alpha}$  denotes the spatial Fourier transform of  $\alpha$  and  $\hat{f}$  denotes the 4-dimensional Fourier transform of  $f$ . The integral on the right hand side is weakly convergent for vectors  $u, v \in \mathcal{D} = \text{ima } J$  (domain of the field operators  $A(\cdot)$ ).

*Proof.* — From the covariance property of  $A(\cdot)$  we obtain

$$A(V_{\mathfrak{a}} f) = U_{\mathfrak{a}} A(f) U_{-\mathfrak{a}},$$

where, as usual,  $(V_{\mathfrak{a}} f)(x) = f(x_0, \mathbf{x} - \mathfrak{a})$ . Further we obtain

$$A\left(\int_{\mathbb{R}^3} \alpha(\mathfrak{a}) V_{\mathfrak{a}} f d\mathfrak{a}\right) = \int_{\mathbb{R}^3} \alpha(\mathfrak{a}) U_{\mathfrak{a}} A(f) U_{-\mathfrak{a}} d\mathfrak{a}.$$

But  $(V_a f)^\wedge(p) = e^{-i(a,p)} \hat{f}(p)$  and

$$\int_{\mathbb{R}^3} \alpha(a)(V_a f)^\wedge(p) da = \int_{\mathbb{R}^3} \alpha(a)e^{-i(a,p)} da \hat{f}(p) = (2\pi)^{3/2} \hat{\alpha}(p) \hat{f}(p).$$

This concludes the proof. ■

The relation (14) implies that the field operators  $A(h(p)\alpha(p))$  are uniquely determined by a single field operator  $A(h(p)\gamma_0(p))$ , where  $\gamma_0$  is multiplicative-generating in  $\mathcal{S}(\mathbb{R}^3)$  (together with the representation  $U_a$ ).

**DEFINITION 3.** — *The function  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  is called multiplicative-generating with respect to  $\mathcal{S}(\mathbb{R}^3)$ , if  $\{\alpha\gamma_0 : \alpha \in \mathcal{S}(\mathbb{R}^3)\}$  is dense in  $\mathcal{S}(\mathbb{R}^3)$ .*

For example, if  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  and  $\gamma_0(p) \neq 0$  for all  $p \in \mathbb{R}^3$ , then  $\gamma_0$  is multiplicative-generating. For example let  $\gamma_0(p) = \exp(-|p|^2)$ .

**PROPOSITION 4.** — *Let  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  be multiplicative-generating. Then  $\{\hat{\alpha}\gamma_0 : \alpha \in \mathcal{S}(\mathbb{R}^3)\}$  is dense in  $\mathcal{S}(\mathbb{R}^3)$  and*

$$(15) \quad A(h(p)\hat{\alpha}(p)\gamma_0(p)) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha(a)U_a A(h(p)\gamma_0(p))U_{-a} da.$$

*Proof.* Obvious. ■

Proposition 1 and Proposition 4 together lead to an explicit description of  $K$ , showing, that  $K$  is uniquely determined by  $A(h(p)\gamma_0(p))$  for some multiplicative-generating function  $\gamma_0$ .

**COROLLARY 5.** — *Let  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  be multiplicative-generating and put  $B_0 := A(h(p)\gamma_0(p))$ . Then  $K \uparrow \text{ima}(J^0 \uparrow \mathcal{L}_0)$  is explicitly given by*

$$(16) \quad K_n \{ S_n(\hat{\alpha}_1\gamma_0 \otimes \hat{\alpha}_2\gamma_0 \otimes \dots \otimes \hat{\alpha}_n\gamma_0) \} \\ = S_n \prod_{j=1}^n \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha_j(a)U_a B_0 U_{-a} da \right\} \omega, \quad n = 1, 2, \dots$$

*Proof.* Obvious. ■

### § 3. CALCULATION OF THE PRE-WAVE OPERATOR

Let  $\gamma_0 \in \mathcal{S}(\mathbb{R}^3)$  be multiplicative-generating and put  $B_0 := A(h(p)\gamma_0(p))$ . The next Proposition calculates the pre-wave operator (11).

**PROPOSITION 6.** — *Let  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{S}(\mathbb{R}^3)$ . Then*

$$(17) \quad (e^{iH} K e^{-iH})_n S_n(\hat{\alpha}_1\gamma_0 \otimes \dots \otimes \hat{\alpha}_n\gamma_0) \\ = \left( S_n \prod_{j=1}^n \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathbf{x}) U_{|-t, \mathbf{x}|} B_0 U_{|t, -\mathbf{x}|} d\mathbf{x} \right\} \right) \omega,$$



where

$$f_j(t, \mathbf{x}) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-it\mu(\mathbf{p}) + i(\mathbf{x}, \mathbf{p})} \hat{\alpha}_j(\mathbf{p}) d\mathbf{p},$$

i. e.  $f_j(t, \mathbf{x})$  is the spatial inverse Fourier transform of  $e^{-it\mu(\mathbf{p})} \hat{\alpha}_j(\mathbf{p})$ , thus  $f_j(t, \mathbf{x})$  is a so-called smooth solution of the Klein-Gordon equation with « negative » frequencies.

*Proof.* — According to Proposition 4 we have the assignment

$$e^{-it\mu(\mathbf{p})} \hat{\alpha}_j(\mathbf{p}) \gamma_0(\mathbf{p}) \mapsto (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathbf{x}) U_{\mathbf{x}} B_0 U_{-\mathbf{x}} d_{\mathbf{x}}.$$

Furthermore,

$$\begin{aligned} e^{-it \sum_{j=1}^n \mu(\mathbf{p}_j)} \hat{\alpha}_1 \gamma_0 \otimes \dots \otimes \hat{\alpha}_n \gamma_0 \\ = \left( \bigotimes_{j=1}^n e^{-it\mu(\mathbf{p}_j)} \hat{\alpha}_j(\mathbf{p}_j) \gamma_0(\mathbf{p}_j) \right) = (e^{-itH_0})_n (\hat{\alpha}_1 \gamma_0 \otimes \dots \otimes \hat{\alpha}_n \gamma_0) \end{aligned}$$

is valid, thus we obtain

$$\begin{aligned} (\mathbf{K} e^{-itH_0})_n S_n (\hat{\alpha}_1 \gamma_0 \otimes \dots \otimes \hat{\alpha}_n \gamma_0) \\ = \left( S_n \prod_{j=1}^n \left\{ (2\pi)^{-3/2} \int_{\mathbb{R}^3} f_j(t, \mathbf{x}_j) U_{\mathbf{x}_j} B_0 U_{-\mathbf{x}_j} d_{\mathbf{x}_j} \right\} \right) \omega. \end{aligned}$$

Therefore we finally obtain (17) for  $\{e^{itH}(\mathbf{K} e^{-itH_0})\}_n S_n (\hat{\alpha}_1 \gamma_0 \otimes \dots \otimes \hat{\alpha}_n \gamma_0)$  because of  $e^{itH} = U_{\{-t, 0\}}$  and  $U_{\{t, 0\}} \omega = \omega$ . ■

Proposition 6 gives a link between the pre-wave operator  $e^{itH} \mathbf{K} e^{-itH_0}$  and the expressions (Haag-Ruelle approximants) used in the Haag-Ruelle scattering theory for Wightman fields. That is, Haag-Ruelle's theory appears as a special part of the abstract two-space scattering theory (see H. Baumgärtel and M. Wollenberg [4]). The basic concept of the abstract scattering theory is given by the pre-wave operator mentioned above. Note that the identification operator  $\mathbf{K}$  in the field-theoretic case is unbounded and not closable but densely defined whereas the identification operators appearing usually (e. g. in the non-relativistic scattering theory) are bounded.

It should be mentioned that the characteristic  $t$ -dependence of the special field operators  $A(e^{it(p_0 - \mu(\mathbf{p}))} f(p))$  occurring in the Haag-Ruelle approximants (see for example K. Hepp [5, p. 96]) suggests the introduction of a pre-wave operator, i. e. of an operator  $\mathbf{K}$ . In fact, one obtains, in an intimate connection with Proposition 6, that

$$\begin{aligned} (e^{itH} \mathbf{K} e^{-itH_0})_n \{ S_n (h(p) \hat{\alpha}_1(\mathbf{p}) \otimes \dots \otimes h(p) \hat{\alpha}_n(\mathbf{p})) \} \\ = \left\{ S_n \prod_{\rho=1}^n A(e^{it(p_0 - \mu(\mathbf{p}))} h(p) \hat{\alpha}_\rho(\mathbf{p})) \right\} \omega \end{aligned}$$

is valid (see for example [2, Lemma 6]). Proposition 6 shows explicitly that the pre-wave operator, i. e. also  $K$ , depends on a single field operator  $A(h(p)\gamma_0(p))$  only.

Under the assumptions listed at the beginning of paragraph 1 the strong limits for  $t \rightarrow \pm \infty$  of (11) exist and they turn out to be isometric (Haag-Ruelle). Because  $K$  is uniquely determined by  $B_0$  the question arises, what properties of  $B_0$  are decisive for the existence and isometry of the wave operators. Some authors (for example A. S. Schwartz [3]) have shown that the (sufficient) assumptions of Haag-Ruelle can be weakened. It would be nice to have some insights what properties of  $K$  resp.  $B_0$  are necessary for the existence and isometry of the wave operators in order to attack the corresponding inverse problem.

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