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Evolution of shock waves  
in relativistic continuum mechanics  

by  

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ABSTRACT. — The compatibility relations are applied in order to study the evolution of shock waves in relativistic continuum mechanics. General results are presented on the damping of shock waves in relativistic fluids when the shock velocity tends to the light speed.

RÉSUMÉ. — On utilise les relations de compatibilité pour étudier l'évolution des ondes de choc en mécanique relativiste des milieux continus. On présente des résultats généraux sur l'amortissement des ondes de choc dans les fluides relativistes lorsque la vitesse du choc tend vers la vitesse de la lumière.

1. INTRODUCTION

The study of shock waves presents several problems in continuum mechanics. One of these is that, once a shock wave is formed (say by the steepening of an acceleration wave in a solid or of an acoustic wave in a fluid), it is difficult to follow its subsequent evolution short of finding an exact solution to the general dynamical equations. In some particular cases heuristic methods can be used [7] but their mathematical validity is doubtful.

In general, however, the only way of tackling the evolution of a shock

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wave is by resorting to a numerical integration scheme. This situation is to be contrasted with that arising in the case of weak discontinuity waves (as acceleration waves, thermal waves, etc.) where the evolution of the wave can be studied independently of the full solution. In fact, if the state ahead of the wavefront is known, it is possible to separate the evolution of the wave from the solution behind the wavefront. In fact the wave decouples from the field behind the wavefront and its evolution can be determined, once the state ahead of the wavefront has been specified, by the knowledge of the initial data for the wave's amplitude [2] [3] [4].

This is not possible in general in the case of a shock wave because the wavefront is subsonic with respect to the flow behind it and therefore the evolution of the shock wave can be influenced by acoustic signals coming from behind the wavefront.

However, although the general program of separating the wave's evolution from the solution behind the wavefront is not feasible for a shock wave, a limited progress can be achieved by using Thomas' method of iterated discontinuities [5]. A general method can be devised which leads to a sort of transport equation for the shock amplitude which, however, is not sufficient to determine completely its subsequent evolution, but still allows some qualitative results to be obtained.

This method has been applied to various situations in classical continuum mechanics, as in non linear elasticity [6] and viscoelasticity [7], and has been proved to be extremely useful in deriving exact qualitative results on the evolution of shock waves.

Relativistic shock waves can occur in various physical situations. Relativistic blast waves can be produced in laboratory plasmas by strong laser beams [8] [9]. Also relativistic electromagnetic shocks could perhaps propagate in polarizable media [10]. Finally in many areas of astrophysics such as supernovae, extragalactic radio sources, and galaxy formation, relativistic shocks seem to be a basic ingredient [11] [12] [13].

For these reasons it is desirable to extend Thomas' method of iterated discontinuities to relativistic shock waves.

A first attempt in this direction was made in [14] [15] [16] where the case of one-dimensional shock waves in a relativistic fluid was treated in detail. The aim of this paper is to develop Thomas' method of iterated discontinuities for relativistic shock waves of arbitrary geometry.

The plan of the paper is the following.

In Sec. 2 the basic formalism of the relativistic compatibility relations is briefly recalled.

In Sec. 3 the main compatibility equations are derived for relativistic shock waves in an arbitrary continuum, drawing only on the conservation equations.

In Sec. 4 a foliation of space-time consisting of space-like hypersurfaces...
is explicitly introduced and the previous equations are rewritten in terms of data on one of these hypersurfaces.

In Sec. 5 the compatibility equations are specialized to the case of a relativistic fluid.

In Sec. 6 some general qualitative results are obtained and the limiting case of an extremely relativistic shock is investigated.

2. THE COMPATIBILITY RELATIONS

Let $\mathcal{M}$ be a space-time, i.e. a 4-dimensional differentiable manifold, oriented, paracompact, endowed with a Lorentz metric $g$ (of signature $+2$) and time-oriented.

Let $\Sigma$ be an orientable hypersurface of $\mathcal{M}$. It is easy to prove [17] that there exist:

i) a differentiable 1-form $l$ such that $l|_{\Sigma} \neq 0$, which is orthogonal to all tangent vectors of $\Sigma$.

ii) three differentiable submanifolds $\Omega$, $\Omega_+$, $\Omega_-$ of $\mathcal{M}$, $\Omega_+$ and $\Omega_-$ having boundaries, such that:

a) $\Omega = \Omega_+ \cup \Omega_-

b) $\Omega$ is an open neighbourhood of $\Sigma$

c) $\Omega_\pm \cap \Sigma = \partial \Omega_\pm$.

A tensor field $T$ of type $(r, s)$, smooth on $\Omega - \Sigma$, is said regularly discontinuous across $\Sigma$ if there exist two tensor fields $\tilde{T}^+, \tilde{T}^-$, of type $(r, s)$, smooth on $\Omega_+$ and $\Omega_-$ respectively, such that

\[ T_{|\Omega_+ \cap \Sigma} = \tilde{T}^\pm_{|\Omega_\pm \cap \Sigma}. \]

In this case the jump of $T$ across $\Sigma$ is defined by

\[ [T] = \tilde{T}^-_{|\Sigma} - \tilde{T}^+_{|\Sigma}. \]  

(1)

Let $\nabla$ denote the riemannian connection of $\mathcal{M}$ and $T$ be regularly discontinuous across $\Sigma$. Then $\nabla T$ is a tensor field of type $(r, s + 1)$ which is regularly discontinuous across $\Sigma$. In fact $\nabla \tilde{T}^\pm$ are well defined and smooth on $\Omega_\pm \cup \Sigma$ and the jump of $\nabla T$ is then given by

\[ [\nabla T] = \nabla \tilde{T}^-_{|\Sigma} - \nabla \tilde{T}^+_{|\Sigma}. \]

Similar remarks apply to the higher covariant derivatives of $T$.

From the previous definition it is apparent that $[T]$ can be extended (in an obviously non unique way) to a tensor field smooth on a neighbourhood of $\Sigma$.

Let now \( \Sigma \) be a time-like hypersurface and let \( n \) denote its unit normal. In local coordinates \( n \) is given by

\[
 n^\mu = \frac{g^{\mu \nu} l_\nu}{(g^{\beta \gamma} l_\beta l_\gamma)^{1/2}}
\]

where \( l_\mu \) are the components of the 1-form \( l \).

The inner covariant derivative \( \tilde{\nabla} \) is defined on the smooth tensor fields of \( \Omega \) defined on \( \Sigma \) as follows \([18]\), in local coordinates,

\[
\tilde{\nabla}_\mu T^\cdot \cdot = h^\cdot \cdot \cdot \mu \nabla \cdot \cdot \cdot T^\cdot \cdot
\]

where \( h^\cdot \cdot \cdot \mu = \delta^\cdot \cdot \cdot \mu - n_\mu n^\cdot \cdot \cdot \) is the projection tensor onto \( \Sigma \). \( \tilde{\nabla}_\mu T^\cdot \cdot \) is a smooth tensor field of \( \Omega \) defined on \( \Sigma \), which depends only on the restriction of \( T^\cdot \cdot \) to \( \Sigma \).

Except when acting on scalar functions, \( \tilde{\nabla} \) is different from the induced riemannian connection of \( \Sigma \) because, in general, \( \tilde{\nabla}_\mu T^\cdot \cdot \) is not tangent to \( \Sigma \) even when \( T^\cdot \cdot \) is tangent to \( \Sigma \).

The definition of the inner covariant derivative given by (2) is local and holds only for non-null hypersurfaces. A global definition which holds also for null hypersurfaces is given in \([19]\).

The compatibility relations restrict the jump \( [\nabla \cdot \cdot \cdot T^\cdot \cdot] \) of the covariant derivative of a tensor field \( T \) regularly discontinuous across \( \Sigma \) and are given by \([18]\) \([19]\)

\[
 [\nabla \cdot \cdot \cdot T^\cdot \cdot] = \tilde{\nabla}_\mu [T^\cdot \cdot \cdot] + [\nabla \cdot \cdot \cdot T^\cdot \cdot \cdot] \otimes n_\mu.
\]

These relations are the natural extension of the classical Hadamard ones \([5]\) to general relativity. A different approach to the compatibility relations is given in \([20]\) where use is made of the theory of tensor distributions. For the aim of this paper, however, the present approach is adequate.

**3. THE GENERAL COMPATIBILITY EQUATIONS**

In this section general compatibility equations are derived on the basis of the conservation equations alone. Therefore they hold for an arbitrary relativistic continuum. Further compatibility equations can be obtained from the constitutive equations once a particular class of media has been selected.

Let \( T^{\cdot \cdot \cdot \cdot} \) be the energy-momentum tensor and \( j^\cdot \) the mass-flux vector of the continuum. Both \( T^{\cdot \cdot \cdot \cdot} \) and \( j^\cdot \) are assumed to be regularly discontinuous across \( \Sigma \). Then, as a consequence of the conservation laws

\[
\nabla \cdot \cdot \cdot T^{\cdot \cdot \cdot \cdot} = 0
\]

\[
\nabla \cdot \cdot \cdot j^\cdot = 0
\]
the following junction conditions hold across $\Sigma$, \([20]\) \([21]\),

\begin{align}
    n_\mu [T^\mu\nu] &= 0 \\
    n_\mu [j^\mu] &= 0
\end{align}

which state that both $[T^\mu\nu]$ and $[j^\mu]$ are tangent to $\Sigma$.

Let

\[
\tilde{T}^{\mu\nu} = h^{\mu}_\alpha h^{\nu}_\beta T^{\alpha\beta} \\
\tilde{j}^\mu = h^{\mu}_\alpha j^\alpha
\]

then the junction conditions can be rewritten in the form

\[
[T^{\mu\nu}] = [\tilde{T}^{\mu\nu}] \\
[j^\mu] = [\tilde{j}^\mu].
\]

From (4), (5), the compatibility relations (3) and the junction conditions, it follows easily

\begin{align}
    \tilde{\nabla}_\mu [T^{\mu\nu}] - n_\mu [n^\sigma \nabla_\sigma T^{\mu\nu}] &= 0 \\
    \tilde{\nabla}_\mu [j^\mu] + n_\mu [n^\sigma \nabla_\sigma j^\mu] &= 0.
\end{align}

Let $x \in \Sigma$ and $U$ be a coordinate neighbourhood around $x$ in $\mathcal{M}$ such that $\bar{U} = U \cap \Sigma$ is a coordinate neighbourhood of $x$ in $\Sigma$. If $(y^0, y^\nu)$ are local coordinates in $U$, then $y^0 = 0$ is the local equation of $\Sigma$ in $U$ and $(y^\nu)$ are local coordinates in $\bar{U}$.

In the following all considerations will be restricted to $U$ and $\bar{U}$ or subsets thereof.

It is easy to see that in $\bar{U}$ there exists an orthonormal basis field $\{k, e_\alpha\}$, $\alpha = 2, 3$, such that $k$ is time-like and future-directed and $e_\alpha$ is space-like.

A way to construct it is the following. $\{\frac{\partial}{\partial y^\nu}\}$ is a basis field in $\bar{U}$ and $h_\mu^\nu = \delta_\mu^\nu - n_\mu n^\nu$ defines a metric of signature $+1$ on $\bar{U}$. Therefore it is possible to orthonormalize the basis vectors $\frac{\partial}{\partial y^\nu}$ with respect to the metric $h_\mu^\nu$ thereby obtaining the orthonormal basis $\{k, e_\alpha\}$. Hence in $\bar{U}$ one has

\[
k_\mu k^\mu = -1, \quad e_\alpha^\mu e_\alpha_\mu = 1, \quad k_\mu e_\alpha^\mu = 0, \quad e_\alpha^\mu e_\beta^\mu = \delta_{\alpha\beta}.
\]

With respect to the basis $\{k, e_\alpha\}$ one has the following decomposition

\begin{align}
    [j^\mu] &= I^1 k^\mu + I^A e_\alpha^\mu \\
    [\tilde{T}^{\mu\nu}] &= \theta^{11} k^\mu k^\nu + \theta^{1A} (k_\mu e_\alpha^\nu + e_\alpha^\mu k^\nu) + \theta^{AB} e_\alpha^\mu e_\beta^\nu.
\end{align}
A simple substitution of (10) and (11) into (8) and (9) yields
\[ k^\mu \nabla_\mu I^1 + I^1 \nabla_\mu k^\mu + e_A^\mu \nabla_\mu I^A + I^A \nabla_\mu e_A^\mu + n_\mu [n^\alpha \nabla_\alpha k^\mu] = 0 \]
(12)
\[ k^\mu \nabla_\mu \theta^{11} + k^\nu \theta^{11} \nabla_\nu k^\mu + k^\mu \theta^{11} \nabla_\nu k^\nu + (k^\nu e_A^\nu + e_A^\mu k^\mu) \nabla_\nu \theta^{1A} + \theta^{1A} (e_A^\nu \nabla_\nu k^\mu + k^\nu \nabla_\nu e_A^\mu + k^\nu \nabla_\nu k^\nu) + e_A^\mu e_B^\nu \nabla_\nu \theta^{AB} + e_B^\nu \theta^{AB} \nabla_\nu e_A^\mu + \theta^{AB} e_A^\mu \nabla_\nu e_B^\nu + n_\mu [n^\alpha \nabla_\alpha T^{\mu \nu}] = 0. \]
(13)

At this stage it is convenient to introduce the induced riemannian connection of \( \nabla \), \( \nabla \), defined on smooth tensor fields \( T^{\nu \gamma} \) tangent to \( \Sigma \) by
\[ \nabla_\mu T^{\nu \gamma} = h_\mu^\sigma h_\nu^\rho h_\gamma^\sigma \nabla T^{\nu \gamma} \]
(14)

Obviously, on scalars, \( \nabla \) coincides with \( \nabla \).
Also, from
\[ \nabla_\mu k^\nu = h_\mu^\sigma h_\nu^\rho V_\sigma k^\rho \]
(15)
and analogous results hold for \( \nabla_\mu e_A^\mu \).

The second fundamental form of \( \Sigma \) is defined by
\[ \chi_{\mu \nu} = h_\mu^\sigma h_\nu^\rho V_\sigma \eta_\rho \]
(16)
and, for any couple of vector fields \( \nu^\mu, z^\mu \), tangent to \( \Sigma \), one has
\[ \nu^\mu \nabla_\mu z^\nu = \nu^\mu \nabla_\mu z^\nu - n^\nu \chi_{\mu \nu} \nu^\nu. \]
(17)

Eq. (12) can be rewritten in the form
\[ k^\mu \nabla_\mu I^1 + I^1 \nabla_\mu k^\mu + e_A^\mu \nabla_\mu I^A + I^A \nabla_\mu e_A^\mu + n_\mu [n^\alpha \nabla_\alpha k^\mu] = 0 \]
(18)

After some manipulations eq. (13), contracting with \( n_\nu \), yields
\[ \theta^{11} \chi_{\nu \beta} k^\nu k^\beta + 2 \theta^{1A} \chi_{\nu \beta} e_A^\beta + \theta^{AB} \chi_{\nu \beta} e_A^\beta - n_\mu [n^\alpha \nabla_\alpha T^{\mu \nu}] = 0. \]
(19)

From (18) it follows that, if \( x \) is a flat point of \( \Sigma \) (i.e. \( \chi_{\mu \nu}(x) = 0 \)), then
\[ [n^\alpha \nabla_\alpha (n_\mu T^{\alpha \nu})]_x = 0. \]

By further contracting eq. (13) with \( k_\nu \) and \( e_C \), respectively one obtains
\[ k^\mu \nabla_\mu \theta^{11} + \theta^{11} \nabla_\mu k^\mu + e_A^\mu \nabla_\mu \theta^{1A} + \theta^{1A} k^\mu e_A^\mu + \theta^{1A} \nabla_\mu e_A^\mu + \theta^{AB} e_A^\mu e_B^\nu \nabla_\nu k^\mu + k^\mu \nabla_\mu \eta = 0 \]
(20)
\[ \theta^{11} e_C k^\nu k^\nu + k^\mu \nabla_\mu (e_C^A k^\mu e_A^\nu + e_C^A e_A^\nu k^\nu) + e_A^\mu \nabla_\mu \theta^{AC} + \theta^{AC} \nabla_\mu e_A^\mu + \theta^{AB} e_C e_A^\mu \nabla_\mu e_B^\nu + e_C e_A^\nu \nabla_\mu k^\nu + e_C \eta = 0. \]

It is interesting to notice that in eqs. (18), (19), (20), the operators
\[ k^\mu \nabla_\mu + \nabla_\mu k^\mu \]
\[ e_A^\mu \nabla_\mu + \nabla_\mu e_A^\mu \]
appear which are suggestive of conservation laws along the lines tangent
to $k^\mu$ and $e_\lambda^\mu$ respectively. The extra terms appearing in these equations are then suggestive of diffusive terms while the jumps of the derivatives would represent the contribution from the flow behind the wavefront.

In eqs. (17-20) the normal jumps of $\nabla_\nu T^{\mu\nu}$ and $\nabla_\nu j^\mu$ appear, which, in general, are unknown and therefore these equations cannot be called «transport equations» in a proper sense. However they would become transport equations after some knowledge of $[n^\mu \nabla_\nu T^{\mu\nu}]$ and $[n^\mu \nabla_\nu j^\nu]$ has been obtained.

4. THE INITIAL DATA

Let the open subset $U$ be endowed with a foliation of space-like hypersurfaces $\mathcal{F}_t$, $t \in [a, b] \subset \mathbb{R}$ and let $N$ be the future-directed time-like vector field of unit normals to $\mathcal{F}_t$. The existence of such a foliation is entailed, for instance, by global hyperbolicity [22].

Denote by $H$ the projection tensor onto $\mathcal{F}_t$, given in local coordinates by

$$H_{\alpha\nu} = \delta_{\alpha\nu} + N^\alpha N_\nu.$$  

The propagation speed $V_\Sigma$ of the hypersurface $\Sigma$ with respect to the family of observers identified with the time-like vector field $N$ is given by

$$(21) \quad \Gamma_\Sigma^2 = \frac{1}{1 - V_\Sigma^2} = 1 + (n_\mu N^\mu)^2$$

where $\Gamma_\Sigma$ is the Lorentz factor of $\Sigma$.

Here the condition

$$n_\mu N^\mu > 0$$

will be assumed, in order to have progressive waves ($V_\Sigma > 0$).

By suitably restricting $U$ it is possible to characterize $\mathcal{F}_t$ by a smooth function $\psi(y^\mu)$, such that

$$\mathcal{F}_t : \psi(y^\mu) = t.$$  

On $\bar{U}$ then one has

$$\psi(0, y^i) = t$$

which represents a 2-surface $\sigma_t$.

On $\bar{U}$ one has a foliation by the 2-surfaces $\sigma_t$.

Let $k$ be the unit normal vector to $\sigma_t$ in $U$,

$$k_\mu = \frac{h_{\mu\nu} N^\nu}{|h_{\nu\rho} N^\nu N^\rho|^{1/2}}, \quad k_\mu k^\mu = -1,$$

and $e_\lambda$ be two orthogonal unit vectors tangent to $\sigma_t$ in $\bar{U}$. Then $\{ k, e_\lambda \}$ is a basis field in $\bar{U}$. Explicitly one has

$$(22) \quad k^\mu = \frac{1}{\Gamma_\Sigma} (N^\mu - \Gamma_\Sigma V_\Sigma n^\mu).$$

In the sequel, instead of using the anholonomic frame \( \{ k, e_A \} \), one could as well use the holonomic basis \( \frac{\partial}{\partial t}, \frac{\partial}{\partial z^\lambda} \) where \( z^\lambda \) are two coordinates on \( \sigma_r \).

Let one of the hypersurfaces \( \mathcal{F}_r \), say \( \mathcal{F}_0 \), be considered as the initial hypersurface, i.e. the data for \( T^{\mu\nu} \) and \( f^\mu \) are given on \( \mathcal{F}_0 \).

On \( \mathcal{F}_0 \) one can define an orthonormal basis field as follows.

In \( U \) \( n \) is a vector field (in fact \( dy^0 \) is a 1-form in \( U \)) and therefore \( n \) is defined on \( \mathcal{F}_0 \). One can then construct the basis field on \( \mathcal{F}_0 \).

\[
\{ b, e_A \}
\]

where

\[
b^\mu = \frac{H^\nu n^\nu}{|H_{20} n^0 n^\beta|^\frac{1}{2}}
\]

and \( e_A \) are two orthogonal unit vectors such that

\[
e_A \cdot b = 0 \quad \text{and} \quad e_A = e_A \quad \text{on} \quad \sigma_0.
\]

Explicitely

\[
b^\mu = \frac{1}{\Gamma_\Sigma} (n^\mu + v_\Sigma \Gamma_\Sigma N^\mu).
\]

Notice that \( b^\mu \), as a vector field, is defined on \( U \).

Now let \( f \) be any quantity which is regularly discontinuous across \( \Sigma \). Then \( f \) has a jump \( [f]_0 \) across \( \sigma_0 \), which can be considered as an initial datum for \( [f] \). Also, if \( f_0 \) is the initial datum for \( f \) on \( \mathcal{F}_0 \), one has, across \( \sigma_0 \)

\[
[f]_0 = [f_0].
\]

Moreover \( [V_{\xi}f]_0 \) can be computed from \( f_0 \), being equal to \( [V_{\xi}f_0] \).

Therefore it is convenient to express \( [V_{\xi}f]_0 \) (which appears in the transport equations of the previous section) in terms of \( [V_{\xi}f]_0 \) (which can be computed from the initial datum).

This can be done by expressing \( n \) in terms of \( b \) and \( k \), as follows

\[
n^\mu = \frac{1}{\Gamma_\Sigma} (b^\mu - \Gamma_\Sigma V_\Sigma k^\mu).
\]

Then one obtains

\[
[V_{\xi}f] = \frac{1}{\Gamma_\Sigma} [V_{\xi}f] - V_\xi [V_{\xi}f],
\]

hence

\[
[V_{\xi}f]_0 = \frac{1}{\Gamma_\Sigma} [V_{\xi}f_0] - V_\xi V_{\xi} [f_0].
\]
After some manipulations eqs. (17)-(20) rewrite, on \( \sigma_0 \),

\[
(26) \quad k^\mu \vec{\nabla}_\mu I^1 + e_A^\mu \vec{\nabla}^A + 1(\vec{\nabla}_\mu k^\mu + V_\Sigma k^\beta k^\gamma) \\
+ I^A(\vec{\nabla}_\mu e_A^\mu + V_\Sigma k^\gamma e_A^\gamma) + \frac{1}{\Gamma} \eta_\mu [b^\alpha \nabla_{\alpha} T^\mu_{\nu}] = 0
\]

\[
(27) \quad \theta^{11} k^\mu \phi A + 2\theta^{11} k^\mu \phi A + \phi^{11} k^\mu \phi A + \phi^{11} k^\mu \phi A - \frac{1}{\Gamma} \eta_\mu \eta_\nu [b^\alpha \nabla_{\alpha} T^\mu_{\nu}] = 0
\]

\[
(28) \quad k^\mu \vec{\nabla}_\mu \phi A + \phi^{11} k^\mu \phi A + e_A^\mu \vec{\nabla}^A + \phi^{11} k^\mu \phi A \\
+ \frac{1}{\Gamma} \eta_\mu \eta_\nu [b^\alpha \nabla_{\alpha} T^\mu_{\nu}] = 0
\]

\[
(29) \quad k^\mu \vec{\nabla}_\mu \phi^{11} k^\mu + e_A^\mu \vec{\nabla}^A + \phi^{11} k^\mu \phi A + \phi^{11} k^\mu \phi A \\
+ \frac{1}{\Gamma} \eta_\mu \eta_\nu [b^\alpha \nabla_{\alpha} T^\mu_{\nu}] = 0
\]

Obviously these equations hold also on any \( \sigma_t \) considered as an initial surface.

It is convenient to view \( \sigma_0 \) as a 2-surface of the pseudo-riemannian manifold \( \mathcal{F}_0 \) and introduce the induced metric on \( \sigma_0 \),

\[
(30) \quad S^\mu _\nu = h^\mu _\nu + k^\mu k_\nu = H^\mu _\nu - b^\mu b_\nu
\]

and its second fundamental form

\[
(31) \quad A_{\mu \nu} = S^\rho _\mu S^\beta _\nu \nabla_\beta b_\beta .
\]

Let \( B_{\mu \nu} \) be the second fundamental form of \( \mathcal{F}_0 \),

\[
(32) \quad B_{\mu \nu} = H^\mu _\nu \beta \phi \nabla_\beta N_\beta ,
\]

then it is easy to prove that

\[
(33) \quad \vec{\nabla}_\mu k^\mu = - \Gamma_M \nabla_\Sigma A^\mu _\mu + \Gamma_M B^\mu _\mu - \Gamma_M b^\alpha b_\mu \nabla_\alpha N_\mu .
\]

In fact

\[
\vec{\nabla}_\mu k^\mu = h^\mu \phi \nabla_\Sigma N_\mu = h^\mu \phi \{ N^\mu \nabla_\Sigma \Gamma_M + \Gamma_M N^\mu - b^\mu \phi (\nabla_\Sigma \Gamma_M) - \Gamma_M \nabla_\Sigma \nabla_\mu \}
\]

Also

\[
\begin{align*}
\eta_\mu N^\mu &= \Gamma_M k^\mu, \\
h^\mu \phi \nabla_\mu b^\mu &= \Gamma_M \nabla_\Sigma k^\mu \\
h^\mu \phi \nabla_\mu b^\mu &= \frac{1}{\Gamma_M} \nabla_\Sigma k^\mu N^\mu \\
h^\mu \phi \nabla_\mu b^\mu &= \frac{1}{\Gamma_M} \nabla_\Sigma k^\mu N^\mu \\
h^\mu \phi \nabla_\mu b^\mu &= \frac{1}{\Gamma_M} \nabla_\Sigma k^\mu N^\mu \\
\end{align*}
\]

whence eq. (33) follows.
With some simple manipulations it is easy to prove the following formulae:

\[
\chi_{ab} k^2 k^d = \Gamma_a^d k^a \nabla_a \Sigma + \frac{1}{V_\Sigma} k^a \nabla_a N_v \\
\chi_{ab} k^2 e_A^d = \frac{1}{V_\Sigma} e_A k^a \nabla_a N_v - \frac{1}{V_\Sigma} e_A k^a \nabla_a k^d
\]

where \( V \) denotes the induced riemannian connection on \( \sigma_0 \).

Therefore eqs. (26-29) can be rewritten in the following form

\[
D1 + e_A \nabla_A I^A + E1 + I^A F_A + \frac{1}{V_\Sigma} n_\mu [b^\tau \nabla_\Sigma b^\tau] = 0
\]

\[
\theta^{11} \left( \Gamma^2 \nabla_\Sigma + \frac{1}{V_\Sigma} k^\tau \nabla_\Sigma \right) + 2 \theta^{1A} \left( \frac{1}{V_\Sigma} e_A \nabla_\Sigma \nabla^\tau - \frac{1}{V_\Sigma} e_A \nabla_\Sigma k^\tau \right) + \theta^{AB} \left( \frac{1}{V_\Sigma} e_A \nabla_\Sigma k^\tau - \frac{1}{V_\Sigma} e_A \nabla_\Sigma \nabla_\Sigma k^\tau \right) = 0
\]

\[
D\theta^{11} + \theta^{11} E + e_A \nabla_\Sigma \theta^{1A} - \theta^{1A} (F_A - e_A \nabla_\Sigma k^\tau)
\]

\[
\theta^{1C} + \theta^{11} e_{AC} \nabla_\Sigma k^\tau + \theta^{1E} + \theta^{1A} (e_{AC} e_A \nabla_\Sigma k^\tau + e_{AC} \nabla_\Sigma k^\tau)
\]

where \( D \equiv k^\mu \nabla_\mu \) is the derivative along \( k \), \( \overline{D} \equiv k^\mu \overline{\nabla}_\mu \) (on scalars \( D \) and \( \overline{D} \) coincide).

\[
E = - \Gamma_\Sigma \Sigma A^\mu + \Gamma_\Sigma B^\mu - \Gamma_\Sigma b^\tau \nabla_\Sigma \nabla_\Sigma + V_\Sigma \nabla_\Sigma \Sigma + \frac{1}{V_\Sigma} k^\tau \Sigma \nabla_\Sigma \Sigma
\]

\[
F_A = \nabla_\Sigma e_A^\mu + \frac{1}{V_\Sigma} e_A \nabla_\Sigma \Sigma
\]

Obviously these equations too hold on any \( \sigma_i \) considered as an initial surface.

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5. THE RELATIVISTIC FLUID

In this section the equations derived in the previous section are specialized to the important case of the relativistic perfect fluid. The energy-momentum tensor is [21]

\[ T^{\mu\nu} = w u^{\mu} u^{\nu} + p g^{\mu\nu} \]

where \( u^{\mu} \) is the fluid 4-velocity, \( w \) and \( p \) are the enthalpy and pressure respectively, both measured in the local rest frame.

The mass-density current is

\[ j^{\mu} = \rho u^{\mu} \]

where \( \rho \) is the proper rest-mass density.

It is convenient to introduce the index \( f \) of the fluid, defined by [18]

\[ f = \frac{w}{\rho} \]

The Rankine-Hugoniot relations express the invariance across \( \Sigma \) of the scalar [18]

\[ m = \rho u^{\mu} n_{\mu} \]

and of the vector

\[ Z^{\mu} = m f u^{\mu} + p n^{\mu} \]

Let \( \tau = \frac{f}{\rho} \) be the dynamical volume [20]. Then it is easy to show that [20]

\[ m^{2}[T^{\mu\nu}] = \left[ \frac{1}{\tau} \right] Z^{\mu} Z^{\nu} - \left[ \frac{p}{\tau} \right] (Z^{\mu} n^{\nu} + n^{\mu} Z^{\nu}) + \left[ \frac{p^{2}}{\tau} \right] n^{\mu} n^{\nu} + m^{2}[p] g^{\mu\nu} \]

\[ m[j^{\mu}] = \left[ \frac{1}{\tau} \right] Z^{\mu} - \left[ \frac{p}{\tau} \right] n^{\mu}. \]

For the sake of simplicity let the fluid ahead of the shock front be at rest with respect to the family of observers defined by \( N \). Then one has, on \( \Sigma \),

\[ u^{\mu}_{+} = N^{\mu} \]

and \( u^{\mu} \) coincides with \( N^{\mu} \) on \( U^{+} = \Omega^{+} \cap \Sigma \).

It follows easily that

\[ Z^{\mu} = m f_{+} N^{\mu} + p_{+} n^{\mu} \]

whence

\[ (52) \quad \Gamma^{1} = - \frac{1}{\tau} f_{+}, \quad \Gamma^{A} = 0 \]

\[ (53) \quad \theta^{11} = \left[ \frac{1}{\tau} \right] f^{2}_{+} \Gamma^{2}_{+} - [p], \quad \theta^{AB} = [p] \delta^{AB}, \quad \theta^{1A} = 0. \]
Therefore eqs. (40-43) can be rewritten as follows

\begin{equation}
D \left( \frac{\Gamma_{\Sigma}}{\tau} f_{+} \left[ \frac{1}{\tau} \right] \right) + E \frac{\Gamma_{\Sigma}}{\tau} f_{+} \left[ \frac{1}{\tau} \right] - \frac{1}{\Gamma_{\Sigma}} n_{\mu} [b^{\mu} \nabla_{\alpha} m] = 0
\end{equation}

\begin{equation}
\left( \frac{1}{\tau} \right) f_{+} 2 \frac{\Gamma_{\Sigma}^{2}}{\tau} - [p] \left( \Gamma_{\Sigma}^{2} D V_{\Sigma} + \frac{1}{V_{\Sigma}} k^{\nu} D N_{\nu} \right)
+ [p] \left( \frac{1}{\Gamma_{\Sigma} V_{\Sigma}} S^{\beta} V_{\Sigma} N_{\beta} - \frac{1}{V_{\Sigma}} \nabla_{\mu} k^{\mu} \right) - \frac{1}{\Gamma_{\Sigma}} n_{\mu} n_{\nu} [b^{\mu} \nabla_{\nu} T^{\mu \nu}] = 0
\end{equation}

\begin{equation}
D \left( \frac{1}{\tau} \right) f_{+} 2 \frac{\Gamma_{\Sigma}^{2}}{\tau} - D [p] + E \frac{1}{\tau} f_{+} 2 \frac{\Gamma_{\Sigma}^{2}}{\tau}
- [p] \left( V_{\Sigma} \frac{\Gamma_{\Sigma}^{2}}{\tau} D V_{\Sigma} + \frac{1}{\Gamma_{\Sigma}} k^{\nu} D N_{\nu} \right) - \frac{1}{\Gamma_{\Sigma}} k_{\lambda} n_{\mu} [b^{\lambda} \nabla_{\nu} T^{\mu \nu}] = 0
\end{equation}

\begin{equation}
\left( \frac{1}{\tau} \right) f_{+} 2 \frac{\Gamma_{\Sigma}^{2}}{\tau} - [p] e_{C} \frac{\partial D}{\partial \tau} + e_{C} \frac{\partial}{\partial \tau} \nabla_{\nu} [p]
+ \frac{1}{\Gamma_{\Sigma}} [p] e_{C} \nabla_{\nu} \mu + \frac{1}{\Gamma_{\Sigma}} e_{C} n_{\mu} [b^{\mu} \nabla_{\nu} T^{\mu \nu}] = 0
\end{equation}

Also, it is easy to see that

\begin{equation}
n_{\mu} [b^{\mu} \nabla_{\alpha} p] = \Gamma_{\Sigma} V_{\Sigma} (b^{\mu} \nabla_{\alpha} p)_{+} - \rho_{+} \Gamma_{\Sigma} V_{\Sigma} (b^{\mu} \nabla_{\alpha} p)_{-}
+ \rho_{+} n_{\mu} b^{\nu} \nabla_{\nu} N_{-} - \rho_{-} n_{\mu} (b^{\nu} \nabla_{\nu} u^{\mu})_{-}
\end{equation}

\begin{equation}
n_{\mu} n_{\nu} [b^{\mu} \nabla_{\alpha} T^{\mu \nu}] = \Gamma_{\Sigma}^{2} V_{\Sigma} (b^{\mu} \nabla_{\alpha} w)_{+} - \left( \frac{\rho_{+}}{\rho_{-}} \Gamma_{\Sigma} V_{\Sigma} \right)^{2} (b^{\nu} \nabla_{\alpha} w)_{-}
+ w_{+} \Gamma_{\Sigma} V_{\Sigma} n_{\mu} b^{\nu} \nabla_{\nu} N_{-} - 2 w_{-} \frac{\rho_{+}}{\rho_{-}} \Gamma_{\Sigma} V_{\Sigma} n_{\mu} (b^{\nu} \nabla_{\nu} u^{\mu})_{-}
+ w_{+} \Gamma_{\Sigma} V_{\Sigma} n_{\mu} b^{\nu} \nabla_{\nu} N_{+} + (b^{\nu} \nabla_{\nu} P)
\end{equation}

\begin{equation}
k_{\mu} n_{\nu} [b^{\mu} \nabla_{\alpha} T^{\mu \nu}] = - \Gamma_{\Sigma}^{2} V_{\Sigma} (b^{\mu} \nabla_{\alpha} w)_{+} + \frac{\rho_{+}}{\rho_{-}} \Gamma_{\Sigma}^{2} V_{\Sigma} f_{+} (b^{\nu} \nabla_{\nu} w)
- w_{+} \Gamma_{\Sigma} n_{\mu} b^{\nu} \nabla_{\nu} N_{-} + \frac{w_{-} \rho_{+}}{\rho_{-}} \Gamma_{\Sigma} n_{\mu} (b^{\nu} \nabla_{\nu} u^{\mu})_{-}
+ w_{+} \Gamma_{\Sigma} V_{\Sigma} k_{\lambda} b^{\nu} \nabla_{\nu} N_{+}
- \frac{w_{-} \rho_{+}}{\rho_{-}} \Gamma_{\Sigma} V_{\Sigma} k_{\lambda} (b^{\nu} \nabla_{\nu} u^{\mu})_{-}
\end{equation}

\begin{equation}
e_{C} n_{\mu} [b^{\mu} \nabla_{\alpha} T^{\mu \nu}] = w_{+} \Gamma_{\Sigma} V_{\Sigma} e_{C} (b^{\nu} \nabla_{\nu} N_{-}) - \frac{w_{-} \rho_{+}}{\rho_{-}} \Gamma_{\Sigma} V_{\Sigma} e_{C} (b^{\nu} \nabla_{\nu} u^{\mu})_{-}
\end{equation}

A better understanding of these equations is obtained in the case of special relativity.

Let \( M \) be the Minkowski space-time, \((t, x^{\mu})\) global inertial cartesian coordinates, where

\[ g_{\mu \nu} = \eta_{\mu \nu} = \text{diag} (-1, 1, 1, 1) \]

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The hypersurfaces \( \mathcal{F}_t \) can be taken to be

\[ t = \text{cost}. \]

Then
\[ N^\mu = (1, 0, 0, 0), \]
\[ n_\mu = \Gamma_\Sigma V_\Sigma - v \]
where \( v \) is a unit 3-vector,
\[ k^\mu = \Gamma_\Sigma (1, V_\Sigma v) \]
\[ b^\mu = (0, -v) \]
\[ e^\mu_\Lambda = (0, i_\Lambda) \]
where \( i_\Lambda \) are two unit 3-vectors orthogonal to \( v \),
\[ i_\Lambda \cdot v = 0. \]

The fluid 4-velocity is in the inertial frame,
\[ u^\mu = \Gamma (1, v) \]
where \( v \) is the velocity 3-vector and \( \Gamma \) is the Lorentz factor.

From the invariance of \( Z^\mu = mf u^\mu + pn^\mu \), it follows that
\[ (V)_- = V_- u. \]

Let \( \frac{\delta}{\delta s} = \frac{1}{\Gamma_\Sigma} \frac{\partial}{\partial \tau} \) (which on scalars coincides with \( \frac{1}{\Gamma_\Sigma} \frac{\partial}{\partial \tau} \)), then eqs. (54-57) yield, with \( \partial_+ = \nu^\mu \partial_\mu \)

\[ \frac{\delta}{\delta s} \left( f_+ \left[ \frac{1}{\tau} \right] \right) + 2 \Gamma_\Sigma^2 V_\Sigma f_+ \left[ \frac{1}{\tau} \right] \frac{\delta V_\Sigma}{\delta s} - V_\Sigma f_+ \left[ \frac{1}{\tau} \right] A^\mu_\mu + \frac{V_\Sigma}{\Gamma_\Sigma^2} (\partial_+ \rho)_+ \]
\[ - \rho_+ \frac{V_\Sigma}{\Gamma_\Sigma^2} (\partial_+ \rho)_+ - \rho_+ \frac{\Gamma^3}{\Gamma_\Sigma^2} (V_- V_\Sigma - 1) v \cdot (\partial_+ V)_- = 0 \]

(62)

(63)

\[ \left( \frac{1}{\tau} \right) f_+^2 \Gamma_\Sigma^2 - [p] \right) \frac{\delta V_\Sigma}{\delta s} + [p] \Gamma_\Sigma A^\mu_\mu + \Gamma_\Sigma V_\Sigma^2 (\partial_+ w)_+ \]
\[ - \frac{\rho^2_+}{\rho_-} \Gamma_\Sigma V_\Sigma (\partial_+ w)_+ + \frac{1}{\Gamma_\Sigma} [\partial_+ (\partial_+ + 2 w)] - \rho_+ \Gamma_\Sigma V_\Sigma (V_\Sigma V_\Sigma V_- - 1) v \cdot (\partial_+ V)_- = 0 \]

(64)

(65)

These equations look rather formidable. However some qualitative information can be obtained from them, as will be seen in the next section.
6. QUALITATIVE RESULTS AND CONCLUSION

From eq. (65) one can obtain at once the following important result. Let $\{p\}$ be uniform all over the shock wavefront $\sigma$, 

$$ e_{c}^{\mu} \nabla_{\mu} \{p\} = 0 $$

and $(\partial_{x} V)_{-}$ be normal to the wavefront, 

$$ i_{c} \cdot (\partial_{x} V)_{-} = 0, $$

then it follows

$$ e_{c}^{\mu} \frac{\delta k^{\tau}}{\delta s} = 0. $$

Now, since $\frac{\delta k^{\tau}}{\delta s}$ lies in the two-plane spanned by $e_{2\tau}, e_{3\tau}$, it follows

$$ \frac{\delta k^{\tau}}{\delta s} = 0 $$

which states that in this case $k^{\mu}$ is a geodesic vector field on $\Sigma$ with respect to the induced connection, and $s$ is an affine parameter on these geodesics.

One might enquire under which conditions $k^{\mu}$ is a geodesic vector field in the full space-time $\mathcal{M}$. The answer is that, beyond (66), $k^{\mu}$ must satisfy

$$ \chi_{i\rho} k^{\tau} k^{\rho} = 0 $$

which, from eq. (34), implies

$$ \frac{\delta V_{x}}{\delta s} = 0, $$

which means a constant amplitude shock.

In the general case one obtains from the condition $e_{c}^{\tau} \frac{\delta k^{\tau}}{\delta s} = 0$, being $k^{\mu} = \Gamma_{2}(1, V_{2\nu})$,

$$ \frac{\delta V_{x}}{\delta s} = 0 $$

which states that the shockfront 2-surfaces $\sigma_{t}$ are parallel surfaces.

This result is analogous to a well known theorem on wave propagation in classical continuum mechanics [7].

Another important result can be obtained from eqs. (63-64). Consider the case of a radiation fluid, for which the equation of state is

$$ w = 4p. $$

A simple analysis of the Rankine-Hugoniot relations (48), (49), shows
that, in the limit of an extremely relativistic shock, \( V_\Sigma \to 1 \), one has the asymptotic relations, to the order \( \Gamma_\Sigma^{-1} \),

\[
p_{-} = \frac{2}{3} \Gamma_\Sigma^2 w_+ , \quad \rho_{-} = 2\sqrt{2} \Gamma_\Sigma \rho_+ ,
\]

\[
f_{-} = \frac{4}{3\sqrt{2}} \Gamma_\Sigma f_+ , \quad \Gamma^2 = \frac{1}{2} \Gamma^2 .
\]

It follows that \( \frac{1}{\tau} \) remains finite in the limit \( V_\Sigma \to 1 \), and in particular one has

\[
\frac{1}{\tau} = -\frac{\rho_+}{f_+} \left( 2 + \frac{9\sqrt{2}}{4 \Gamma_\Sigma} \right).
\]

Therefore, asymptotically, one has,

\[
\frac{\delta}{\delta s} \left[ \frac{1}{\tau} \right] = -2 \frac{\delta}{\delta s} \left( \frac{\rho_+}{f_+} \right) + \frac{9\sqrt{2}}{4} \frac{\rho_+}{f_+} \Gamma_\Sigma \frac{\delta V_\Sigma}{\delta s}
\]

to the order \( \Gamma_\Sigma^{-1} \).

Hence from eq. (64) it follows, asymptotically,

\[
(68) \quad \frac{16}{3} \rho_+ f_+ \frac{\delta V_\Sigma}{\delta s} = \frac{3}{8 \Gamma_\Sigma^6} (\partial_\Sigma w)_- - \frac{5}{3} \frac{\rho_+ f_+}{\Gamma_\Sigma^2} \frac{\delta}{\delta s} \cdot (\partial_\Sigma V)_- .
\]

Similarly, from eq. (63) one obtains, asymptotically,

\[
(69) \quad 2 \rho_+ f_+ \frac{\delta V_\Sigma}{\delta s} = \frac{1}{8 \Gamma_\Sigma^6} (\partial_\Sigma w)_- + \frac{\rho_+ f_+}{\Gamma_\Sigma^2} \frac{\delta}{\delta s} \cdot (\partial_\Sigma V)_- .
\]

Now one can assume that \( \gamma \cdot (\partial_\Sigma V)_- \) remains finite in the limit \( V_\Sigma \to 1 \). Then, from eqs. (68)-(69) it is easy to show that, in the limit \( V_\Sigma \to 1 \),

\[
(70) \quad \frac{\delta V_\Sigma}{\delta s} = 0(\Gamma_\Sigma^{-2}) .
\]

This result which is at striking variance with the behaviour of shock waves in classical fluiddynamics had already been conjectured by Liang and Baker [23]. The first rigorous proof was given by Anile, Miller and Motta [14] in the case of a plane shock wave propagating into a constant state in a relativistic barotropic fluid and confirmed in more general cases by numerical calculations [13].

In this section the asymptotic behaviour (70) has been proved for a shock wave of arbitrary geometry propagating into an arbitrary state in a radiation fluid. By using a similar method it should be possible to prove the asymptotic behaviour (70) without any restriction on the equation of state.
In this section the first order compatibility relations (62-65) have been investigated and some qualitative results have been obtained. Further work could be done on the following two lines. The first is to envisage an approximation method for weak shocks which would close the system (62-65) thereby obtaining a system of propagation equations. The second is to postulate special relationships among \( \left( \partial_\tau \rho \right)_- \), \( \nabla \cdot \left( \partial_\tau \mathbf{V} \right)_- \) and \( \left( \partial_\tau \rho \right)_- \), drawn from the requirements of self-similarity, which would also permit to obtain proper propagation equations for the shock amplitude.

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