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Self-adjointness of Lattice Yang-Mills Hamiltonians and Kato's Inequality with Indefinite Metric

by

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ABSTRACT. — The Hamiltonian for a Yang-Mills quantum field theory in Feynman gauge on a periodic lattice is shown to be essentially self-adjoint in the sense of Krein. The Hamiltonian also satisfies a Kato inequality.

RÉSUMÉ. — On démontre que, dans le cadre de la théorie quantique des champs l'hamiltonien de Yang-Mills sur un réseau périodique est essentiellement auto-adjoint au sens de Krein. L'hamiltonien satisfait aussi une inégalité de Kato.

1. INTRODUCTION

There are several proposals for starting a mathematically rigorous construction of a continuum non-abelian quantum gauge theory. The lattice version of the Euclidean (imaginary time) framework put forward by Wilson [*10*] has received most attention as it maintains exact gauge invariance throughout the regularization and may be studied by the methods of statistical mechanics. Ultra-violet stability, established by Balaban [*2*] in three (space-time) dimensions for this regularization, indicates how Wilson's form for the counter-terms in the Yang-Mills theory [*11*] leads to boundedness below of the Hamiltonian in the physical sector. The proposal in [3] employs a fixed gauge and thus requires construction of non-gauge invariant sectors of the theory as well as the physical one. In this way, one can learn how physical observables depend upon the non-physical local fields used in the framing of dynamical quantum field theories. For a fixed gauge, however, the Hamiltonian is no longer bounded below as an operator in the entire Hilbert space so the issue of boundedness below must be examined. This is undertaken in this paper for a fixed lattice cut-off and requires studying self-adjointness with respect to an indefinite (Krein) metric for an unbounded linear operator. This is of independent mathematical interest even though the context is somewhat special.

Throughout this paper we shall use the formalism of quantum field theory in which s indicates the number of space dimensions and V a finite s-dimensional periodic lattice with volume $|\mathrm{V}|$. Points $x \in \mathrm{V}$ have the form $x = n\delta$, $\delta > 0$; n an integer s-tuple. The cyclic group dual to V is denoted by Γ in the Fourier transform

$$f_\mu(x) = |\mathrm{V}|^{-\frac{1}{2}} \Sigma_{k\in\Gamma_0} \hat{f}_\mu(k) e^{ikx},$$

with $k\cdot x = \Sigma_{j=1}^{s} (\pi n_j m_j \delta / \mathrm{L}_j)$; $\mu = 0, 1, \ldots, s$; while

$$\mathrm{L}^2(\Gamma_0) \cong \mathrm{L}_0^2(\mathrm{V}) = \{ f \in \mathrm{L}^2(\mathrm{V}) \mid \Sigma_{x\in 2\,\mathrm{period}}\ f_\mu(x) = 0 \}.$$

The time-zero lattice gauge fields $\mathrm{A}_\mu^{(a)}(x)$; $a = 1, 2, \ldots, \mathrm{N}$; satisfy the conventions in [3] with minor modification. The fields are realized on a Fock space (infinite symmetric tensor product space over $\mathrm{L}_0^2(\mathrm{V})$) denoted by $\mathscr{H}$ appropriate for an irreducible cyclic representation of the canonical commutation relations

$$[\mathrm{A}_\mu^{(a)}(x), \pi_\nu^{(b)}(y)] = i\delta_{ab} g_{\mu\nu} \delta(x-y) \tag{1.1}$$

in the Feynman gauge (Gupta-Bleuler). The Hilbert space $\mathscr{H}$ is a Krein space [1] with respect to the Gupta-Bleuler indefinite metric η:

$$\{ \Phi, \Psi \} = (\Phi, \eta\Psi) \qquad \eta^2 = 1, \qquad \eta^* = \eta \tag{1.2}$$

For this lattice regularization, the Yang-Mills Hamiltonian [3, 11] is a densely defined, closable Krein (+) symmetric operator:

$$\mathrm{H} + \mathrm{H}_0 + \mathrm{V}_{\text{magnetic}} + \mathrm{V}_{\text{electric}} \tag{1.3}$$

The free Hamiltonian H_0 defines a positive self-adjoint operator on $\mathscr{H}$ given by

$$\begin{aligned} \mathrm{H}_0 &= \frac{1}{2} \sum_{x\in\mathrm{V}} \delta^s : \pi_k^2(x) - \pi_0^2(x) - \partial_k \mathrm{A}_\mu \partial_k \mathrm{A}^\mu(x) : \\ &= - \sum_{p\in\Gamma_0} \omega(p) a_\mu^+(p) g_{\mu\nu} a_\nu(p), \end{aligned}$$

where $\omega(p) = \Sigma_{j=1}^{s} 4\delta^{-2} \sin^2 (k_j\delta/2)$ and a_μ, a_μ^+ are the Fock annihilation and creation operators (see appendix). For the interaction terms:

$$V_{\text{magnetic}} = \sum_{x \in V} \delta^s[\lambda \partial_k^F A_l \, . \, A_k \times A_l(x) + \lambda^2/4(A_k \times A_l(x))^2] \, .$$

$$V_{\text{electric}} = \lambda \sum_{x \in V} \delta^s[\pi_k(x) + \partial_k^F A_0(x)] \, . \, A_0 \times A_k(x) \, .$$

The summation convention is used for repeated indices k, $l = 1, 2, \ldots, s$ and for color index summations which are suppressed with the conventions $(A_0 \times A_k)^{(a)} = c_{abc} A_0^{(b)} A_k^{(c)}$, $A_k \cdot A_l = A_k^{(a)} A_l^{(a)}$; etc., with completely antisymmetric structure constants c_{abc} for the Lie gauge group of the Yang-Mills theory. The coupling constant λ is real and $\partial_k^F f_\mu(x) = \delta^{-1} [f_\mu(x + e_k\delta) - f_\mu(x)]$ indicates the « forward » lattice derivative in the kth direction. The « backward » and « midpoint » derivative would serve equally well.

For a closable, densely defined operator T on $\mathscr{H}$, the Krein adjoint T^+ and the Hilbert adjoint T^* are related by $T^+ = \eta T^* \eta$ while for the minimal closure $\tilde{T} = T^{**} = T^{++}$. Let D_F denote vectors in $\mathscr{H}$ containing finitely many particles then H is densely defined on D_F. We prove in section two that

$$\widetilde{(H_{\restriction D_F})} = (H_{\restriction D_F})^+ \, .$$

This answers a long standing question of Jaffe, Lanford and Wightman [7]. H is not symmetric in the Hilbert sense and fails to be bounded below due to the singular nature of V_{electric} and due to the indefinite metric used to quantize $A_\mu^{(a)}$ « covariantly ». Our proof relates H by a Krein unitary transformation to an operator H′ which has a (quasi) maximal accretive closure. In section three, we make use of the functional integral derived in [3] to obtain a Feynman-Kac representation for the semigroup generated by H′ in terms of Brownian motion. It then follows immediately that H′ satisfies a Kato inequality by dominating an operator formed by setting all « magnetic » terms equal to zero in H′. It would appear that while the structure of indefinite metric Yang-Mills quantum fields is more complicated than for theories of massive scalar fields, it is still natural and mathematically accessible.

2. KREIN SELF-ADJOINTNESS

A Krein unitary operator U is a densely defined, closed linear operator in $\mathscr{H}$ such that

$$\{ U\Phi, U\Psi \} = \{ \Phi, \Psi \} \qquad \forall \Phi, \quad \Psi \in D(U) \, .$$

It is not required that U be bounded. Define

$$\mathrm{S} = \sum_{x\in\mathrm{V}} \delta^s \partial_k^{\mathrm{F}} \mathrm{A}_0^{(a)}(x)\mathrm{A}_k^{(a)}(x) = -\sum_{x\in\mathrm{V}} \delta^s \mathrm{A}_0^{(a)}(x)\partial_k^{\mathrm{B}}\mathrm{A}_k^{(a)}(x)\,, \qquad (2.1)$$

for which $\mathrm{S}_{\restriction \mathrm{D_F}}$ is skew-symmetric and $+$-symmetric. A short calculation shows vectors in $\mathrm{D_F}$ are analytic vectors for S and there exist constants c_0, c_1 depending upon Φ such that

$$\Sigma_{n=0}^{\infty} \frac{\| z^n \mathrm{S}^n \Phi \|}{n\,!} \leq \frac{c_0 \| \Phi \|}{1 - c_1 | z |} \qquad c_1 | z | < 1, \quad \Phi \in \mathrm{D_F} \qquad (2.2)$$

LEMMA 2.1. — The operator $\widetilde{\mathrm{S}_{\restriction \mathrm{D_F}}}$ is skew-adjoint and $+$ self-adjoint. Moreover, exp $[iz\tilde{\mathrm{S}}]$ is self-adjoint and $+$-unitary for real z, while for complex z satisfies

$$e^{-iz\tilde{\mathrm{S}}}\tilde{\mathrm{H}}e^{iz\tilde{\mathrm{S}}}\Phi = (\mathrm{H} + iz[\mathrm{H},\mathrm{S}] + (iz)^2/2\,[[\mathrm{H},\mathrm{S}],\mathrm{S}])\Phi, \quad \forall\Phi\in\mathrm{D_F} \quad (2.3)$$

It follows that H is $+$-unitary to a (quasi) accretive operator.

Proof. — Notice $\eta \mathrm{S} \eta = -\mathrm{S}$ on $\mathrm{D_F}$, so by closure this relation extends to $\tilde{\mathrm{S}}$. By Nelson's theorem on analytic vectors, $\eta e^{iz\mathrm{S}}\eta = e^{-iz\mathrm{S}}$ on the appropriate domain. This proves the first part.

Suppose $\Phi \in \mathrm{D_F}$ and put

$$\Phi_{\mathrm{N}} = \Sigma_{n=0}^{\mathrm{N}} (iz)^n \mathrm{S}^n \Phi / n\,! \qquad | z | \leq z_1(\Phi).$$

By (2.2), $\Phi_{\mathrm{N}} \xrightarrow{s} e^{iz\tilde{\mathrm{S}}}\Phi$ and H is closable; so if we write

$$\mathrm{H}\Phi_{\mathrm{N}} = \Sigma_{n=0}^{\mathrm{N}} (iz)^n [\mathrm{H}, \mathrm{S}^n]\Phi/n\,! + \Psi_{\mathrm{N}},$$

then $\Psi_{\mathrm{N}} \xrightarrow{s} e^{iz\tilde{\mathrm{S}}}\mathrm{H}\Phi$ for $| z | \leq z_2(\Phi)$. Now using the commutation relations (1.1) with (2.1), $[[\mathrm{H},\mathrm{S}]\mathrm{S}]$ commutes with S. Consequently,

$$\mathrm{H}\Phi_{\mathrm{N}} \xrightarrow{s} e^{iz\tilde{\mathrm{S}}}(\mathrm{H} + iz[\mathrm{H},\mathrm{S}] + (iz)^2/2\,[[\mathrm{H},\mathrm{S}],\mathrm{S}])\Phi \quad \text{for} \quad | z | \leq z_3(\Phi).$$

This means

$$\tilde{\mathrm{H}}e^{iz\tilde{\mathrm{S}}}\Phi = e^{iz\tilde{\mathrm{S}}}\,(\mathrm{H} + iz[\mathrm{H},\mathrm{S}] + (iz)^2/2\,[[\mathrm{H},\mathrm{S}],\mathrm{S}])\Phi\,;$$

and, in particular, $\tilde{\mathrm{H}}e^{iz\tilde{\mathrm{S}}}\mathrm{D_F} \subset \mathrm{Ran}\,(e^{iz\tilde{\mathrm{S}}})$. Applying the spectral theorem for $\tilde{\mathrm{S}}$ shows $(\exp\,[iz\tilde{\mathrm{S}}])^{-1} = \exp\,[-iz\tilde{\mathrm{S}}]$ whereupon (2.3) is valid for $| z | \leq z_{\min}(\Phi)$. The right hand side of this relation is an entire function of z and defines an analytic continuation of the left hand side to all complex z.

The canonical transformation using S enables us to control the highly singular second term in $\mathrm{V_{electric}}$. If we denote $\mathrm{U} = \exp\,[i\tilde{\mathrm{S}}]$ and $\tilde{\mathrm{H}}' = \mathrm{U}^{-1}\tilde{\mathrm{H}}\mathrm{U}$, then H′ is given by the right hand side of (2.3) and is (quasi) accretive.

To see this use (2.3) and properties of our lattice regularization, particularly the identity

$$\sum_{x\in V} \delta^s \partial_k^F A_l(x) \partial_l^F A_k(x) = \sum_{x\in V} \delta^s (\partial_k^B A_k(x))^2 \tag{2.4}$$

which is a consequence of periodicity, to obtain an expression

$$H' = R + K \tag{2.5}$$

The operator R is essentially self-adjoint on D_F and bounded below by $-E_0 = -(s+1)/2\Sigma_{p\in\Gamma_0}\omega(p)$. It is given by

$$R = \sum_{x\in V} \delta^s[\pi_k^2(x)/2 - \pi_0^2(x)/2 + (F_{kl}(x))^2/4] - E_0 \tag{2.6}$$

in which $F_{kl}(x) = \partial_k^F A_l(x) - \partial_l^F A_k(x) + \lambda A_k \times A_l(x)$. The operator K is skew-symmetric and involves momentum dependent interactions between the fields:

$$K = \sum_{x\in V} \delta^s[\lambda\pi_k\cdot A_0 \times A_k(x) + \pi_0\partial_k^B A_k(x) - \pi_k\partial_k^F A_0(x)] \tag{2.7}$$

REMARK 2.1. — The complete « magnetic » portion of the Yang-Mills interaction appears in the expression for the operator R; while the « electric » portion resides in the essentially skew-adjoint operator K and permits recovery of part of the positivity inherent in the heuristic Lagrangean density $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ from physics.

For Krein symmetric operators which are also accretive, Krein self-adjointness is equivalent to being maximal accretive. For completeness we give the simple proof.

LEMMA 2.2. — Let T be a densely defined, closed linear accretive operator with $T \subset T^+$. Then $T = T^+$ if and only if T is maximal accretive.

Proof. — If T is maximal accretive and $v \in D(T^+)$ there exists $u \in D(T)$ for which $(T^+ + \bar\lambda)v = (T + \bar\lambda)u$ for some complex λ with $\operatorname{Re}\lambda < 0$. As $T \subset T^+$ then $(T^+ + \bar\lambda)(v - u) = 0$; but, $N(T^+ + \bar\lambda) = \eta\operatorname{Ran}(T+\lambda)^\perp = \{0\}$ hence $v = u$ and $D(T^+) = D(T)$.

Suppose $T = T^+$ and let T_1 be a maximal accretive extension of T [8], then $T_1^+ \subset T^+ = T \subset T_1$. T_1^* is maximal accretive and $u \in D(T_1^+)$ requires $\eta u \in D(T_1^*)$, where $\operatorname{Re}(u, T_1^+, u) = \operatorname{Re}(\eta u, T_1^*\eta u) \geq 0$ and T_1^+ is accretive. Consider $v \in \mathscr{H}$ with $0 = (v, (T_1^+ + I)u) = (\eta v, (T_1^* + I)\eta u)$ for all $u \in D(T_1^+)$. Clearly $v = 0$ and T_1^+ is maximal accretive so by the first part of the proof $T_1 = T_1^+ = T$.

COROLLARY. — *i*) T and T* accretive if and only if T and T^+ are accretive, whence T is maximal accretive.

ii) T is maximal accretive if and only if T^+ is maximal accretive.

One of the most useful criteria for maximal accretivity is due to M. Krein and is just the proof given for Lemma 2.2.

PROPOSITION 2.3 (Krein). — Let T be a densely defined, closed linear accretive operator. Then T* accretive implies T is maximal accretive.

In order to show $H_{\restriction D_F}$ is +-self-adjoint, we see from Lemmas 2.1, 2.2 and Proposition 2.3 it is enough to prove that H'* is accretive. To do this we exploit a method from the theory of elliptic partial differential operators centered around the study of Kato's inequality carried out by A. Devinatz [4]. For quantum fields on a finite lattice obeying Bose-Einstein statistics, J. von Neumann's theorem on equivalence of the Heisenberg and Schrödinger versions of quantum mechanics relates H and H' to second order elliptic operators. Conventions for these « harmonic oscillator » coordinates are listed briefly in an appendix. If $q = (q', q'') \in \mathbb{R}^\nu$, $\nu = \nu_1 + \nu_2$, where $q' \in \mathbb{R}^{\nu_1}$ are position coordinates associated with $\{ A_k^{(a)}(x) \mid x \in V ; k = 1, \ldots, s ; a = 1, \ldots, N \}$ and $q'' \in \mathbb{R}^{\nu_2}$ those similarly associated with the fields $A_0^{(a)}(x)$, we find from (2.6) and (2.7)

$$H' = -\Delta/2 + \vec{a}\cdot\vec{\nabla} + \mathscr{V}(q') \tag{2.8}$$

Here Δ is the ν-dimensional Laplace operator and $\mathscr{V}$ is a quartic polynomial in q' which is bounded below. The vector fields $\vec{a} = \vec{b} + \vec{c} + \vec{d}$ in which

$$\vec{a}\cdot\vec{\nabla} = \Sigma_{i=1}^{\nu_1} [b_i(q', q'') + d_i(q'')] \frac{\partial}{\partial q_i'} + \Sigma_{j=1}^{\nu_2} c_j(q') \frac{\partial}{\partial q_j''} \tag{2.9}$$

with the individual components satisfying

$$\operatorname{div} \vec{a} = 0 \qquad b_i q_i' = 0 \tag{2.10a}$$

$$b_i = 0(|q|^2) \qquad c_j = 0(|q'|) \qquad d_i = 0(|q''|) \tag{2.10b}$$

These relations are readily checked using the q-coordinates in the appendix.

THEOREM 2.4. — Set $\tilde{H}' = U^{-1}\tilde{H}U$. $\tilde{H}'$ is a +-symmetric and (quasi) maximal accretive operator and hence $\tilde{H}'$ is +-self-adjoint. Thus $\tilde{H}$ is +-self-adjoint.

Proof. — Under the change to q-coordinates leading to (2.8), the domain D_F is mapped into the Hermite functions on $L^2(\mathbb{R}^\nu)$. As $C_0^\infty(\mathbb{R}^\nu)$ is a core for $H'_{\restriction D_F}$ it is enough to show $(H'_{\restriction C_0^\infty(\mathbb{R}^\nu)})^*$ is accretive. In fact, $H'^* = (-\Delta/2 - \vec{a}\cdot\vec{\nabla} + \mathscr{V})_{\max}$ is the maximal differential operator defined on $D(H') = \{ u \in L^2(\mathbb{R}^\nu) \mid (-\Delta/2 - \vec{a}.\vec{\nabla} + \mathscr{V})_{\text{distribution}}\, u \in L^2 \}$. In the following we assume $\mathscr{V} \geq 0$.

Let χ be a positive C^∞-function of compact support on $\mathbb{R}_+$ which is one for $0 \leq t \leq 1$ and zero for $t \geq 2$. Set $\theta(q) = \chi(|q|^2/K^2)$, $K > 0$. Further denote $\psi_1(q) = \chi(|q|^2/4N^2)$, $\psi_2(q) = \chi(|q|^2/25N^2)$ for $N > K$ and define $H_1'^* = \psi_1 H'^*$, $u^{(2)} = \psi_2 u$. Notice for $u \in D(H'^*)$, $H_1'^* u^{(2)} = \psi_1 H'^* u = H'^* u$ on supp θ. $H_1'^*$ is a degenerate elliptic second order operator with bounded C^∞-coefficients. For such an operator, Devinatz [4] has shown $\widetilde{H_1'^* \upharpoonright_{C_0^\infty}} = (H_1'^*)_{\max}$ so that $H_1'^*$ is maximal (quasi) accretive with a core C_0^∞. There is then a sequence $\{u_n\} \subset C_0^\infty$ such that $u_n \xrightarrow{s} u^{(2)}$ and $H_1'^* u_n \xrightarrow{s} H_1'^* u^{(2)}$ for $u \in D(H'^*)$.

Consider now the expression

$$\int_{|q| \leq 4N} dq \theta \bar{u}_n H'^* u_n = \int_{|q| \leq 4N} dq \theta \bar{u}_n [-\Delta/2 - a_i \partial_i + \mathscr{V}] u_n .$$

Integrating by parts and discarding surface terms leads to

$$2 \operatorname{Re} \int_{|q| \leq 4N} dq \theta \bar{u}_n H'^* u_n \geq \int_{|q| \leq 4N} dq \, |u_n|^2 [-\Delta\theta/2 + \vec{a} \cdot \vec{\nabla}\theta] .$$

First let $n \to \infty$ and then let $N \to \infty$ using dominated convergence for u, $H'^* u$ in L^2 to arrive at

$$2 \operatorname{Re} \int dq \theta \bar{u} H'^* u \geq \int_{K \leq |q| \leq 2K} dq \, |u|^2 [-2|q|^2 \chi''/K^4 - \nu\chi/K^2 + 2\vec{a} \cdot \vec{q} \chi'/K^2] \tag{2.11}$$

Without loss of generality we may require χ', χ'' uniformly bounded and $\theta \to 1$ boundedly as $K \to \infty$. The bounds in (2.10) imply the right hand side of (2.11) tends to zero as $K \to \infty$ again by dominated convergence. The left hand side converges to $2 \operatorname{Re}(u, H'^* u) \geq 0$ and hence H'^* is (quasi) accretive. ■

The dynamics for the Yang-Mills Hamiltonian $\tilde{H}$ may be given in terms of the exponentially bounded, strongly continuous semigroup

$$S(t) = \exp[-t\tilde{H}'] \qquad t \geq 0 \tag{2.12}$$

by a Krein unitary transformation

$$T(t) = U S(t) U^{-1} \qquad t \geq 0 . \tag{2.13}$$

$T(t)$ has been defined as an (unbounded) semigroup on $D(U^{-1})$ and satisfies the functional integral representation given in [3] which exhibits the gauge invariance present in (1.1), (1.2) and (1.3).

REMARK 2.2. — The additional cut-off (M cut-off) used in [3] to give an easy proof of convergence of the Dyson series for $T(t)$ by reducing the growth of the number of particles in V_{electric} is readily removed by using (2.12), (2.13) and Theorem 3.1 of the next section.

3. A FEYNMAN-KAC FORMULA

The semigroup $T(t)$ may be represented as a Euclidean path integral with respect to the Feynman gauge Euclidean process associated with A_μ. In this context, V_{electric} required the use of a stochastic integral to represent the momentum term which we now show how to transform into a simple Feynman-Kac formula for the bounded semigroup $S(t)$. This second representation depends entirely upon a martingale decomposition which we used in [3] and extend further here.

Consider a Gaussian random process $B_\mu^{(a)}(t, x)$ on $\mathbb{R} \times V$ with mean zero and covariance

$$\langle B_\mu^{(a)}(t, x) B_\nu^{(b)}(s, y) \rangle = \frac{\delta_{ab}\delta_{\mu\nu}}{2V} \Sigma_{p\in\Gamma_0} \frac{e^{-\omega|t-s|+ip(x-y)}}{\omega(p)}$$
$$= \frac{1}{2} \delta_{ab}\delta_{\mu\nu} \exp\left[-(-\Delta_s)^{\frac{1}{2}} \mid t-s \mid\right](-\Delta_s)^{-\frac{1}{2}}\delta(x-y). \quad (3.1)$$

Choose a continuous separable version of this process in terms of a probability space $(\Omega, \mathcal{F}, \mu_0)$. For example, Ω could be a set of paths and $\mathcal{F}$ the σ-algebra generated by $\{ B_\mu^{(a)}(t, x) \}$. The σ-algebras $\mathcal{F}_t$ generated by $\{ B_\mu^{(a)}(s, x) \mid 0 \leq s \leq t \}$ form a filtration of $\mathcal{F}$. By a « Brownian motion » with respect to $(\mathcal{F}, \mu_0)$ and $\mathcal{F}_t$ we mean an $\mathcal{F}_t$ measurable stochastic process $\beta_\mu^{(a)}(t, x)$ satisfying

$$i) \quad \beta_\mu^{(a)}(0, x) = 0 \qquad (ii) \quad \langle \beta_\mu^{(a)}(t, x) \rangle = 0$$
$$iii) \quad \langle \beta_\mu^{(a)}(t, x)\beta_\nu^{(b)}(s, y) \rangle = \delta_{\mu\nu}\delta_{ab}\delta(x - y)t \wedge s \quad (3.2)$$

in which expectations $\langle \, . \, \rangle$ are taken with respect to μ_0. The process $B_\mu^{(a)}(t, x)$ is a quasimartingale [3, 6] given as

$$B_\mu^{(a)}(t, x) = B_\mu^{(a)}(0, x) + \beta_\mu^{(a)}(t, x) - \int_0^t (-\Delta_s)^{\frac{1}{2}} B_\mu^{(a)}(u, x)du$$
$$= \exp\left[-(-\Delta_s)^{\frac{1}{2}}t\right]\left\{ B_\mu^{(a)}(0, x) + \int_0^t e^{(-\Delta_s)^{\frac{1}{2}}u} d\beta_\mu^{(a)}(u, x) \right\}, \qquad t \geq 0. \quad (3.3)$$

In (3.1), (3.3) and throughout this section Δ_s denotes the finite difference Laplacean on V. In [3, Proposition 2.3], we proved the existence of a linear embedding

$$I_t : \mathcal{H} \to \operatorname{Ran}(I_t) \subset L^2(\Omega, \mathcal{F}, \mu_0)$$

with

$$I_t^* I_s = e^{-|t-s|H_0},$$

such that I_t is isometric for the Feynman gauge in both the Krein and Hilbert metrics. When I_t is applied to vectors in D_F the Dyson series for $T(t)$ converges strongly by means of the correspondence:

Minkowski	*Euclidean*
$I_s A_0^{(a)}(0, x) I_s^*$	$i B_0^{(a)}(s, x)$
$I_s \pi_0^{(a)}(0, x) I_s^* ds$	$d B_0^{(a)}(s, x)$ with a term $-\delta_{ab}\delta(t-s)\delta(x-y)$ for each $\pi_0^{(a)}\pi_0^{(b)}$ contraction
$I_s A_k^{(a)}(0, x) I_s^*$	$B_k^{(a)}(s, x)$
$I_s \pi_k^{(a)}(0, x) I_s^* ds$	$- i d B_k^{(a)}(s, x)$ with a term $\delta_{ab}\delta_{kl}\delta(t-s)\delta(x-y)$ for each $\pi_k^{(a)}\pi_l^{(b)}$ contraction... (3.4)

REMARK 3.1. — This correspondence is valid also for I_s^* replaced by the Krein adjoint I_s^+ as proved in [3]. The same calculation leads to (3.4).

The Feynman-Kac formula for $S(t)$ will be stated in terms of a system of diffusion measures $\{ P_{B_\mu(0,x)} \}$ on $(\Omega, \mathscr{F})$ indexed by the initial condition for the stochastic differential equation

$$d\tilde{B}_\mu(t, x) = d\tilde{\beta}_\mu(t, x) + [(-\lambda \tilde{B}_0 x \tilde{B}_\mu + \partial_\mu^F \tilde{B}_0)(t, x)\delta_{\mu k} - \partial_l^B \tilde{B}_l(t, x)\delta_{\mu 0}]dt$$
$$\tilde{B}_\mu(0, x) = B_\mu(0, x). \tag{3.5}$$

The process $\tilde{B}_\mu$ for $t \geq 0$ is defined as the unique strong solution to (3.5) with respect to a Brownian process $\tilde{\beta}_\mu$ for a second probability measure $\tilde{\mu}$ on $(\Omega, \mathscr{F})$. The relation between μ_0 and $\tilde{\mu}$ is given explicitly in the proof of Theorem 3.1 below. A transition probability for the diffusion in (3.5) is defined by

$$P(t; B_\mu(0, x); \Gamma) = P_{B_\mu(0,x)}(B_\mu(t, x) \in \Gamma), \qquad \Gamma \in \mathscr{B}(\mathbb{R}^\nu) \tag{3.6}$$

and

$$\tilde{\mu}(A) = \int d\mu_0(B_\mu(0, x)) P_{B_\mu(0,x)}(A), \qquad A \in \mathscr{F} \tag{3.7}$$

Expectations taken with respect to the diffusion measures in (3.6) are denoted by $E(\cdot)$. The Fock space $\mathscr{H}$ may be identified with

$$\mathrm{Ran}\,(I_0) = L^2(\Omega, \mathscr{F}_0, \mu_0)$$

and vectors $\Psi \in \mathscr{H}$, regarded as functions of A_μ, map into functions $I_0\Psi$ of $B_\mu(0, x)$.

THEOREM 3.1. — Consider $\Psi \in \mathscr{H}$ as a function of A_μ, the time-zero quantum field. Then, if Ω_0 is the Fock vacuum for (1.1),

$$I_0(e^{-E_0 t} S(t)\Psi) = E\left\{ e^{-\frac{1}{4}\int_0^t du \sum_{x \in V} \delta^s [F_{kl}(\tilde{B}(u,x))]^2} (I_t\Omega_0)(I_t\Psi)(\tilde{B}(t, x)) \right\}.$$

The measures μ_0 and $\tilde{\mu}$ are absolutely continuous when restricted to $\mathscr{F}_t$; in fact, $d\tilde{\mu}/d\mu_{0\restriction\mathscr{F}_t}$ is an exponential L'-martingale.

Proof. — Suppose Φ, Ψ are polynomials in the time-zero fields A_μ applied to the Fock vacuum Ω_0. Using Theorem 5.1 of [3] together with the correspondences in (3.4),

$$(\Phi, S(t)\Psi) = \int_\Omega d\mu_0(\overline{I_0\Phi})(I_t\Psi)e^{-U}\Gamma(t) \tag{3.8}$$

The integrand is

$$U = \frac{1}{4}\int_0^t du \sum_{x\in V} \delta^s[F_{kl}(B(u, x))]^2$$

$$\Gamma(t) = \Gamma_1(t)\Gamma_2(t)\Gamma_3(t);$$

in which

$$\Gamma_1(t) = \exp\Big[-\int_0^t \sum_{x\in V} \delta^s(\lambda\{dB_k(u, x) - \partial_k^F B_0(u, x)du\}\cdot B_0 \times B_k(u, x) + \lambda^2/^2(B_0 \times B_k(u, x))^2 du)\Big]$$

$$\Gamma_2(t) = \exp\Big[\int_0^t \sum_{x\in V} \delta^s\{dB_k \cdot \partial_k^F B_0(u, x) - dB_0 \cdot \partial_k^B B_k(u, x)\}\Big]$$

$$\Gamma_3(t) = \exp\frac{1}{2}\Big[\int_0^t du \sum_{x\in V} \delta^s\{(\partial_k^F B_l(u, x))^2 - (\partial_k^B B_k(u, x))^2\}\Big].$$

The last term in the exponential for Γ_3 results from periodicity in B_μ in x; see for example (2.4). The desired formula for $S(t)$ is obtained by a series of drift transformations on (3.8).

The relation (3.3) may be written as $B_\mu(t, x) = B_\mu(0, x) + \beta_\mu^{(1)}(t, x)$ with $\beta_\mu(t, x) = \beta_\mu^{(1)}(t, x) + \int_0^t du(-\Delta_s)^{\frac{1}{2}}B_\mu(u, x)$ in which $\beta_\mu^{(1)}$ is a Brownian motion with respect to $(\mathscr{F}, \mu_0^{(1)})$. The derivative $d\mu_0 = M(t)d\mu_{0\restriction\mathscr{F}_t}^{(1)}$ is determined by an exponential martingale

$$M(t) = \exp\Big[-\int_0^t \sum_{x\in V} \delta^s\{(-\Delta_s)^{\frac{1}{2}}B_\mu(u, x)\cdot dB_\mu(u, x) + \frac{1}{2}((-\Delta_s)^{\frac{1}{2}}B_\mu(u, x))^2 du\}\Big].$$

Applying Itô's theorem for stochastic differentials to the process

$$-\frac{1}{2}\sum_{x\in V} \delta^s B_\mu(t, x)(-\Delta_s)^{\frac{1}{2}}B_\mu(t, x),$$

M(t) may be rewritten as

$$\mathrm{M}(t)=e^{\mathrm{E}_0 t}\exp\left[-\frac{1}{2}\int_0^t du\sum_{x\in \mathrm{V}}\delta^s\{(-\Delta_s)^{\frac{1}{2}}\mathrm{B}_\mu(u,x)\}^2\right]\ \text{(times)}$$
$$\exp\left[-\frac{1}{2}\sum_{x\in \mathrm{V}}\delta^s\{(-\Delta_s)^{\frac{1}{2}}\mathrm{B}_\mu(t,x)\}^2+\frac{1}{2}\sum_{x\in \mathrm{V}}\delta^s\{(-\Delta_s)^{\frac{1}{2}}\mathrm{B}_\mu(0,x)\}^2\right].$$

Summations over color indices continue to be suppressed. Noticing

$$\sum_{x\in \mathrm{V}}\delta^s\{(-\Delta_s)^{\frac{1}{2}}\mathrm{B}_\mu(t,x)\}^2=\sum_{x\in \mathrm{V}}\delta^s[\{\partial_k^{\mathrm{F}}\mathrm{B}_0(t,x)\}^2+\{\partial_k^{\mathrm{F}}\mathrm{B}_l(t,x)\}^2],\tag{3.9}$$

one learns the measure $\mu_0^{(1)}$ is a representation in terms of B_μ of a Wiener measure in the q-coordinates with an initial distribution for $\mathrm{B}_\mu(0, x)$ given by (3.1).

Remove the factor exp ($\mathrm{E}_0 t$) and combine Γ_2, Γ_3 with (3.9) in the first term in M(t) to produce two exponential martingales

$$\Gamma^{(2)}(t)=\exp\left[\int_0^t\sum_{x\in \mathrm{V}}\delta^s\left\{d\mathrm{B}_k\cdot\partial_k^{\mathrm{F}}\mathrm{B}_0(u,x)-\frac{1}{2}(\partial_k^{\mathrm{F}}\mathrm{B}_0(u,x))^2du\right\}\right],$$
$$\Gamma^{(3)}(t)=\exp\left[-\int_0^t\sum_{x\in \mathrm{V}}\delta^s\left\{d\mathrm{B}_0\,.\,\partial_k^{\mathrm{B}}\mathrm{B}(u,x)+\frac{1}{2}(\partial_k^{\mathrm{B}}\mathrm{B}_k(u,x))^2du\right\}\right];$$

where upon (3.8) becomes

$$(\Phi,\mathrm{S}(t)\Psi)e^{-\mathrm{E}_0 t}=\int_\Omega d\mu_0^{(1)}(\overline{\mathrm{I}_0\Phi})(\mathrm{I}_0\Omega_0)^{-1}(\mathrm{I}_t\Psi)(\mathrm{I}_t\Omega_0)e^{-\mathrm{U}}\Gamma_1(t)\Gamma^{(2)}(t)\Gamma^{(3)}(t)\tag{3.10}$$

Each of the martingales $\Gamma^{(2)}$, $\Gamma^{(3)}$ gives rise to a drift transformation of $\beta_\mu^{(1)}$ by

$$\beta_\mu^{(2)}(t,x)=\beta_\mu^{(1)}(t,x)-\int_0^t du\{\partial_k^{\mathrm{F}}\mathrm{B}_0(t,x)\delta_{\mu k}-\partial_l^{\mathrm{B}}\mathrm{B}_l(u,x)\delta_{\mu 0}\}$$

and a measure $d^{\mu(2)}=\Gamma^{(2)}(t)\Gamma^{(3)}(t)d\mu^{(1)}_{\restriction\mathcal{F}_t}$ for which $\beta_\mu^{(2)}$ is a Brownian motion. Finally the martingale $\Gamma_1(t)$ may be absorbed by one more drift transformation

$$\tilde{\beta}_\mu(t,x)=\beta_\mu^{(2)}(t,x)-\lambda\delta_{\mu k}\int_0^t du\mathrm{B}_0\times\mathrm{B}_k(u,x)$$

into a measure $d\tilde{\mu}=\Gamma_1(t)d\mu^{(2)}_{\restriction\mathcal{F}_t}$. Γ_1 is certainly a local martingale but due to the quadratic growth of the coefficient of $\beta_\beta^{(2)}$ a further test is needed to show $\tilde{\mu}(\Omega)=1$. In terms of the q-coordinates, Γ_1 corresponds to a diffusion with generator $-\Delta/2 + \vec{b}.\vec{\nabla}$. Relation (2.10 *a*) used in Hashiminsky's test, [*6*, p. 375], shows this diffusion is non-explosive. The Feynman-Kac formula (3.10) now reduces to

$$(\Phi, S(t)\Psi)e^{-E_0t} = \int_\Omega d\tilde{\mu}(\overline{I_0\Phi})(I_0\Omega_0)^{-1}(I_t\Psi)(I_t\Omega_0)e^{-U}$$

and it remains to observe that $(I_0\Omega_0)^2$ is the density for the initial distribution of $B_\mu(0, x)$. Hence removing this distribution leads to the diffusion measures (3.6) parametrized by the initial condition in (3.5).

The semigroup formula extends to general $\Psi \in \mathscr{H}$ by a limiting argument using the density of polynomials in $\mathscr{H}$, the martingale property for $\Gamma(t)$ and

$$|(\Phi, S(t)e^{-E_0t}\Psi)| \leq \left[\int d\mu_0 \, |I_0\Phi|^2\Gamma(0)\right]^2\left[\int d\mu_0 \, |I_t\Psi|^2 E\{\Gamma(t)\,|\,\mathscr{F}_t\}\right]^2$$
$$= \|\Phi\|^2\,\|\Psi\|^2. \quad \blacksquare$$

The expression in Theorem 3.1 shows S(t) to be positivity preserving as an operator on $L^2(\Omega, \mathscr{F}_0, \mu_0)$ for those $\Psi \in \mathscr{H}$ with $I_0\Psi \geq 0$. There are many such vectors; for example, $\Psi = P(A)\Omega_0$ where the polynomial $P(q) \geq 0$ for all $q \in \mathbb{R}^\nu$. Given $\Psi \in \mathscr{H}$ we may define $|\Psi| \in \mathscr{H}$ by $|\Psi| = I_0^*|I_0\Psi|$ as $I_0I_0^*(\cdot) = E\{\cdot\,|\,\mathscr{F}_0\}$. Let $W(t)$, $t \geq 0$, be the contraction semigroup generated by $\tilde{H}''_{\restriction D_F}$ with $H'' = \sum_{x \in V} \delta^s[\pi_k(x)^2 - \pi_0(x)^2]/2 - E_0 + K$. Theorem 3.1 then has an immediate consequence.

COROLLARY (Kato Inequality). — Choose $\Psi \in D(\tilde{H}')$, $\Phi \in \mathscr{H}$ such that $|\Phi| \in D(\tilde{H}'')$ with $(\Phi, \Psi) = (|\Phi|, |\Psi|)$. Then

$$\operatorname{Re}(\Phi, \tilde{H}'\Psi) \geq (\tilde{H}''^* |\Phi|, |\Psi|).$$

Proof. — From Theorem 3.1

$$\operatorname{Re}(\Phi, \tilde{H}''\Psi)$$
$$\geq \lim_{t \downarrow 0} \operatorname{Re} t^{-1} \int_\Omega d\tilde{\mu} \, |I_0\Phi| \, (I_0\Omega_0)^{-1}[(I_0\Omega_0)\,|I_0\Psi| - I_0I_0^*\,|I_t\Psi|\,(I_t\Omega_0)e^{tE_0}]$$
$$= \lim_{t \downarrow 0} t^{-1}(|\Phi|, \{1 - W(t)\}|\Psi|)$$
$$= \lim_{t \downarrow 0} t^{-1}(\{1 - W(t)^*\}|\Phi|, |\Psi|)$$
$$= (H''^*|\Phi, |\Psi|).$$

REMARK 3.2. — The necessity of the relation between Φ and Ψ has been pointed out by H. Hess et al. [*5*]. A weak form of Kato's inequality follows by taking Φ and Ψ to be positive polynomials applied to Ω_0.

REMARK 3.3. — The spectrum of $\tilde{H}$ appears to be complicated. Consider the self-adjoint operator $-\Delta'/2 + \mathscr{V}(q') - E_0$ in (2.8). This operator has discrete spectrum with a ground state corresponding to an eigenvalue E_{min}. To see this notice that $\mathscr{V}(q') + E_0$ is a homogeneous quartic polynomial in the q'-coordinates, thus if $r = |q'|$, $\partial(\mathscr{V} + E_0)/\partial r = 2(\mathscr{V} + E_0)/r \geq 0$. Then $\mathscr{V}$ restricted to the surface $r = |q'|$ is an analytic function of the spherical angles. From this it is easy to see $\mathscr{V} + E_0$ vanishes on the sphere only on a set of spherical measure zero. Consequently $\mathscr{V} + E_0$ increases almost everywhere in $\mathbb{R}^{\nu_1}$ and Rellich's criterion applies [*9*, p. 247-249]. The effect of Δ'' is to add continuous spectrum beginning with each eigenvalue while $\vec{a} \cdot \vec{\nabla}$ contributes continuous spectrum in imaginary directions and thus in the half-space $\operatorname{Re} z \geq E_{min} - E_0$. It would be very interesting to know how the spectrum of H' is related to the physical spectrum of H taken on states with zero color charge?

REMARK 3.4. — Do the results of Theorems 2.4 and 3.1 extend to the Yang-Mills part of the Hamiltonian when A_μ is in one of the Rideau gauges, see [*3*]?

APPENDIX

In terms of Fock annihilation and creation forms $c_\mu(k)$, $c_\mu^*(k)$ for which

$$[c_\mu(k),\ c_\nu^*(k')] = \omega(k)\delta_{\mu\nu}\delta_{k,k'}\ ;$$

see [3, Remark 2.2], a representation of (1.1) is provided by

$$A_\mu(x) = (2\,|\,V\,|\,)^{-\frac{1}{2}} \sum_{k\in\Gamma_0} \{\, g_{\mu\nu}c_\nu^*(k)e^{-ikx} - c_\mu(k)e^{ikx}\,\}/\omega(k)$$

$$\pi_\mu(x) = -\, i(2\,|\,V\,|\,)^{-\frac{1}{2}} \sum_{k\in\Gamma_0} \{\, g_{\mu\nu}c_\nu^*(k)e^{-ikx} + c_\mu(k)e^{ikx}\,\}\ .$$

The q-coordinates are defined by

$$A_\mu(x = (2\,|\,V\,|\,)^{-\frac{1}{2}} \sum_{k\in\Gamma_0'} \{\, q_{1,\mu}(k)\cos(k\cdot x) + q_{2,\mu}(k)\sin(k\cdot x\,\}$$

$$\pi_\mu(x) = (2\,|\,V\,|\,)^{-\frac{1}{2}} \sum_{k\in\Gamma_0'} \{\, p_{1,\mu}(k)\cos(k\cdot x) + p_{2,\mu}(k)\sin(k\cdot x)\}$$

where Γ_0' indicates those momenta remaining after using the relations

$$q_{j,\mu}(k) = (-\,1)^{j+1}q_{j,\mu}(-\,k), \qquad p_{j,\mu}(k) = (-\,1)^{j+1}p_{j,\mu}(-\,k)\,.$$

For these momenta (1.1) requires $[q_{j,\mu}(k), p_{l,\nu}(k')] = i\delta_{jl}g_{\mu\nu}\delta_{k,k'}$. Each $q_{j,l}$ and $p_{j,l}$ are symmetric but in the Feynman gauge $q_{j,0}$ and $p_{j,0}$ are skew-symmetric. J. von Neumann's theorem on irreducible representations of the Heisenberg commutation relations for finitely many degrees of freedom provides $\mathscr{H} \cong L^2(\mathbb{R}^{\nu_1+\nu_2})$ with $q_{j,l}(k)$ a multiplication operator by the appropriate coordinate with $p_{j,l} = i\partial/\partial q_{j,l}$. Similarly $-\,iq_{j,0}(k)$ is a multiplication operator and $p_{j,0} = -\,i\partial/\partial q_{j,0}$. Substitution of the Fourier series for $A_\mu(x)$, $\pi_\nu(x)$ into (2.6) and (2.7) leads to (2.8), (2.9) and (2.10).

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