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Equilibrium fluctuations in relativistic thermoelectric fluids

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ABSTRACT. — The formalism of the extended thermodynamics is applied to a relativistic thermoelectric fluid. Two new terms appear in the generalised Gibbs equation. Their physical meaning is studied by means of the analysis of the fluctuations of the dissipative fluxes. The constitutive equations as well as the fluctuation-dissipation theorem are obtained.

RÉSUMÉ. — On applique le formalisme d'une thermodynamique généralisée (extended thermodynamics) à un fluide thermoélectrique relativiste. Nous soulignons la présence de deux nouveaux termes dans l'équation de Gibbs généralisée, dont la signification physique est mise en évidence à l'aide de la théorie des fluctuations des fluides dissipatifs. On déduit aussi le théorème de fluctuation-dissipation et les équations constitutives du fluide.

1. INTRODUCTION

For many years the role played by the relaxation proper times in the description of dissipative phenomena has not been properly taken into account. Recently however causality requirements have thrown them into an outstanding level in classical [*1*] as well as in relativistic [*2*] [*3*] formulations of these phenomena. Usually the mentioned proper times are introduced into the theory through a non-equilibrium entropy function, whose physical meaning in some specific situations has been explored by two of us [4] [5].

In this note we analyse the simultaneous process of heat and electric conduction in a relativistic inviscid fluid as well as the fluctuations of both fluxes around equilibrium. In our study the relaxation proper times of the process under consideration play a main role. Let us imagine for instance a thermoelectric fluid submitted to a temperature gradient; consequently a heat and an electric flux will appear. If the temperature difference responsible for the temperature gradient is suddenly removed, both fluxes will not vanish immediately but after a finite time. This reflection suggests that the relaxation proper times must enter on their own right in the thermodynamical description of the involved processes.

We start from the relativistic version of extended irreversible thermodynamics [3]. The ensuing constitutive equations reduce to the earlier obtained by us in the limits of non heat [6] and non electric [7] conducting fluids respectively. Likewise in both limits the second moments for the fluctuations of the dissipative fluxes go into the expressions derived in a previous paper [8].

The outline of this note is as it follows. In Section 2 we present the phenomenological description of the relativistic thermoelectric fluid including besides a relation holding between the different relaxation proper times. In Section 3 we analyse the equilibrium fluctuations of dissipative fluxes. Lastly, Section 4 is devoted to some final remarks.

As it is customary $\Delta^{\mu\nu}$ denotes the spatial projector $g^{\mu\nu} + u^\mu u^\nu$, with $g^{\mu\nu}$ the metric tensor of signature $(-, +, +, +)$ and u^μ the world velocity normalized according to $u^\mu u_\mu = -1$. Derivation along u^μ will be indicated by means of en upper dot.

2. RELATIVISTIC DESCRIPTION OF THERMOELECTRIC FLUIDS

Let us assume a relativistic inviscid fluid capable of heat and electric conduction, submitted to an external electromagnetic field $F^{\mu\nu}$ and with a momentum-energy tensor given by

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} + 2c^{-1}q^{(\mu}u^{\nu)}, \tag{1}$$

where q^μ stands for the heat flux and ε for the internal specific energy. The evolution of this latter quantity along the world line is afforded by the balance equation

$$\rho(\dot{\varepsilon} + p\dot{v}) + c^{-1}(q^\mu{}_{,\mu} + q^\mu \dot{u}_\mu) = -I^\mu u^\nu F_{\mu\nu}, \tag{2}$$

v being the specific volume ($v = 1/\rho$), and I^μ the electric conduction current.

By virtue of the conservation of the electric charge, this vector obeys to the relation

$$\rho \dot{z} = - \mathrm{I}^{\mu}{}_{,\mu}, \tag{3}$$

z being the specific electric charge, which yields the evolution of this quantity along u^{μ}. Moreover I^{μ} shares with q^{μ} the well-known geometric property $u_{\mu}\mathrm{I}^{\mu} = u_{\mu}q^{\mu} = 0$; i. e. both vectors are of spatial type. As we will see, this fact makes easy the mathematical treatment of our problem.

Following the lines of the extended relativistic thermodynamics we assume that the dissipative quantities enter besides the equilibrium variables in the non-equilibrium specific entropy function η as well as in the entropy flux $\mathrm{I}^{\mu}_{(\eta)}$. Thus, we postulate on the one hand the generalized entropy $\eta = \eta(\varepsilon, v, z, q^{\mu}, \mathrm{I}^{\mu})$ whose Gibbs equation can be written as

$$\mathrm{T}\dot{\eta} = \dot{\varepsilon} + p\dot{v} - \mu_e z^{\cdot} + v(\alpha_{11}q^{\mu} + \alpha_{12}\mathrm{I}^{\mu})\dot{q}_{\mu} + v(\alpha_{21}q^{\mu} + \alpha_{22}\mathrm{I}^{\mu})\dot{\mathrm{I}}_{\mu} \tag{4}$$

and on the other hand, the following expression for the entropy flux vector

$$\mathrm{I}^{\mu}_{(\eta)} = \beta_1 q^{\mu} + \beta_2 \mathrm{I}^{\mu}. \tag{5}$$

These two expressions are the most general that can be constructed up to second order in the dissipative fluxes for an isotropic fluid. The chemical-like potential μ_e is defined by means of the equation of state $(\partial\eta/\partial z)' = -\mu_e/\mathrm{T}$, where an upper prime denotes that all quantities but z are to be kept constant during the derivation. The coefficients α_{ij} as well as the β_i $(i, j = 1, 2)$ are functions of ε, v and z; the former being relaxation parameters which will be identified later. Comparison of (5) with the classical expressions enables us to identify β_1 and β_2 as $1/c\mathrm{T}$ and $-\mu_e/\mathrm{T}$ respectively. Since $\dot{\eta}$ must be a perfect differential, the Maxwell relation $\alpha_{12} = \alpha_{21}$ is satisfied. Also, the following set of restrictions exists on the α_{ij}

$$\alpha_{11} \leqslant 0, \qquad \alpha_{22} \leqslant 0, \qquad 4\alpha_{11}\alpha_{22} \geqslant (\alpha_{12} + \alpha_{21})^2, \tag{6}$$

which arise from the requirement of η being maximum at equilibrium state.

By combining the entropy balance equation $\rho\dot{\eta} + \mathrm{I}^{\mu}_{(\eta),\mu} = \sigma$ with (2)-(5) we get for the entropy production

$$\sigma = \frac{1}{\mathrm{T}}\{ q^{\mu}[\mathrm{X}_{\mu}]_{\perp} + \mathrm{I}^{\mu}[\mathrm{Y}_{\mu}]_{\perp} \}, \tag{7}$$

with $[\mathrm{X}_{\mu}]_{\perp} = \Delta^{\nu}_{\mu}\mathrm{X}_{\nu}$, $[\mathrm{Y}_{\mu}]_{\perp} = \Delta^{\nu}_{\mu}\mathrm{Y}_{\nu}$ and where the generalized thermodynamical forces are given according to

$$\mathrm{X}_{\mu} = -\frac{1}{c\mathrm{T}}(\mathrm{T}_{,\mu} + \mathrm{T}\dot{u}_{\mu}) + \alpha_{11}\dot{q}_{\mu} + \alpha_{12}\dot{\mathrm{I}}_{\mu}, \tag{8}$$

$$\mathrm{Y}_{\mu} = -u^{\nu}\mathrm{F}_{\mu\nu} - \mathrm{T}\left(\frac{\mu_e}{\mathrm{T}}\right)_{,\mu} + \alpha_{21}\dot{q}_{\mu} + \alpha_{22}\dot{\mathrm{I}}_{\mu}, \tag{9}$$

respectively.

In order to get the transport equations we expand both conjugate forces in function of the dissipative fluxes up to second order in these variables; thus one has

$$[\mathrm{X}_\mu]_\perp = a_{11} q_\mu + a_{12} \mathrm{I}_\mu , \tag{10}$$

$$[\mathrm{Y}_\mu]_\perp = a_{21} q_\mu + a_{22} \mathrm{I}_\mu , \tag{11}$$

where the coefficients a_{ij} $(i, j = 1, 2)$ depend on the equilibrium variables only, and they are also submitted to the restrictions

$$a_{11} \geqslant 0, \qquad a_{22} \geqslant 0, \qquad 4a_{11}a_{22} \geqslant (a_{12} + a_{21})^2 , \tag{12}$$

which proceed from the semipositive definite character of σ, as it may readily checked by inserting (10) and (11) into (7). From equations (8)-(11) the phenomenological relations

$$\alpha_{11}\dot{q}_\mu = a_{11}q_\mu + a_{12}\mathrm{I}_\mu + \Delta_\mu^\nu \left\{ \frac{1}{c\mathrm{T}}(\mathrm{T}_{,\nu} + \mathrm{T}\dot{u}_\nu) - \alpha_{12}\dot{\mathrm{I}}_\nu \right\} + \alpha_{11} u_\mu q_\nu \dot{u}^\nu , \tag{13}$$

$$\alpha_{22}\dot{\mathrm{I}}_\mu = a_{21}q_\mu + a_{22}\mathrm{I}_\mu + \Delta_\mu^\nu \left\{ u^\lambda \mathrm{F}_{\nu\lambda} + \mathrm{T}\left(\frac{\mu_e}{\mathrm{T}}\right)_{,\mu} - \alpha_{21}\dot{q}_\nu \right\} + \alpha_{22} u_\mu \mathrm{I}_\nu \dot{u}^\nu \tag{14}$$

arise. They form a set of equations giving the evolution of the dissipative fluxes along the world line of each particle mass element. In this way (13) and (14) together (2) and (3) and the continuity equation $(\rho u^\mu)_{,\mu} = 0$ describe the behaviour of the fluid as it evolves along u^μ. Obviously (13) and (14) recover respectively the transport equations already deduced in two previous papers [*7*] [*6*].

It remains to determine the parameters a_{ij} as well as the α_{ij}. This can be done by especializing (13) and (14) at the comoving frame — in it one has $\Delta_\mu^\nu = \mathrm{diag}\,(0, 1, 1, 1)$ — then by comparing them, in the stationary case, with the well-known equations of the classical literature [*9*]. Thus we get

$$a_{11} = 1/kc\mathrm{T}, \quad a_{12} = -(1/k\mathrm{T})(\theta + \mu_e), \quad a_{21} = (1/k\mathrm{T})(\mathscr{H}\mathrm{T} - \mu_e),$$
$$a_{22} = c\,\{ (1/\chi) + (1/k\mathrm{T})(\mu_e - \mathscr{H}\mathrm{T})(\theta + \mu_e) \}, \tag{15}$$

where k and χ stand for the heat and the electric conductivity of the fluid respectively, whereas $\mathscr{H}$ and θ are respectively the differential thermoelectric power and the heat at uniform temperature per unit electric current. Once related the a_{ij} to measurable quantities the α_{ij} can be obtained if (13) and (14) are compared, again in the comoving frame, and for the non-stationary case, with the relaxed classical equations. This gives rise to

$$\alpha_{11} = -c a_{11}\tau_{11}, \qquad \alpha_{12} = -(c/\mathrm{T})a_{21}\tau_{12},$$
$$\alpha_{21} = -(c/\mathrm{T})a_{12}\tau_{21}, \qquad \alpha_{22} = -c a_{22}\tau_{22}. \tag{16}$$

Here the $\tau_{ij}(\geqslant 0)$ are the relaxation proper times of the involved processes. The afore-mentioned Maxwell relation, $\alpha_{12} = \alpha_{21}$, together with the second and third equations of (16), implies $\tau_{12}(\mu_e - \mathscr{H}\mathrm{T}) = \tau_{21}(\theta + \mu_e)$. Moreover, if the Onsager relation $a_{12} = a_{21}$ is satisfied it follows the more stringent condition $\tau_{12} = \tau_{21}$. Actually, its validity can be assured in the equilibrium only since the Onsager relations may not hold outside equilibrium [10]. At any rate, there exists another restriction on the τ_{ij} which proceeds from the third equation of (6) and it reads

$$\tau_{11}\tau_{22}\left\{\frac{1}{x} + \frac{1}{k\mathrm{T}}(\mu_e - \mathscr{H}\mathrm{T})(\theta + \mu_e)\right\} \geqslant \frac{\tau_{12}^2}{k\mathrm{T}}\left(\frac{\mathscr{H}\mathrm{T} - \mu_e}{\mathrm{T}}\right). \tag{17}$$

In some circumstances τ_{11} and τ_{22} may be measured, and then the last inequality would be employed to set upper limits to both τ_{12} and τ_{21}.

3. FLUCTUATIONS OF THE FLUXES AROUND EQUILIBRIUM

As it is well-known, in a thermoelectric fluid heat fluctuations originate electric current fluctuations and *vice versa*. Therefore, in dealing with such a medium both kind of fluctuations must be considered together. On the other hand, since both of them are mutually implied, it is reasonable to expect a non-vanishing correlation amongst them.

Let us assume a thermoelectric fluid at equilibrium. In such a case we have $\mathrm{F}^{\mu\nu} = 0$, and the average values of q^μ and I^μ vanish, but however both vectors can fluctuate around that value. In addition the acceleration must vanish also since otherwise, as a consequence of the inertia of heat, a heat flux would arise. In order to obtain the probability of a given spontaneous fluctuation δq^μ or $\delta \mathrm{I}^\mu$ we resort to the Einstein-Boltzmann expression largely used in the literature

$$\mathrm{Pr} \sim \exp\{(\mathrm{M}/2k_\mathrm{B})\delta^2\eta\} \tag{18}$$

where M and k_B stand for the mass of the system and the Boltzmann constant respectively. By combining the generalized Gibbs equation (4) with (18) it follows

$$\mathrm{Pr}(\delta q^\lambda, \delta \mathrm{I}^\lambda) \sim \exp\left\{\frac{\mathrm{V}}{2k_\mathrm{B}\mathrm{T}}[\alpha_{11}\delta q^\mu \delta q_\mu + \alpha_{12}\delta \mathrm{I}^\mu \delta q_\mu + \alpha_{21}\delta q^\mu \delta \mathrm{I}_\mu + \alpha_{22}\delta \mathrm{I}^\mu \delta \mathrm{I}_\mu]\right\}, \tag{19}$$

and from this last expression the second moments calculated in a given event point, say x^σ, read

$$\Lambda^\mu{}_\nu[\delta q] \equiv \langle \delta q^\mu(x^\sigma)\delta q_\nu(x^\sigma)\rangle = k_\mathrm{B}\mathrm{T}\alpha_{22}\alpha^*\Delta^\mu{}_\nu, \tag{20}$$

$$\Lambda^\mu{}_\nu[\delta q, \delta \mathrm{I}] \equiv \langle \delta q^\mu(x^\sigma)\delta \mathrm{I}_\nu(x^\sigma)\rangle = -k_\mathrm{B}\mathrm{T}\alpha_{12}\alpha^*\Delta^\mu{}_\nu, \tag{21}$$

$$\Lambda^\mu{}_\nu[\delta \mathrm{I}] \equiv \langle \delta \mathrm{I}^\mu(x^\sigma)\delta \mathrm{I}_\nu(x^\sigma)\rangle = k_\mathrm{B}\mathrm{T}\alpha_{11}\alpha^*\Delta^\mu{}_\nu, \tag{22}$$

α^* being $(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})^{-1}$ and V the volume of the system. These three equations are relativistic versions of the fluctuation-dissipation theorem relating the second moment of fluctuations to the phenomenological coefficients. Note that in the limit when $\tau_{12} \to 0$ equations (20) and (22) reduce to their counterparts deduced in [8] and, on the other hand, $\Lambda^{\mu}{}_{\nu}[\delta q, \delta I] \to 0$. Since, on physical grounds this latter quantity must be different from zero, relation (21) may be regarded as an indirect corroboration of the presence of the crossed terms $v\alpha_{12}I^{\mu}\dot{q}_{\mu}$ and $v\alpha_{21}q^{\mu}\dot{I}_{\mu}$ in the generalized Gibbs equation. Notwithstanding, it is convenient to remind that (4) is only an approximation up to second order in the dissipative fluxes.

Because of the spatial character of δq^{μ} and δI^{μ} the restrictions

$$\Pr(\delta q^0) = \Pr(\delta I^0) = 0$$

must be satisfied by (19) for a comoving observer. Next we show that (19) bears such a property. Effectively, for such an observer we have from (20)-(22) the relations $\Lambda^0{}_0[\delta q] = \Lambda^0{}_0[\delta q, \delta I] = \Lambda^0{}_0[\delta I] = 0$, and as a consequence (19) fulfills the above restrictions.

The correlation functions, $\tilde{\Lambda}^{\mu}{}_{\nu}[\delta ....]$, for the fluctuations between two very close event points, say x^{σ} and $x^{\sigma} + \Delta x^{\sigma}$, which lie on the same world line can be obtained from (20)-(22) and the evolution equation for the fluctuations. These later follow from (13) and (14) respectively, and they read

$$\alpha_{11}\delta\dot{q}_{\mu} = a_{11}\delta q_{\mu} + a_{12}\delta I_{\mu} - \Delta^{\nu}_{\mu}\alpha_{12}\delta\dot{I}_{\nu}, \tag{23}$$

$$\alpha_{22}\delta\dot{I}_{\mu} = a_{21}\delta q_{\mu} + a_{22}\delta I_{\mu} - \Delta^{\nu}_{\mu}\alpha_{21}\delta\dot{q}_{\nu}. \tag{24}$$

In deducing them we have neglected the fluctuations of the gradients $T_{,\nu}$ and $(\mu_e/T)_{,\nu}$ since they fluctuate much slower than the heat flux and the electric current density. After some manipulations the set of equations becomes

$$\delta\dot{q}_{\mu} = M_{11}\delta q_{\mu} + M_{12}\delta I_{\mu}, \tag{25}$$

$$\delta\dot{I}_{\mu} = M_{21}\delta q_{\mu} + M_{22}\delta I_{\mu}, \tag{26}$$

with the matrix **M** of the coefficients M_{ij} defined through $\mathbf{M} = \boldsymbol{\alpha}^{-1} \cdot \boldsymbol{a}$. Since the matrix $\boldsymbol{\alpha}^{-1}$ is negative semidefinite and $\boldsymbol{a}$ is positive semidefinite, **M** is negative semidefinite.

Integration of the above set of equations between x^{σ} and $x^{\sigma} + \Delta x^{\sigma}$ yields

$$\delta q_{\mu}(x^{\sigma} + \Delta x^{\sigma}) = A_{11}\delta q_{\mu}(x^{\sigma}) + A_{12}\delta I_{\mu}(x^{\sigma}), \tag{27}$$

$$\delta I_{\mu}(x^{\sigma} + \Delta x^{\sigma}) = A_{21}\delta q_{\mu}(x^{\sigma}) + A_{22}\delta I_{\mu}(x^{\sigma}), \tag{28}$$

where the A_{ij} terms are given by $A_{ij} = \exp(M_{ij}s)$, s being the arc lenght measured along the world line. The negative semidefinite character of **M** guarantees the decay of both fluctuations $\delta q_\mu(x^\sigma+\Delta x^\sigma)$ and $\delta I_\mu(x^\sigma+\Delta x^\sigma)$. Finally from (20)-(22) and (27) (28) we get

$$\tilde{\Lambda}^\mu{}_\nu[\delta q] = A_{11}\Lambda^\mu{}_\nu[\delta q] + A_{12}\Lambda^\mu{}_\nu[\delta q, \delta I], \tag{29}$$

$$\tilde{\Lambda}^\mu{}_\nu[\delta q, \delta I] = A_{21}\Lambda^\mu{}_\nu[\delta q] + A_{22}\Lambda^\mu{}_\nu[\delta q, \delta I], \tag{30}$$

$$\tilde{\Lambda}^\mu{}_\nu[\delta I] = A_{21}\Lambda^\mu{}_\nu[\delta q, \delta I] + A_{22}\Lambda^\mu{}_\nu[\delta I], \tag{31}$$

As it can be seen, the second moments in two separated event points can be expressed, in this simple way, as a linear combination of the correlations $\Lambda^\mu{}_\nu[\delta ...]$.

4. CONCLUDING REMARKS

The form of the non-equilibrium part of the entropy,

$$v T^{-1}\{(\alpha_{11}q^\mu + \alpha_{12}I^\mu)\dot{q}_\mu + (\alpha_{21}q^\mu + \alpha_{22}I^\mu)\dot{I}_\mu\},$$

of the generalized Gibbs equation (4) was proposed, in principle, in basis to mathematical requirements only. Later, we found on the one hand the meaning of the α_{ij} and their implication — causality — on the transport equations and, on the other hand, we had observed that the α_{ij} cannot vanish since in that case the correlations $\Lambda^\mu{}_\nu[\delta ...]$ would vanish also, which is against physical sense. Moreover, on physical grounds, we know that the relaxation proper times, although generally small, are finite and hence the α_{ij} cannot be set to zero.

To summarize, we se that the τ_{ij} enter not only in the macroscopic description of the irreversible processes (transport equations), but also in the mesoscopic one (fluctuations). Between both of them it exists a bridge, namely the fluctuation-dissipation theorem, whose relativistic version for thermoelectric fluids has been analysed here in a relaxation time approximation.

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