Erik Balslev
Erik Skibsted

Boundedness of two- and three-body resonances


<http://www.numdam.org/item?id=AIHPA_1985__43_4_369_0>
Boundedness of two- and three-body resonances

by

Erik BALSLEV and Erik SKIBSTED
Aarhus Universitet, Matematisk Institut,
Ny Munkegade, DK-8000, Aarhus C, Danmark

Abstract. — We prove that the set of dilation-analytic resonances is bounded for two-body Schrödinger operators with dilation-analytic multiplicative potentials having short-range boundary values.

For the three-body problem with dilation-analytic multiplicative short-range interactions we prove that resonances are bounded in any strip between cuts associated with consecutive two-body thresholds. For pair potentials going to zero faster than $r^{-2}$ at $\infty$ an upper bound on the real part of resonance energies is obtained.

Résumé. — On démontre que l’ensemble des résonances, définies par la méthode d’analyticité par rapport aux dilatations, est borné pour les opérateurs de Schrödinger à deux corps avec des potentiels multiplicative analytiques, à valeurs au bord de courte portée.

Pour le problème à trois corps avec des potentiels à deux corps multiplicative, analytiques, à courte portée, on démontre que l’ensemble des résonances est borné dans chaque bande entre deux coupures associées aux seuils, soit des valeurs propres, soit des résonances des problèmes à deux corps associés. Pour des potentiels décroissant plus vite que $r^{-2}$ à l’infini les énergies de résonance sont bornées.

Resonances are defined in the dilation-analytic theory as discrete eigenvalues of a complex-dilated Hamiltonian [6], [13]. In the two-body case and in the many-body case above thresholds it is easy to show that the set of resonances is bounded in any angle smaller than the maximal sector.
defined by the potential. This leaves the question open, whether the resonances remain bounded up to the limiting half line \( e^{-2i\omega R^+} \) (\( S_{\alpha} \) is the analyticity sector of the potential) and between cuts in the many-body case.

In the two-body case examples of the type \( \gamma r^\alpha e^{-r^\beta} \), \( \alpha > 0 \) are given in [14], which show that the set of dilation-analytic resonances may be bounded \( (-2 < \beta < -1) \) or unbounded \( (\beta = 1, \alpha = 2) \). These examples suggest that if the boundary operator \( V(ia) \) is defined by a short-range potential \( \sim e^{-2\pi r/\varepsilon} \), then the set of dilation-analytic resonances is bounded. This is in fact proved in Theorem 2.1 for dilation-analytic, multiplicative potentials with \( \Delta \)-form-compact, short range boundary values.

In the many-body case boundedness of resonances in strips parallel to and at a positive distance from cuts was proved in [3] for pair potentials going to 0 at \( \infty \) faster than \( r^{-\varepsilon} \), \( \varepsilon > 0 \). Here we restrict the discussion to the three-body problem, where resonances are identical with singular points of the analytically continued Faddeev matrix \( A(z, \zeta) \) \( (z \) dilation parameter, \( \zeta \) energy variable) [4]. It is further shown in [4] that for short-range potentials \( A(z, \zeta) \) has continuous boundary values \( A_\pm(z, \zeta) \) in a suitably weighted \( L^2 \)-space—on the cuts associated with two-body thresholds. This together with the two-body result suggests that three-body resonances are bounded in any closed strip bounded by two consecutive cuts (except the one starting from 0) for dilation-analytic, multiplicative, \( \Delta \)-form-compact, short range pair potentials. This is precisely our main result, formulated in Theorem 3.1. It is proved by showing that \( ||A^2(z, \zeta)|| \) \( \to 0 \) for \( \zeta \to \infty \) within any such strip in a certain weighted \( L^2 \)-space. This result could also be formulated so as to include the boundary values \( A_\pm(z, \zeta) \), and hence Theorem 3.1 includes any resonances that might lie on the cuts bounding the given strip (Remark 4.7). This is useful for a possible extension of the local scattering theory of [4] to a global theory. We also note that, under a weak additional condition on the potentials, the estimate of \( ||A^2(z, \zeta)|| \) is uniform in the dilation angle \( \phi \), away from 0 and \( \pi \) (Theorems 4.2 and 4.4).

For pair potentials going to zero faster than \( r^{-2-\varepsilon} \) at \( \infty \) stronger results are obtained. The zero-channel is included (Theorem 5.1), and the uniform estimates hold for \( \phi \) up to 0 (Theorem 5.6). This means that there are no three-body resonances above a certain energy. For this result we have to assume that the two-body systems have no zero-energy resonance or positive energy bound states. Related results for such potentials are given in [15].

We emphasize, that the results of Theorems 2.1 and 3.1 are concerned only with dilation-analytic resonances. The set of resonances, which may
be studied by other methods of analytic continuation, may be unbounded. In the two-body case it is actually proved [14] for the potentials \( \gamma r^\beta e^{-\alpha r} \), \(-2 < \beta < -1, \alpha > 1\), that the resonances converge to \( \infty \) along \( e^{-i \theta} \mathbb{R}^+ \) outside the dilation-analytic sector. In the three-body case with dilation-analytic potentials decaying faster than any exponential, the Faddeev matrix and the resolvent have analytic continuations to a much larger Riemann surface, extending to \( \infty \) on all sheets, than the one defined by dilation-analyticity [5], leaving the possibility open of an unbounded set of resonances on this Riemann surface. It is tempting to conjecture that for a three-body problem with the above Gaussian type of pair potentials, resonances are unbounded and converge from the outside to the limiting dilation cuts.

1. DEFINITIONS AND NOTATIONS

For \( t \in \mathbb{R} \) we introduce the following Hilbert spaces and Banach spaces of complex-valued functions on \( \mathbb{R}^3 \):

\[
\mathcal{H}_t = L^2_{1/2}(\mathbb{R}^3) = \left\{ g \mid \| g \|^{1/2}_t = \int_{\mathbb{R}^3} |g(x)|^2(1 + |x|)^t dx < \infty \right\}; \quad \mathcal{H} = \mathcal{H}_0
\]

\[
L^1_t(\mathbb{R}^3) = \left\{ g \mid \| g \|^{1/1}_t = \int_{\mathbb{R}^3} |g(x)|((1 + |x|)^t dx < \infty \right\}; \quad L^1(\mathbb{R}^3) = L^1(\mathbb{R}^3)
\]

\[
L^\infty_t(\mathbb{R}^3) = \left\{ g \mid \| g \|^{1/\infty}_t = \text{ess sup}_{x \in \mathbb{R}^3} |g(x)(1 + |x|)^t| < \infty \right\}; \quad L^\infty(\mathbb{R}^3) = L^\infty(\mathbb{R}^3)
\]

\[
R = \left\{ g \mid \| g \|^{1/\infty}_R = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |g(x)| |g(y)| |x - y|^{-2} dxdy < \infty \right\}
\]

\[
H^m(\mathbb{R}^3) = \left\{ g \mid \| g \|^{1/m}_{H^m} = \sum_{|x| \leq m} \| D_x^2 g \|^{1/2} < \infty \right\}, \quad m = 1, 2.
\]

The Sobolev space \( H^{-1}(\mathbb{R}^3) \) of order \(-1\) is the dual of \( H^1(\mathbb{R}^3) \). The weight function \( f_s \) is defined for \( s > 1 \) by

\[
f_s(x) = (1 + |x|)^{-s}, \quad x \in \mathbb{R}^3.
\]

For \( s > 1 \) and \( p > \frac{3}{2} \) we consider the function spaces

\[
M_s = R \cap L^1_{-s} + L^\infty_s
\]

\[
M^p_s = L^p \cap L^1_{-s} + L^\infty_s
\]

We note that \( R \subseteq L^1_{-s} \) for \( s \geq 3 \) and \( M^p_s \subseteq M_s \).

We shall make use of the following property of functions in \( M_s \) and \( M^p_s \).
LEMMA 1.1. — Let $V \in \mathcal{M}_s$. Then for every $\delta > 0$ there exist $V_1^\delta \in \mathbb{R} \cap L^{1-s}_1$ and $V_2^\delta \in L^{\infty}_s$ such that

$$V = V_1^\delta + V_2^\delta, \quad \|V_1^\delta\|_{\mathbb{R}} < \delta, \quad \|V_1^\delta\|_{L^{1-s}_1} < \delta.$$ 

Let $V \in \mathcal{M}^p_s$. Then for every $\delta > 0$ there exist $V_1^\delta \in L^p \cap L^s_1$ and $V_2^\delta \in L^{\infty}_s$ such that

$$V = V_1^\delta + V_2^\delta, \quad \|V_1^\delta\|_p < \delta, \quad \|V_1^\delta\|_{L^{1,s}_1} < \delta.$$ 

Proof. — For a given decomposition $V = V_1 + V_2$ let

$$V_{1,n} = \chi_{\{|x| \leq n\}} \cdot \chi_{\{|x| \leq n\}} V_1.$$ 

If $V_1 \in \mathbb{R}(L^{1-s}_1, L^p, L^s_1)$, by Lebesgue's dominated convergence theorem $V_{1,n} \to V_1$ in $\mathbb{R}(L^{1-s}_1, L^p, L^s_1)$, and the Lemma follows.

By $\{ U(\rho) | \rho \in \mathbb{R}^+ \}$ we mean the dilation group on $A$. We fix throughout this paper $a$, $0 < a \leq \frac{\pi}{2}$. It is emphasized that the restriction $a \leq \frac{\pi}{2}$ is only for convenience, the results can be extended to $a > \frac{\pi}{2}$. By $S_a$ we mean $\{ z \neq 0 | \text{Arg} z < a \}$, $\overline{S}_a = \{ z \neq 0 | | \text{Arg} z | \leq a \}$.

In the two-body case the masses of the two particles (1 and 2) are denoted by $m_1$ and $m_2$ and their reduced mass by $\mu = (m_1^{-1} + m_2^{-1})^{-1}$. Then the free Hamiltonian in the center-of-mass system is $h_0 = -\frac{\Delta}{2\mu}$, and for $V \in \mathbb{R} + L^{\infty}(\mathbb{R}^3)$ the total Hamiltonian $h$ is constructed in a standard way [12] [13] using the closed quadratic form

$$\int_{\mathbb{R}^3} (2\mu)^{-1} |\nabla g|^2 + V |g|^2, \quad g \in H^1(\mathbb{R}^3).$$

It is assumed $V$ is $\overline{S}_a$-dilation-analytic [13], i.e. the $\mathbb{C}(H^1, H^{-1})$-valued function $V(\rho) = U(\rho)VU(\rho^{-1})$, $\rho \in \mathbb{R}^+$, has a continuous extension $V(z)$ to $\overline{S}_a$ such that $V(z)$ is analytic in $S_a$. Furthermore we assume that $V(e^{i\theta}) \in \mathcal{M}_s$ for some $s > 1$. For the construction of the dilated Hamiltonians $h(z) = z^{-2}h_0 + V(z), z \in \overline{S}_a$, we refer to [13]. The essential spectrum of $h(z)$ is $z^{-2}\mathbb{R}^+$ and the non-real, discrete spectrum $r_0$ of $h(z)$ ($z = \rho e^{i\theta}$) lies in the sector between $\mathbb{R}^+$ and $e^{-2i\phi}\mathbb{R}^+$ and is otherwise $z$-independent [6] [13].

It is easy to see that $r_a = \bigcup_{0 < \phi < a} r_\phi$.

Since the result and all proofs in the two-body case are concerned with...
a fixed operator $h(e^{i\alpha})$, we omit for simplicity of notation the variable $e^{i\alpha}$, making the following change of notation.

$$-e^{-2i\alpha} \frac{\Delta}{2\mu} \to h_0, \quad V(e^{i\alpha}) \to V, \quad -e^{-2i\alpha} \frac{\Delta}{2\mu} + V(e^{i\alpha}) \to h,$$

$$\left(-e^{-2i\alpha} \frac{\Delta}{2\mu} - \zeta\right)^{-1} \to r_0(\zeta) \quad \text{for} \quad \zeta \notin e^{-2i\alpha} \mathbb{R}^+, \quad \left(-e^{-2i\alpha} \frac{\Delta}{2\mu} + V(e^{i\alpha}) - \zeta\right)^{-1} \to r(\zeta) \quad \text{for} \quad \zeta \notin e^{-2i\alpha} \mathbb{R}^+ \cup r_a \cup \sigma(h(1)).$$

If we need to specify the variable $\varphi$, we write $V(\varphi)$ for $V(e^{i\varphi})$.

We factorize $V$ as $V = AB$, where $A = |V|^{\frac{1}{2}} \text{sign } V$, $B = |V|^{\frac{1}{2}}$.

The symmetrized resolvent equation is

$$r(\zeta) = r_0(\zeta) - r_0(\zeta)A(1 + Br_0(\zeta)A)^{-1}Br_0(\zeta) \quad (1.1)$$

valid for all $\zeta$ such that $(1 + Br_0(\zeta)A)^{-1}$ exists. It is easy to see that if $V \in M_\phi$, this is the case precisely for $\zeta \in \rho(h)$. In fact, the map $\phi \to B\phi$ is an isomorphism of $\mathcal{N}(h - \zeta)$ onto $\mathcal{N}(1 + Br_0(\zeta)A)$ ($Br_0(\zeta)A \in \mathcal{B}(h)$). We shall return to the operator-function $Br_0(\zeta)A$ in Section 2, where it is proved that for $V \in M_\phi$, $||Br_0(\zeta)A||_{\mathcal{B}(h)} \to 0$ for $\zeta \to \infty$, $\zeta \notin e^{-2i\alpha} \mathbb{R}^+$, implying boundedness of $r_a$.

For the 3-body problem we use the following standard notations. Let particles 1, 2, 3 have masses $m_1$, $m_2$, $m_3$ and denote pairs $(i, j)$ by $\alpha$, $\beta$, etc. If $\alpha = (1, 2)$, for example,

$$m_1x_1 + m_2x_2 = m_3x_3 - \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.$$  

The conjugate momenta are denoted by $p_1$, $p_2$.

Note that for $\alpha \neq \beta$, the change of variables is given by

$$\begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix} = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} \quad (1.2)$$

where $t_2 = \pm 1$ and $t_i = t_4(\alpha, \beta) \neq 0$ for $i = 1, 3, 4$.

$\mathcal{H} = H^0(\mathbb{R}^6) = L^2(\mathbb{R}^6)$ and $H^2(\mathbb{R}^6)$ are the Sobolev-spaces of order 0 and 2 respectively.

The free Hamiltonian $H_0$ is given for each $\alpha$ by the operator $-\frac{\Delta x_\alpha}{2m_\alpha} - \frac{\Delta y_\alpha}{2n_\alpha}$ on the domain $H^2(\mathbb{R}^6)$.

The pair potentials $V_\alpha = V_\alpha(x_\alpha)$ are assumed to be real-valued functions in $M^p_\phi$ for some $p > \frac{3}{2}$, $s > 1$ and $S_\alpha$-dilation-analytic for some $a \leq \frac{\pi}{2}$.

Moreover, we assume $V_\alpha(e^{i\varphi}) \in M^p_\phi$ for every $\varphi \in (-a, a)$.  

Vol. 43, n° 4-1985.
The Hamiltonian $H$ is defined (cf. [13]) through the quadratic form

$$H[F] = \int_{\mathbb{R}^6} \left( \frac{|\nabla x_a F|^2}{2m_x} + \frac{|\nabla y_a F|^2}{2n_x} + \sum_a V_a(0) |F|^2 \right) dx_a dy_a$$

and similarly for the operators $H_x = H_0 + V_x$.

$\{U(\rho)\}_{\rho \in \mathbb{R}^+}$ now denotes the dilation group on $H$ and the dilated operators $H(\rho e^{i\omega})$ and $H_0(\rho e^{i\omega})$ are defined as in [9].

As in the 2-body case we then fix $z = e^{i\varphi}$, $0 < \varphi < \pi$ and omit the variable $\varphi$, using the short-hand notation

$$-e^{-2i\varphi} \frac{(\Delta x_a + \Delta y_a)}{2m_x} \to H_0, \quad V_x(\rho e^{i\omega}) \to V_x, \quad -e^{-2i\varphi} \frac{(\Delta x_a + \Delta y_a)}{2m_x} \to H_x,$$

$$-e^{-2i\varphi} \frac{\Delta x_a}{2m_x} + V_x(\rho e^{i\omega}) \to h_a, \quad R_0(\zeta) = (H_0 - \zeta)^{-1}, \quad R_a(\zeta) = (H_x - \zeta)^{-1}, \quad r_a(\zeta) = (h_a - \zeta)^{-1}$$

for $\zeta$ in the resolvent sets of the various operators, indicated above.

To further simplify the presentation we assume that each two-body operator has exactly one 1-dimensional, negative eigenvalue $\lambda_a$. The extension to the general case is straightforward (cf. [3]), the basic estimates are the same. The eigenfunction of $h_a$ corresponding to $\lambda_a$ is taken to be $
abla \phi_a = \phi_a(\varphi)$, where $\phi_a(0)$ is a real normalized eigenfunction of $-\frac{\Delta x_a}{2m_x} + V_x(0)$ associated with the eigenvalue $\lambda_a$.

We let $\mathcal{H}_a = \mathcal{L}_2(\mathbb{R}^3_{y_a}), \mathcal{H}_{a,s} = \mathcal{L}_2(s,\mathbb{R}^3_{y_a})$.

The relative free Hamiltonian $\tilde{h}_{a0} = -e^{-2i\varphi} \frac{\Delta y_a}{2n_x} + \lambda_a$ in $\mathcal{H}_a$ has resolvent

$$\tilde{r}_a(\zeta) = (\tilde{h}_{a0} - \zeta)^{-1} \in \mathcal{B}(\mathcal{H}_a) \text{ for } \zeta \neq \lambda_a + e^{-2i\varphi} \mathbb{R}^+.$$

The essential spectrum of $H$ is

$$\bigcup_{\lambda} \{ \lambda + e^{-2i\varphi} \mathbb{R}^+ \},$$

where $\lambda$ ranges over zero and all discrete eigenvalues and resonances of the two-body operators. The non-real discrete spectrum of $H$, denoted by $\mathcal{R}_\varphi$, is confined between the half-lines $\{ \lambda_c + \mathbb{R}^+ \}$ and $\{ \lambda_c + e^{-2i\varphi} \mathbb{R}^+ \}$, where $\lambda_c$ is the smallest negative threshold.

To prove boundedness of $\mathcal{R}_\varphi$ along cuts associated with two-body resonances we have to make the restrictive assumption that these two-body resonances are simple poles of the corresponding resolvents. We shall call such poles simple resonances.

The set $\mathcal{R}_\varphi$ is identical with the set of singular points of the symmetrized
Faddeev matrix $A(\zeta)$, defined as follows (cf. [4]). Here we restrict the discussion to the negative eigenvalues $\lambda_a$. The case of thresholds defined by simple resonances is very similar.

We decompose $R_a(\zeta)$ as

$$R_a(\zeta) = R_a^0(\zeta) + |\phi_a| < \tilde{r}_a(\zeta) < |\phi_a|$$

where

$$\langle \chi | g(y) \rangle = \int_{\mathbb{R}^3_x} \chi(x)g(x, y)dx \quad \text{for} \quad g \in \mathcal{H}, \quad \chi \in L^2(\mathbb{R}^3_x).$$

$$\langle | \chi \rangle | \sigma \rangle(x, y) = \chi(x, y) \sigma(y) \quad \text{for} \quad \chi \in L^2(\mathbb{R}^3_x), \quad \sigma \in \mathcal{H}_x.$$

We factorize $V_a$ as $V_a = A_aB_a, A_a = |V_a|^{1/2} \text{sign} V_a, B_a = |V_a|^{1}$. Set for $\alpha \neq \beta$

$$Q_{ab}(\zeta) = \left\{ \begin{array}{c}
B_aR^0_b(\zeta)A_{\beta} \\
\tilde{r}_a(\zeta) < |\phi_a| < V_aR^0_b(\zeta)A_{\beta} \\
|\tilde{r}_a(\zeta) < |\phi_a| < V_a| \phi_\beta| > \end{array} \right\} \quad (1.3)$$

$$A(\zeta) = \left[ \begin{array}{cc}
Q_{ab}(\zeta) & \bar{Q}_{x\gamma}(\zeta) \\
Q_{x\gamma}(\zeta) & \bar{Q}_{x\gamma}(\zeta)
\end{array} \right] \quad (1.4)$$

We introduce the auxiliary spaces $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}_{-s}$ defined by

$$\tilde{\mathcal{H}} = \sum_{\alpha} (\mathcal{H} \oplus \mathcal{H}_\alpha), \quad \tilde{\mathcal{H}}_{-s} = \sum_{\alpha} (\mathcal{H} \oplus \mathcal{H}_{\alpha,-s})$$

It will be proved in Lemmas 3.3-3.6 that if $V_a \in M^p$ then $A(\zeta) \in \mathcal{B}(\tilde{\mathcal{H}})$ for $\zeta \in \mathbb{C} \setminus \sigma_a(H)$. Using this we shall now sketch a proof of the fact that $\mathcal{H}_\varphi$ is the set of singular points of $1 + A(\zeta)$.

**Lemma 1.2.** — Assume that $V_a \in M^p$ for some $p > \frac{3}{2}, s > 1$ and all $\alpha$, and let $\zeta \in \mathbb{C} \setminus \sigma_a(H)$. Then $\mathcal{N}(H - \zeta)$ and $\mathcal{N}(1 + A(\zeta))$ ($A(\zeta) \in \mathcal{B}(\tilde{\mathcal{H}})$) are isomorphic via the maps

$$\mathcal{N}(H - \zeta) \ni \psi \rightarrow \mathcal{L}\psi = \sum_{\alpha} (u_\alpha \oplus \sigma_\alpha) \in \mathcal{N}(1 + A(\zeta)),$$

where

$$u_\alpha = B_\alpha(1 + R_\alpha(\zeta)V_\alpha)\psi, \quad \sigma_\alpha = \tilde{r}_a(\zeta) < |\phi_a| \cdot A_\alpha u_\alpha$$

and

$$\mathcal{N}(1 + A(\zeta)) \ni \Phi = \sum_{\alpha} (u_\alpha \oplus \sigma_\alpha) \rightarrow \psi = K\Phi \in \mathcal{N}(H - \zeta)$$

where

$$K\Phi = - \sum_{\alpha} R_\alpha(\zeta)A_\alpha u_\alpha.$$

Vol. 43, n° 4-1985.
It is easy to see that $L$ maps $H^1$ into $\tilde{H}$ and $K$ maps $\tilde{H}$ into $H^1$. The algebraic verification that $L\tilde{\psi} \in \mathcal{N}(1 + A(\zeta))$ and $K\Phi \in \mathcal{N}(H - \zeta)$ as well as $KL\psi = \psi$ and $LK\Phi = \Phi$ is carried out in [4].

For $V_\delta \in M^p_\delta$ we denote by $V^\delta_{s_1}$ and $V^\delta_{s_2}$ functions chosen in accordance with Lemma 1.1, such that

$$V_\delta = V^\delta_{s_1} + V^\delta_{s_2}, \quad \|V^\delta_{s_1}\|_{L^p} < \delta, \quad \|V^\delta_{s_2}\|_{L^1} < \delta, \quad V^\delta_{s_2} \in L^\infty_\delta.$$ 

For $\varepsilon > 0$, $0 < \phi < \alpha$, we define the half-planes $S_\varepsilon$ and $S_\varepsilon^-$ by

$$S_\varepsilon = \{ \zeta = s + e^{-2i\phi t} \mid s \geq \varepsilon, t \in \mathbb{R} \}$$
$$S_\varepsilon^- = \{ \zeta = s + e^{-2i\phi t} \mid s \leq -\varepsilon, t \in \mathbb{R} \}$$

We set $R_{\phi, -\varepsilon} = R_{\phi} \cap S_{-\varepsilon}$, $R_{\phi, \varepsilon} = R_{\phi} \cap S_\varepsilon$ for $\varepsilon > 0$, $0 < \phi < \alpha$.

In the 3-body case we need the following assumption on the 2-body eigenfunctions $\phi$ and resonance functions $\psi_\alpha$. We formulate this as a condition on the two-body system with potential $V$ for any eigenfunction $\phi$ associated with a discrete eigenvalue.

**CONDITION A.** — For some $C$, $k > 0$

$$\sup_{x \in \mathbb{R}^3} | e^{k|x|} \phi(x) | < C$$

If $V$ is $\Delta$-compact, condition A is always satisfied. In this case $\phi \in H^2(\mathbb{R}^3)$, and the standard boost-analytic argument yields $e^{k|x|} \phi \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ for $k$ small, positive. Since $V \in M^p_\delta$ for some $p \geq 2$, $s > 1$ implies $V \in L^2(\mathbb{R}^3) + L^\infty_\delta(\mathbb{R}^3)$, we have

(i) If $V \in M^p_\delta$ for some $p \geq 2$, then A is satisfied for all discrete eigenfunctions and resonance functions.

In the case of negative eigenvalues, let $I$ be a closed interval contained in $(-\alpha, \alpha)$; then clearly $k = k_I$ and $C = C_I$ can be chosen such that

$$\sup_{\phi \in I} \sup_{x \in \mathbb{R}^3} | e^{k_I|x|} \phi(\phi, x) | < C_I$$

where $\phi(\phi, x)$ is the dilation-analytic extension of the eigenfunction $\phi$. The same holds with suitably chosen $I$ in the case of resonance thresholds.

If $V$ is only known to be in $M^p_\delta$ for some $p$, $\frac{3}{2} < p < 2$, $s > 1$, then $V$ is $\Delta$-form-compact, $\phi \in H^1(\mathbb{R}^3)$, and it is not known whether A holds in general. For radial potentials, however, one can prove the following result using ordinary differential equations techniques.

(ii) If $V$ is radial and $V \in M^p_\delta$ for some $p$, $\frac{3}{2} < p < 2$, $s > 1$, then every discrete eigenfunction $\phi$ satisfies condition A. Moreover, a uniform estimate as above can be obtained for the dilation-analytic extension $\phi(\phi)$ of $\phi$. Similar results hold for resonance functions.
For non-radial potentials the following condition is given in [11], vol. IV, p. 200:

(iii) If $\tilde{V} \in L^1 + L^d$ for some $q$, $2 < q < 3$, then $A$ is satisfied for all discrete eigenfunctions. We note that the proof also works for resonance functions.

2. THE TWO-BODY CASE

The main result is the following

**Theorem 2.1.** — Suppose $V = V(a) \in M_s$ for some $s > 1$. Then the set $r_a$ is bounded.

We introduce four Lemmas and then prove Theorem 2.1.

**Lemma 2.2.** — For every $s > 1$ there exists $K = K(s)$, such that

$$
\| f_s^\perp r_0(\zeta) f_s^\perp \|_{\mathcal{H}(a)} < K |\zeta|^{-\frac{1}{2}} \quad \text{for} \quad |\zeta| > 1, \quad \zeta \notin e^{-2i\varphi_R/2}.
$$

**Proof.** — This follows from [11], III, p. 443, keeping track of the $\zeta$-dependence of the various constants at all stages of the proof of this Lemma.

**Lemma 2.3.** — For every $s > 1$ there exists $K < \infty$ such that

$$
\sup_{y \in \mathbb{R}^d} \left\{ (1 + |y|)^{s-1} \int_{\mathbb{R}^d} |y - x|^{-2} f(x) dx \right\} = K^2_s
$$

**Proof.** — A straightforward exercise using the decomposition

$$
\int_{\mathbb{R}^d} = \int_{|x| < |y|/2} + \int_{|x| > |y|/2}
$$

**Lemma 2.4.** — Suppose $f, g \in \mathbb{R}$. Then

$$
\int_{\mathbb{R}^6} |f(x)g(y)| |x - y|^{-2} dx dy
\leq \left( \int_{\mathbb{R}^d} |f(x)g(y)| |x - y|^{-2} dx dy \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |f(x)g(y)| |x - y|^{-2} dx dy \right)^{\frac{1}{2}}.
$$

**Proof.** — See [12] (1.10, p. 14).

**Lemma 2.5.** — Suppose $V \in M_s$, $s > 1$. Then

$$
\| Br_0(\zeta) A \|_{\mathcal{H}(a)} \to 0 \quad \text{for} \quad \zeta \to \infty, \quad \zeta \notin e^{-2i\varphi_R/2}.
$$

**Proof.** — Let $\epsilon > 0$ be given. Since

$$
|V|^{\frac{1}{2}} = |V_1^{\delta} + V_2^{\delta}|^{\frac{1}{2}} \leq |V_1^{\delta}|^{\frac{1}{2}} + |V_2^{\delta}|^{\frac{1}{2}} \quad \text{for every} \quad \delta > 0,
$$

Vol. 43, n° 4-1985.
we have for all $\delta_1, \delta_2 > 0$

$$
\| Br_0(\zeta)A \| = \| Br_0(\zeta)B \|
\leq \| V_{11}^{\delta_1} r_0(\zeta) V_{11}^{\delta_1} \| + \| V_{12}^{\delta_1} r_0(\zeta) V_{12}^{\delta_1} \| + \| V_{21}^{\delta_2} r_0(\zeta) V_{21}^{\delta_2} \| + \| V_{22}^{\delta_2} r_0(\zeta) V_{22}^{\delta_2} \| \quad (2.1)
$$

We now estimate each of the four terms on the right hand side of (2.1), where $\delta_1$ and $\delta_2$ have to be chosen successively. First choose $\delta_1 > 0$ such that $(K_s$ given as in Lemma 2.3)

$$
\delta_1^2 < \varepsilon \pi \min \{ \| V_{11}^{\delta_1} \|^{-\frac{1}{2}}, K_s^{-1} \| V_{22}^{\delta_1} \|^{-\frac{1}{2}} \}. \quad (2.2)
$$

Then, because $\| V_{11}^{\delta_1} \| L^\infty \leq \| V_{11}^{\delta_1} \|^{\delta_1}, \| V_{11}^{\delta_1} \|_{L^1} < \delta_1^2$, we get using at the last step Lemma 2.4

$$
\| V_{11}^{\delta_1} r_0(\zeta) V_{11}^{\delta_1} \| \leq \| V_{11}^{\delta_1} r_0(\zeta) V_{11}^{\delta_1} \|_{h.s.} \leq (4\pi)^{-1} \left\{ \int |V_{11}^{\delta_1}(x)| |V_{11}^{\delta_1}(y)| x - y |^{-2} \text{d}x \text{d}y \right\}^{\frac{1}{2}} \leq (4\pi)^{-1} \| V_{11}^{\delta_1} \| L^\infty \| V_{11}^{\delta_1} \|_{L^1} < \frac{\varepsilon}{4}
$$

Also

$$
\| V_{21}^{\delta_2} r_0(\zeta) V_{21}^{\delta_2} \| \leq \| V_{21}^{\delta_2} r_0(\zeta) V_{21}^{\delta_2} \|_{h.s.} \leq (4\pi)^{-1} \left\{ \int |V_{21}^{\delta_2}(y)(1 + | y |)^{-1-s} \right. \left. |V_{21}^{\delta_2}(x)f_s^{-1}(x) (1 + | y |)^{s-1} f_s(x) | x - y |^{-2} \text{d}x \text{d}y \right\}^{\frac{1}{2}} \leq (4\pi)^{-1} \| V_{21}^{\delta_2} \|_{L^1} \| V_{21}^{\delta_2} \|_{L^\infty} K_s \leq \frac{\varepsilon}{4} \quad (2.3)
$$

Now choose $\delta_2 > 0$ such that

$$
\delta_2^2 < \varepsilon \pi K_s^{-1} \| V_{22}^{\delta_2} \|^{-\frac{1}{2}}. \quad (2.4)
$$

Then

$$
\| V_{22}^{\delta_2} r_0(\zeta) V_{22}^{\delta_2} \| \leq \| V_{22}^{\delta_2} r_0(\zeta) V_{22}^{\delta_2} \|_{h.s.} \leq (4\pi)^{-1} \left\{ \int |V_{22}^{\delta_2}(y)(1 + | y |)^{-1-s} \right. \left. |V_{22}^{\delta_2}(x)f_s^{-1}(x) (1 + | y |)^{s-1} f_s(x) | x - y |^{-2} \text{d}x \text{d}y \right\}^{\frac{1}{2}} \leq (4\pi)^{-1} \| V_{22}^{\delta_2} \|_{L^1} \| V_{22}^{\delta_2} \|_{L^\infty} K_s \leq \frac{\varepsilon}{4}
$$

The estimates (2.2), (2.3) and (2.4) hold true for all $\zeta$. Finally, by Lemma 2.2 there exists $R_0 > 0$ such that

$$
\| f_s^2 r_0(\zeta)f_s^2 \| \leq \frac{\varepsilon}{4} \| V_{22}^{\delta_2} \|^{-\frac{1}{2}} \| V_{22}^{\delta_2} \|_{L^\infty} \quad \text{for} \quad |\zeta| > R_0
$$

Annales de l'Institut Henri Poincaré - Physique théorique
and hence
\[ \| \mathbf{V}_2 \|_{1,\infty,\sigma}^\frac{1}{2} \leq \| \mathbf{V}_2 \|_{\infty,\sigma}^\frac{1}{2} \cdot \| f^{\frac{1}{2}} r_0(\zeta) f^{\frac{1}{2}} \| \leq \frac{\varepsilon}{4} \text{ for } |\zeta| > R_0 \] (2.5)

By (2.1)-(2.5)
\[ \| \mathbf{B}r_0(\zeta)A \| \leq \varepsilon \text{ for } |\zeta| > R_0 \]
and the Lemma is proved.

Proof of Theorem 2.1. — By Lemma 2.5 there exists $R_0 > 0$ such that
\[ \| \mathbf{B}r_0(\zeta)A \| < 1 \text{ for } |\zeta| > R_0. \]
Consequently (cf. (1.1)) the set of resonances is confined to \{ |\zeta| \leq R_0 \}.

Remark 2.6. — If $V(z)$ is known to have an analytic continuation from $S_a$ to $S_b$ for some $b > a$, then Theorem 2.1 is well known [6].

Remark 2.7. — Suppose we drop the assumption that $V$ is $S_a$-dilation-analytic and only require that $V$ be $S_a$-dilation-analytic. Then Theorem 2.1 is not true in the sense that $r = \bigcup_{0 < \varphi < a} r_\varphi$ is bounded. For a counterexample we refer to [14].

Also an assumption like $V(ia) \in M_s$ seems necessary. For instance if it is only known that $V(ia) \in C(H^1, H^{-1})$, then we do not expect Theorem 2.1 to hold true. (We believe that $V(r) = r^\beta e^{-r^\alpha}$, $-\frac{1}{2} < \beta < 0$, $\frac{2}{3} \beta + \frac{4}{3} \alpha > 1$, represent counter-examples [14 (4.4)].

Remark 2.8. — In the case $V \in R$, Lemma 2.5 is well-known (cf. [10], p. 274-276 and [12], Theorem I.23). The Lemma implies that the first Born approximation is good in the high-energy limit, that is $(1 + \mathbf{B}r_0(\zeta)A)^{-1} \simeq 1 - \mathbf{B}r_0(\zeta)A$.

3. THE THREE-BODY CASE

We formulate the main result:

Theorem 3.1. — Assume that $V_\varphi = V_a(\varphi) \in M_\varphi^s$ for some $p > \frac{3}{2}$, $s > 1$, $|\varphi| < a$ and all $\varphi$. Let $\varepsilon > 0$ be given and suppose that every two-body eigenfunction $\phi_\varphi$ associated with an eigenvalue $\lambda_\varphi \leq -\varepsilon$ satisfies A. Then $\mathcal{R}_{\varphi, -\varepsilon}$ is bounded.

If all two-body resonances in $S_\varphi$ are simple and every two-body eigenfunction $\phi_\varphi$ associated with a positive eigenvalue $\lambda_\varphi \geq \varepsilon$ or a resonance $\lambda_\varphi \in S_\varphi$ satisfies $A$, then $\mathcal{R}_{\varphi, \varepsilon}$ is bounded.
The proof consists in showing that $\| A^2(\zeta) \| \to 0$ for $\zeta \to \infty$ in $S_{-\varepsilon}$ and $S_{\varepsilon}$ respectively. We make the simplifying assumption of Section 1, that each two-body system has exactly one 1-dimensional eigenvalue in $S_{-\varepsilon}$. The following Lemmas (to be proved later) contain the basic estimates of the operators $Q_{\alpha\beta}(\zeta)$ ($\alpha \neq \beta$) given by (1.3), constituting the Faddeev matrix (1.4). Using these Lemmas we shall prove Theorem 3.1, focusing on the case of $S_{-\varepsilon}$. The case of $S_{\varepsilon}$ is then briefly discussed.

Remark 3.2. — In the case of negative eigenvalues, the estimates of $\| A^2(\zeta) \|$ can be obtained uniformly for $\phi$ in a closed interval $I \subseteq (0, a)$, provided all $V_{\alpha}(\phi)$ have decompositions $V_{\alpha}(\phi) = V_{\alpha}(\phi^+) + V_{\alpha}(\phi^-)$ with $V_{\alpha}(\phi^-)$ a continuous $L^p$-valued ($L^1$-valued) and $V_{\alpha}(\phi)$ a continuous $L^\infty$-valued function on $(-a, a)$. We shall discuss this in Section 4. This extends in an obvious way to the case of simple resonance thresholds.

Lemma 3.3. — For $\zeta \in S_{-\varepsilon}$
\[ B_{2\alpha}R_0^0(\zeta)A_{\beta}, \quad B_{\beta}R_0^0(\zeta)A_{\alpha}, \quad B_{2\alpha}R_0(\zeta)A_{\beta} \in \mathcal{B}(\mathcal{H}) \]
and
\[ B_{2\alpha}R_0^0(\zeta)A_{\beta} = B_{2\alpha}R_0(\zeta)(1 - |\phi_{\beta} > < \phi_{\beta}|)A_{\beta} - B_{2\alpha}R_0(\zeta)A_{\beta} \cdot B_{\beta}R_0^0(\zeta)A_{\beta} \]
Moreover
1) $\| B_{2\alpha}R_0^0(\zeta)A_{\beta} \|_{\mathcal{B}(\mathcal{H})}, \quad \| B_{\beta}R_0^0(\zeta)A_{\beta} \|_{\mathcal{B}(\mathcal{H})} \to 0$ for $\zeta \to \infty$, $\zeta \in S_{-\varepsilon}$
2) $\| B_{\beta}R_0^0(\zeta)A_{\beta} \|_{\mathcal{B}(\mathcal{H})} < C < \infty$ for $\zeta \in S_{-\varepsilon}$.

Lemma 3.4. —
\[ B_{\alpha} | \phi_{\beta} > \in \mathcal{B}(\mathcal{H}_{\beta, -\varepsilon}, \mathcal{H}) \]

Lemma 3.5. — For $\zeta \in S_{-\varepsilon}$, $\zeta \notin \lambda + e^{-2i\phi R^+}$
\[ \tilde{r}_d(\zeta) < |\phi_{\alpha} | V_{\alpha} R_0^0(\zeta)A_{\beta} |_{\mathcal{B}(\mathcal{H}, \mathcal{H}_d)} \in \mathcal{B}(\mathcal{H}, \mathcal{H}_d) \]
Moreover
\[ \| \tilde{r}_d(\zeta) < |\phi_{\alpha} | V_{\alpha} R_0^0(\zeta)A_{\beta} \|_{\mathcal{B}(\mathcal{H}, \mathcal{H}_d)} \to 0 \text{ for } \zeta \to \infty \text{ in } S_{-\varepsilon}. \]

Lemma 3.6. — For $\zeta \in S_{-\varepsilon}$, $\zeta \notin \lambda + e^{-2i\phi R^+}$
\[ \tilde{r}_d(\zeta) < |\phi_{\alpha} | V_{\alpha} | \phi_{\beta} > \in \mathcal{B}(\mathcal{H}_{\beta}, \mathcal{H}_d) \cap \mathcal{B}(\mathcal{H}_{\beta, -\varepsilon}, \mathcal{H}_d, -\varepsilon) \]
Moreover
\[ \| \tilde{r}_d(\zeta) < |\phi_{\alpha} | V_{\alpha} | \phi_{\beta} > \|_{\mathcal{B}(\mathcal{H}_{\beta, -\varepsilon}, \mathcal{H}_d, -\varepsilon)} \to 0 \text{ for } \zeta \to \infty \text{ in } S_{-\varepsilon}. \]

Proof of Theorem 3.1. — By Lemma 1.2 for any resonance $\lambda$ there exists $0 \neq \Phi \in N(1 + A(\lambda))(A(\lambda) \in \mathcal{B}(\mathcal{H}))$. Then $\Phi \in \mathcal{H}_{-\varepsilon}$, and
\[ \Phi \in N(1 - A^2(\lambda))(A^2(\lambda) \in \mathcal{B}(\mathcal{H})) \]

Annales de l’Institut Henri Poincaré - Physique théorique
By Lemmas 3.3-3.6 there exists $R_0 > 0$ such that $\| A^2(\zeta) \|_{\mathcal{B}(\mathcal{H}_-, \mathcal{H}_+)} < 1$ for $\zeta \in S_{-\varepsilon}, |\zeta| > R_0$. Hence $1 - A^2(\zeta)$ is invertible and there are no resonances for $\zeta \in S_{-\varepsilon}, |\zeta| > R_0$.

In the general case, where each $h_a$ may have possibly infinitely many eigenvalues accumulating at 0, the proof is similar. For $\varepsilon > 0$ there is a finite number of finite-dimensional eigenvalues of the operators $h_a$ below $-\varepsilon$. For each $\alpha$ we choose an orthonormal basis of the total eigenspace with energies below $-\varepsilon$ and obtain a Faddeev-matrix containing as elements a finite number of operators of the type treated in Lemmas 3.3-3.6 (cf. [4] [5]). Also the case of $S_{-\varepsilon}$, where the cuts start from two-body resonances or positive eigenvalues, is treated in a very similar way. The assumption that all two-body resonances in $S_{\varepsilon}$ are simple is required to prove the analogues of Lemmas 3.3 and 3.5. It is clear from the proofs of Lemmas 3.3 and 3.5 (to be given later) why this assumption is necessary.

**Remark 3.7.** — It is not difficult to sharpen the results in Lemmas 3.3, 3.5, 3.6 as follows:

<table>
<thead>
<tr>
<th>Lemma</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td>$B_{\alpha}R_{\beta}(\zeta)A_{\beta} \in \mathcal{C}(\mathcal{H})$</td>
</tr>
<tr>
<td>3.5</td>
<td>$\tilde{r}<em>{\alpha}(\zeta) &lt; \phi</em>{\alpha} \mid V_{\alpha}R_{\beta}(\zeta)A_{\beta} \in \mathcal{C}(\mathcal{H}, \mathcal{H}_a)$</td>
</tr>
<tr>
<td>3.6</td>
<td>$\tilde{r}<em>{\alpha}(\zeta) &lt; \phi</em>{\alpha} \mid V_{\alpha} \mid \phi_{\beta} &gt; \in \mathcal{C}(\mathcal{H}_B, \mathcal{H}_a)$</td>
</tr>
</tbody>
</table>

Using these facts we find that $A^2(\zeta) \in \mathcal{C}(\mathcal{H})$ and hence the singularities of $(1 + A(\zeta))^{-1} = (1 - A(\zeta)(1 - A^2(\zeta))^{-1}$ are isolated poles in $\mathcal{C} \setminus \{ \lambda + e^{-2i\phi R^+} \}$ where $\lambda$ ranges over 0 and all discrete eigenvalues of two-body operators $h_{\alpha}(e^{i\sigma})$. This of course is well known [13].

For the purpose of proving Lemmas 3.3-3.6 we need the following Lemmas (3.8-3.16).

**Lemma 3.8.** — For every $s > 1$ there exists $K = K(s)$ such that

$$
\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}}r_0(\zeta)f_s^{\frac{1}{2}}(\cdot + y) \|_{\mathcal{B}(\mathcal{H})} < K |\zeta|^{-\frac{s}{2}} \quad \text{for} \quad |\zeta| > 1.
$$

**Proof.** — Compare with Lemma 2.2. We use the proofs of Lemmas 3, 4, 5 in [11], vol. III, p. 442.

**Lemma 3.9.** — Let $s > 1, p > \frac{3}{2}$ and $V_{\alpha} \in M_{s'}$ be given. Then

$$
\sup_{y \in \mathbb{R}^3} \| f_s^{\frac{1}{2}}r_0(\zeta)A_{\alpha}(\cdot + y) \|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{for} \quad \zeta \to \infty, \quad \zeta \notin e^{-2i\phi R^+}
$$

**Proof.** — Let $\varepsilon > 0$ be given. We choose $\delta > 0$ such that $\delta^{\frac{s}{2}} < 2\varepsilon \pi K^{-1}_s$ ($K_s$ defined in Lemma 2.3).
Since
\[ |A_\tau| = |V_\tau|^\frac{1}{2} = |V_{\tau1} + V_{\tau2}|^\frac{1}{2} \leq |V_{\tau1}|^\frac{1}{2} + |V_{\tau2}|^\frac{1}{2} \]
we have
\[ \|f_{\tau}^\delta r_0(\zeta)A_{\tau}(\cdot + y)\| \leq \|f_{\tau}^\delta r_0(\zeta)|V_{\tau1}(\cdot + y)|^\frac{1}{2}\| + \|f_{\tau}^\delta r_0(\zeta)|V_{\tau2}(\cdot + y)|^\frac{1}{2}\| \]
Because \(\|V_{\tau1}(\cdot + y)\|_{L^1(\mathbb{R}^3)} < \delta\), we have for all \(y \in \mathbb{R}^3\)
\[ \|f_{\tau}^\delta r_0(\zeta)|V_{\tau1}(\cdot + y)|^\frac{1}{2}\| \leq \|f_{\tau}^\delta r_0(\zeta)|V_{\tau1}(\cdot + y)|^\frac{1}{2}\|_{H.S.} \]
\[ \leq (4\pi)^{-1}\int_{\mathbb{R}^3} f_{\delta}(x)|V_{\tau1}(x + y)| |x - z|^{-\frac{3}{2}} dx dz \]
\[ \leq (4\pi)^{-1}K_\delta \|V_{\tau1}(\cdot + y)\|_{L^1(\mathbb{R}^3)} < (4\pi)^{-1}K_\delta \delta \leq \frac{\epsilon}{2} \]
Furthermore, by Lemma 3.8 there exists \(R_0 > 0\) such that
\[ \sup_{y \in \mathbb{R}^3} \|f_{\tau}^\delta r_0(\zeta)f_{\tau}^\delta(\cdot + y)\|_{H(\mathbb{R})} < \frac{\epsilon}{2} \|V_{\tau2}\|_{\frac{1}{2},\infty,s} \text{ for } |\zeta| > R_0 \]
Hence, if \(|\zeta| > R_0\) we have for all \(y \in \mathbb{R}^3\)
\[ \|f_{\tau}^\delta r_0(\zeta)|V_{\tau2}(\cdot + y)|^\frac{1}{2}\| \leq \|f_{\tau}^\delta r_0(\zeta)f_{\tau}^\delta(\cdot + y)\| \cdot \|V_{\tau2}\|_{\frac{1}{2},\infty,s} \leq \frac{\epsilon}{2} \]
By (3.1)-(3.3)
\[ \sup_{y \in \mathbb{R}^3} \|f_{\tau}^\delta r_0(\zeta)g(\cdot + y)\| < \epsilon \text{ for } |\zeta| > R_0, \]
and the Lemma is proved.

**Lemma 3.10.** — For \(\zeta \notin \{e^{-2i\phi \frac{p^2}{2m} + \frac{\lambda_x - \zeta}{2m_x}}\} \cup \{\lambda + e^{-2i\phi \frac{p^2}{2m} + \frac{\lambda_x - \zeta}{2m_x}}\} \)
\[ \tilde{r}_\tau(\zeta)R_0(\zeta) = \left( -e^{-2i\phi \frac{\Delta x}{2m_x} + \lambda_x} \right)^{-1} \left[ \tilde{r}_\tau(\zeta) - R_0(\zeta) \right]. \]

**Proof.** — In momentum representation
\[ \tilde{r}_\tau(\zeta)R_0(\zeta) = \left( e^{-2i\phi \frac{p^2}{2n_x} + \lambda_x - \zeta} \right)^{-1} \left( e^{-2i\phi \frac{k^2}{2m_x} + e^{-2i\phi \frac{p^2}{2n_x} + \zeta}} \right) \]
and
\[ \left( e^{-2i\phi \frac{k^2}{2m_x} + \lambda_x} \right)^{-1} [\tilde{r}_\tau(\zeta) - R_0(\zeta)] \]
\[ = \left( e^{-2i\phi \frac{k^2}{2m_x} + \lambda_x} \right)^{-1} \left[ \left( e^{-2i\phi \frac{p^2}{2n_x} + \lambda_x - \zeta} \right)^{-1} - \left( e^{-2i\phi \frac{k^2}{2m_x} + e^{-2i\phi \frac{p^2}{2n_x} - \zeta}} \right)^{-1} \right] \]
**Lemma 3.11.** Let $g \in L^2(\mathbb{R}^3_{\alpha \beta})$. Then

$$< g | \in \mathcal{B}(\mathcal{H}, \mathcal{K}_g) \quad \text{and} \quad \| < g | \|_{\mathcal{B}(\mathcal{H}, \mathcal{K}_g)} \leq \| g \|_{L^2(\mathbb{R}^3_{\alpha \beta})}$$

**Proof.** Fubini's theorem and Cauchy-Schwarz' inequality.

**Lemma 3.12** [8]. Assume that $V_\alpha \in L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for some $p > \frac{3}{2}$ and all $\alpha$. Then $B_\alpha R_\alpha(\zeta)A_\beta \in \mathcal{B}(\mathcal{H})$ and there exists $C_p < \infty$ (depending on $p$ and $m_i$) such that

$$\| B_\alpha R_\alpha(\zeta)A_\beta \| \leq C_p \max \{ \| V_\alpha \|_{L^p}, \| V_\beta \|_{L^p}, \| V_\alpha \|_{L^1}, \| V_\beta \|_{L^1} \}$$

for $\zeta \notin e^{-2i\phi_{R_{\mathbb{R}^3}}}$ and all pairs $\alpha, \beta$.

**Lemma 3.13.** Let $V_\alpha$ and $C_p$ be as in Lemma 3.12 and assume

$$\vartheta(\zeta) = | \text{Im} (e^{2i\phi_{\zeta}}) | > 0.$$ 

Then

$$\| B_\alpha R_\alpha(\zeta) \| \leq \varepsilon^{-\frac{1}{2}}(\zeta)C_p^\frac{1}{2} \max \{ \| V_\alpha \|_{L^p}, \| V_\alpha \|_{L^1} \}$$

**Proof.**

$$\| B_\alpha R_\alpha(\zeta) \| = \| B_\alpha R_\alpha(\zeta)(R_\alpha(\zeta))^*B_\alpha^* \|$$

$$= \left\| \frac{1}{2} \varepsilon^{-1}(\zeta)B_\alpha [R_\alpha(\zeta)(R_\alpha(\zeta))^*]B_\alpha^* \right\|$$

$$\leq \varepsilon^{-\frac{1}{2}}(\zeta) \| B_\alpha R_\alpha(\zeta)B_\alpha \|$$

$$\leq \varepsilon^{-\frac{1}{2}}(\zeta)C_p^\frac{1}{2} \max \{ \| V_\alpha \|_{L^p}, \| V_\alpha \|_{L^1} \}$$

where we have used Lemma 3.12.

**Lemma 3.14.** Let $\varepsilon > 0$, $s > 1$ and $\alpha \neq \beta$ be given. Then

$$\| (1 + | x_\lambda |)^{-s/2}R_\alpha(\zeta)(1 + | x_\beta |)^{-s/2} \| \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}.$$ 

**Proof.** By a result of Agmon [1] we have for some $b$ satisfying $\frac{1}{4} < b < \frac{1}{2}$ and a constant $C$ for $| \zeta | > 1$, $\zeta \notin e^{-2i\phi_{R_{\mathbb{R}^3}}}$,

$$\| (1 + | x_\lambda |)^{-s/2}R_\alpha(\zeta)(1 + | x_\beta |)^{-s/2} \| < C \| R_\alpha(\zeta)(1 + | x_\lambda |)^{-b}(1 + | x_\beta |)^{-s/2} \|.$$ 

Moreover, introducing $\vartheta(\zeta) = | \text{Im} (e^{2i\phi_{\zeta}}) |$

$$\| R_\alpha(\zeta)(1 + | x_\lambda |)^{-b}(1 + | x_\beta |)^{-s/2} \|

\leq \| R_\alpha(\zeta)(1 + | x_\lambda |)^{-2b}(1 + | x_\beta |)^{-s} \| \| R_\alpha(\zeta) \|$$

$$\leq \frac{1}{2} \varepsilon^{-1}(\zeta) \{ (1 + | x_\lambda |)^{-2b}(1 + | x_\beta |)^{-s} [(R_\alpha(\zeta))^* - R_\alpha(\zeta)] \}

(1 + | x_\lambda |)^{-2b}(1 + | x_\beta |)^{-s} \| R_\alpha(\zeta) \|$$

$$\leq \varepsilon^{-3/4}(\zeta) \| (1 + | x_\lambda |)^{-2b}(1 + | x_\beta |)^{-s} R_\alpha(\zeta)(1 + | x_\lambda |)^{-2b}(1 + | x_\beta |)^{-s} \|^{1/4}$$ 

(3.5)
Since \( \inf_{\zeta \in S_{-\kappa}} \varepsilon(\zeta) = \varepsilon \sin 2\varphi > 0 \) and \( 2b, s > \frac{1}{2} \), we conclude from (3.4) and (3.5), using Lemma 2.2 (with \( \mathbb{R}^3 \) replaced by \( \mathbb{R}^d \)) that the Lemma holds true.

**Lemma 3.15.** — Let \( \varepsilon > 0, p > \frac{3}{2}, s > 1 \) and \( \alpha \neq \beta \) be given. Suppose \( V_\alpha, V_\beta \in L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \).

Then \( B_\alpha R_0(\zeta)A_\beta \in \mathcal{B}(\mathcal{H}) \), and

\[
\| B_\alpha R_0(\zeta)A_\beta \| \to 0 \text{ for } \zeta \to \infty \text{ in } S_{-\kappa}.
\]

**Proof.** — Since

\[
B_\gamma = \| V_\gamma \| = \| V_{\gamma 1} \| + \| V_{\gamma 2} \| \leq \| V_{\gamma 1} \| + \| V_{\gamma 2} \|
\]

for all pairs \( \gamma \) and every \( \delta > 0 \), we have for all \( \delta_1, \delta_2 > 0 \),

\[
\| B_\alpha R_0(\zeta)A_\beta \| \leq \| V_{\alpha 1} \| \| V_{\beta 1} \| + \| V_{\alpha 2} \| \| V_{\beta 2} \|
\]

where \( \delta = \varepsilon \sin 2\varphi = \inf_{\zeta \in S_{-\kappa}} \varepsilon(\zeta) \) and \( C_p \) is given in Lemma 3.12.

Then by Lemma 3.12

\[
\| V_{\alpha 1} \| \| V_{\beta 1} \| \leq C_p \max \{ \| V_{\alpha 1} \|_{L^p}, \| V_{\beta 1} \|_{L^p}, \| V_{\alpha 2} \|_{L^1}, \| V_{\beta 2} \|_{L^1} \}
\]

\[
< \frac{C_p \delta_1^4}{4} \leq \frac{\varepsilon_0}{4}
\]

Also, by Lemma 3.13

\[
\| V_{\alpha 2} \| \| V_{\beta 2} \| \leq \| V_{\alpha 2} \| \| V_{\beta 2} \|_{L^\infty}
\]

\[
< \left( \frac{C_p \delta_1^4}{4} \varepsilon_1^{-\frac{1}{2}} \right)^{\frac{1}{2}} \leq \frac{\varepsilon_0}{4}
\]

Now choose \( \delta_2 > 0 \) such that

\[
\delta_2^4 < \frac{\varepsilon_0}{4} \varepsilon_1^{-\frac{1}{2}} C_p \delta_1^4
\]

Then by Lemma 3.13

\[
\| V_{\alpha 2} \| \| R_m(\zeta) \| \leq \| V_{\alpha 2} \| \| R_m(\zeta) \| \| \zeta \| \leq \frac{\varepsilon_0}{4}
\]

Annales de l'Institut Henri Poincaré - Physique théorique
Finally, by Lemma 3.14 there exists $R_0 > 0$ such that
\[ \| (1 + |x_\alpha|)^{-s/2} R_0(\zeta) (1 + |x_\beta|)^{-s/2} \| < \frac{\epsilon_0}{4} \| V_{x_2}^{\beta_1} \|_{\infty,s} \| V_{\beta_2}^{\beta_2} \|_{\infty,s} \]
for $\zeta \in S_{-\varepsilon}$ and $|\zeta| > R_0$.
Hence
\[ \| V_{x_2}^{\beta_1} R_0(\zeta) V_{\beta_2}^{\beta_2} \| \leq \| V_{x_2}^{\beta_1} \|_{\infty,s} \| V_{\beta_2}^{\beta_2} \|_{\infty,s} \| (1 + |x_2|)^{-s/2} R_0(\zeta) (1 + |x_\beta|)^{-s/2} \| < \frac{\epsilon_0}{4} \text{ for } \zeta \in S_{-\varepsilon} \text{ and } |\zeta| > R_0 \] (3.10)
By (3.6)-(3.10)
\[ \| B_\alpha R_0(\zeta) A_\beta \| < \epsilon_0 \text{ for } \zeta \in S_{-\varepsilon}, |\zeta| > R_0 \]
and the Lemma is proved.

**Lemma 3.16.** — For $\epsilon$, $\zeta$ and $R_0^0(\zeta)$ as in Lemma 3.3, $B_\alpha R_0^0(\zeta) A_\alpha \in \mathcal{B}(\mathcal{H})$ and there exists $K$ such that
\[ \| B_\alpha R_0^0(\zeta) A_\alpha \| < K \text{ for all } \zeta \in S_{-\varepsilon} \]

**Proof.** — By (1.1)
\[ B_\alpha r_\alpha(\zeta) A_\alpha = q(\zeta) - q(\zeta)(1 + q(\zeta))^{-1} q(\zeta) \] (3.11)
where $q(\zeta) = B_\alpha r_\alpha(\zeta) A_\alpha$, valid for all $\zeta$ such that $(1 + q(\zeta))^{-1}$ exists.
By Lemma 2.5,
\[ \| q(\zeta) \| \to 0 \text{ for } \zeta \to \infty \text{ in } S_{-\varepsilon} \] (3.12)
Together with the assumption that $h_\alpha$ has exactly one 1-dimensional eigenvalue $\lambda_\alpha < 0$, (3.11) and (3.12) imply that
\[ \| B_\alpha r_\alpha(\zeta) A_\alpha \| \leq C_1 < \infty \text{ for } \zeta \in S_{-\varepsilon} \setminus D \] (3.13)
where
\[ D = \left\{ \zeta \mid |\zeta - \lambda_\alpha| \leq \delta = \frac{\epsilon}{2} \sin 2\varphi \right\} . \]
We write $r_\alpha(\zeta)$ as
\[ r_\alpha(\zeta) = |\phi_\alpha > (\lambda_\alpha - \zeta)^{-1} < \phi_\alpha| + r_\alpha^0(\zeta) . \]
It is easy to see that the pole term of $r_\alpha(\zeta)$ satisfies
\[ \| B_\alpha |\phi_\alpha > (\lambda_\alpha - \zeta)^{-1} < \phi_\alpha| A_\alpha \| \leq C_2 < \infty \text{ for } \zeta \in S_{-\varepsilon} \setminus D \] (3.14)
For $|\zeta - \lambda_\alpha| < \frac{3\delta}{2}$, $B_\alpha r_\alpha^0(\zeta) A_\alpha$ is given by the norm-convergent integral
\[ B_\alpha r_\alpha^0(\zeta) A_\alpha = -\frac{1}{2\pi i} \int_{|\zeta - \lambda_\alpha| = \frac{3\delta}{2}} (\zeta - \zeta')^{-1} B_\alpha r_\alpha(\zeta') A_\alpha d\zeta' \] (3.15)
Hence $B_\alpha r^0_2(\zeta)A_\alpha$ is analytic for $|\zeta - \lambda_\alpha| < \frac{3\delta}{2}$. This implies that
\[
\| B_\alpha r^0_2(\zeta)A_\alpha \| \leq C_3 < \infty \quad \text{for} \quad \zeta \in D
\] (3.16)

By (3.13), (3.14) and (3.16)
\[
\| B_\alpha r^0_2(\zeta)A_\alpha \| \leq \max(C_1 + C_2, C_3) \quad \text{for} \quad \zeta \in S_{-\varepsilon}
\] (3.17)

In momentum representation
\[
R^0_2(\zeta) = r^0_2(\zeta - \frac{z^2 p^2_\alpha}{2n_\alpha})
\]
and hence for all $\zeta \in S_{-\varepsilon}$
\[
B_\alpha R^0_2(\zeta)A_\alpha = B_\alpha r^0_2(\zeta - \frac{z^2 p^2_\alpha}{2n_\alpha})A_\alpha.
\] (3.18)

The Lemma follows from (3.17) and (3.18).

We now proceed to the proof of Lemmas 3.3-3.6

**Proof of Lemma 3.3.** --- By the 2nd resolvent equation

\[
B_\alpha R^0_0(\zeta)A_\beta
= B_\alpha R^0_0(\zeta)(1 - |\phi_\beta\rangle\langle \phi_\beta|)A_\beta - B_\alpha R^0_0(\zeta)A_\beta \cdot B_\beta R^0_0(\zeta)A_\beta \quad \text{for} \quad \zeta \in S_{-\varepsilon}
\] (3.19)

Because $B_\alpha R^0_0(\zeta)A_\beta$, $B_\beta R^0_0(\zeta)A_\beta \in \mathcal{B}(\mathcal{H})$ by Lemmas 3.15 and 3.16, it follows from (3.19) that $B_\alpha R^0_0(\zeta)A_\beta \in \mathcal{B}(\mathcal{H})$. Furthermore, by Lemma 3.15
\[
\| B_\alpha R^0_0(\zeta)A_\beta \|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon}
\]
and by Lemma 3.16
\[
\| B_\beta R^0_0(\zeta)A_\beta \|_{\mathcal{B}(\mathcal{H})} < C \quad \text{for} \quad \zeta \in S_{-\varepsilon}
\]

Hence by (3.19), since $\phi_\beta$ satisfies condition A,
\[
\| B_\alpha R^0_0(\zeta)A_\beta \|_{\mathcal{B}(\mathcal{H})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\varepsilon},
\]
and the Lemma is proved.

**Proof of Lemma 3.4.** --- First we prove that $B_\beta |\phi_\beta \rangle \in \mathcal{B}(\mathcal{H})$. Let $g \in \mathcal{H}_\beta$. Then by (1.2) (assuming for simplicity $t_2 = 1$; $t_2 = -1$ is similar)
\[
\| B_\beta |\phi_\beta \rangle \|_{\mathcal{H}_\beta}^2 = \int dx_\beta dy_\beta |B_\beta(y_\beta + t_1 x_\beta)\phi_\beta(x_\beta)g(y_\beta)|^2 \leq
\]
\[
\leq \| g \|_{\mathcal{H}_\beta}^2 \sup_{y_\beta \in \mathcal{R}^3} \int dx_\beta |B_\beta(y_\beta + t_1 x_\beta)\phi_\beta(x_\beta)|^2 = K^2 \| g \|_{\mathcal{H}_\beta}^2
\]

Annales de l'Institut Henri Poincaré - Physique théorique
where $K^2 < \infty$ by Condition A. Hence $B_\sigma | \phi_\beta > \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H})$.

We shall now prove that $B_\sigma | \phi_\beta > \in \mathcal{B}(\mathcal{H}_{\mu,-s}, \mathcal{H})$. Let $g \in \mathcal{H}_{\mu,-s}$. Then

$$
\| B_\sigma | \phi_\beta > g \|^2 \leq \int dx_\beta dy_\beta | V_{a_1}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) g(y_\beta) |^2 \\
+ \int dx_\beta dy_\beta | V_{a_2}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) g(y_\beta) |^2 \\
\leq \| g \|_{\mathcal{H}_{\mu,-s}} \left\{ \sup_{x_\beta} \int dx_\beta | V_{a_1}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \right\} \\
+ \sup_{x_\beta} \int dx_\beta | V_{a_2}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \right\}
$$

(3.20)

We set

$$
C_\beta = \sup_{x_\beta,y_\beta} f_s(y_\beta + t_1 x_\beta) f_s^{-1}(y_\beta) | \phi_\beta(x_\beta) |^2.
$$

By Condition A, $C_\beta < \infty$ and hence

$$
\sup_{y_\beta} \int dx_\beta | V_{a_1}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \leq \\
\leq C_\beta \sup_{y_\beta} \int dx_\beta | V_{a_1}^1(y_\beta + t_1 x_\beta) | f_s^{-1}(y_\beta + t_1 x_\beta) \\
= C_\beta \| t_1 \|^{-3} | V_{a_1}^1 \|_{L_1(\mathbb{R}^3)} < C_\beta \| t_1 \|^{-3}
$$

(3.21)

Furthermore, we set

$$
C = \sup_{x_\beta,y_\beta} f_s^{-1}(y_\beta) f_s(x_\beta) f_s(y_\beta + t_1 x_\beta).
$$

Then for all $x_\beta, y_\beta \in \mathbb{R}^3$

$$
| V_{a_2}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \\
= | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \cdot | V_{a_2}^1(y_\beta + t_1 x_\beta) | f_s^{-1}(y_\beta + t_1 x_\beta) \\
\cdot f_s^{-1}(y_\beta) f_s(x_\beta) f_s(y_\beta + t_1 x_\beta) \leq | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) C \| V_{a_2}^1 \|_{\infty}.
$$

Hence

$$
\sup_{y_\beta} \int dx_\beta | V_{a_2}^1(y_\beta + t_1 x_\beta) | \cdot | \phi_\beta(x_\beta) |^2 f_s^{-1}(y_\beta) \leq \| \phi_\beta \|_{2,-s} C \| V_{a_2}^1 \|_{\infty} < \infty,
$$

(3.22)

where we have used Condition A.

From (3.20)-(3.22) we obtain

$$
B_\sigma | \phi_\beta > \in \mathcal{B}(\mathcal{H}_{\mu,-s}, \mathcal{H})
$$

and the Lemma is proved.

**Proof of Lemma 3.5.** — Using Lemma 3.3 it suffices to prove

$$
\tilde{\gamma}_{a}(\zeta) < \tilde{\phi}_{a} | V_\sigma R_0(\zeta) A_\beta \in \mathcal{B}(\mathcal{H}, \mathcal{H}_a) \quad \text{for} \quad \zeta \in S_{-\epsilon} \{ \lambda_\sigma + e^{-2i\theta} \mathbb{R}^+ \}
$$

Vol. 43, no 4-1985.
and
\[ \| \tilde{r}_a(\zeta) - \tilde{\phi}_a \|_{L^2(\mathcal{H})} \rightarrow 0 \] for \( \zeta \rightarrow \infty \) in \( S_{-\varepsilon} \) and

Because
\[ \tilde{r}_a(\zeta) - \tilde{\phi}_a \|_{L^2(\mathcal{H})} \rightarrow 0 \] for \( \zeta \rightarrow \infty \) in \( S_{-\varepsilon} \).

We now prove the last part of the Lemma.

From Lemma 3.10 we obtain
\[ \tilde{r}_a(\zeta) - \tilde{\phi}_a \|_{L^2(\mathcal{H})} \rightarrow 0 \] for \( \zeta \rightarrow \infty \) in \( S_{-\varepsilon} \).

First we prove that
\[ f_{\hat{s}}^2(\zeta)A_\beta, \quad f_{\hat{s}}^4R_0(\zeta)A_\beta \in \mathcal{B}(\mathcal{H}) \]
and
\[ \| f_{\hat{s}}^2(\zeta)A_\beta \|_{L^2(\mathcal{H})}, \quad \| f_{\hat{s}}^4R_0(\zeta)A_\beta \|_{L^2(\mathcal{H})} \rightarrow 0 \] for \( \zeta \rightarrow \infty \) in \( S_{-\varepsilon} \).

Clearly \( f_{\hat{s}}^2R_0(\zeta)A_\beta \in \mathcal{B}(\mathcal{H}) \), and from the proofs of Lemmas 3.12-3.15 we find that
\[ \| f_{\hat{s}}^2R_0(\zeta)A_\beta \|_{L^2(\mathcal{H})} \rightarrow 0 \] for \( \zeta \rightarrow \infty \) in \( S_{-\varepsilon} \).

Let \( F \in L^2(\mathbb{R}^3_x \oplus \mathbb{R}^3_y) \). Then for all \( \zeta \in S_{-\varepsilon}, \zeta \neq \lambda_x + e^{-2i\varphi\mathbb{R}^2} \) (we assume \( t_2 = 1 \))
\[ \| f_{\hat{s}}^1(\zeta)A_\beta F \| \leq K(\zeta) \int dx \int dy | F(x, y) |^2 \]
where (setting \( x = t_1x_2 \))
\[ K(\zeta) = \sup_{x \in \mathbb{R}^3} \| f_{\hat{s}}^1(\zeta)A_\beta (\cdot + x) \|_{L^2(\mathcal{H})} \]
By Lemma 3.9 $K(\zeta) \to 0$ for $\zeta \to \infty$ in $S_{-\epsilon}\backslash\{\lambda_{\alpha} + e^{-2i\varphi_{\alpha}}\}$.

From (3.23)-(3.25) and Lemma 3.11 we find that

$$\tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} R_{0}(\zeta) \alpha_{\beta} \|_{B(\mathcal{H}, H_{a_{1}a_{2}})} \quad \text{for} \quad \zeta \in S_{-\epsilon}$$

and

$$\| \tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} R_{0}(\zeta) \alpha_{\beta} \|_{B(\mathcal{H}, H_{a_{1}a_{2}})} \to 0 \quad \text{for} \quad \zeta \to \infty \quad \text{in} \quad S_{-\epsilon}.$$ 

This concludes the proof of the Lemma.

**Proof of Lemma 3.6.** — To prove that $\tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} \| \alpha_{\beta} \in B(\mathcal{H}_{\beta}, H_{a})$

we remark that by Condition A $g_{a} := V_{a}(\overline{\phi_{a}}) \in L^{1}(\mathbb{R}_{x}^{3})$ and hence it suffices to prove that

$$\langle g_{a} \| \alpha_{\beta} \rangle \in B(\mathcal{H}_{\beta}, H_{a}). \quad (3.26)$$

Let $f \in \mathcal{H}_{\beta}$ be given. Then (assuming $t_{2} = 1$)

$$\| \langle g_{a} \| \alpha_{\beta} \rangle f \|_{a_{a}}^{2} = \int_{y_{a}} \int_{x_{a}} g_{a}(x') \overline{\phi_{a}}(y_{a} + t_{1}x') \int f(t_{4} y_{a} + t_{3}x') dx' \cdot \int \overline{g_{a}}(x'') \phi_{a}(y_{a} + t_{1}x'') f(t_{4} y_{a} + t_{3}x'') dx''.$$ 

We use Fubini’s theorem and Cauchy-Schwarz’ inequality and find that

$$\| \langle g_{a} \| \alpha_{\beta} \rangle f \|_{a_{a}}^{2} \leq |t_{4}|^{-3} \| \int f \|_{a_{a}}^{2} \sup_{x_{a} \in \mathbb{R}^{3}} \| \phi_{a}(x) \| \| g_{a} \|_{L^{1}(\mathbb{R}^{3})}^{2}.$$ 

Hence we have proved (3.26).

To prove that $\tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} \| \alpha_{\beta} \in B(\mathcal{H}_{\beta_{1}, \beta_{2}}, H_{a_{1}, a_{2}})$ we remark that by Condition A

$$V_{a}(\overline{\phi_{a}}) \in L^{1}(\mathbb{R}_{x}^{3}), \quad \text{where} \quad g_{a} \in L^{1}(\mathbb{R}_{x}^{3}). \quad (3.27)$$

Let $f \in \mathcal{H}_{\beta_{1}, \beta_{2}}$ be given. Then

$$\| \tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} \| \alpha_{\beta} \|_{a_{1}, a_{2}}^{2} = \| f_{\| s \| a_{a}}^{2} \tilde{r}_{a}(\zeta) < \overline{\phi_{a}} \| V_{a} \| \alpha_{\beta} \| f_{\| s \| a_{a}}^{2} = \| \langle g_{a} \| \alpha_{\beta} \rangle F \|_{a_{a}}^{2},$$

where $g_{a}$ is given by (3.27) and $F : \mathbb{R}_{x}^{3} \oplus \mathbb{R}_{x}^{3} \to \mathbb{C}$ is given by

$$F(x_{a}, y_{a}) = f_{\| s \| a_{a}}^{2} f_{\| s \| a_{a}}^{2} f_{\| s \| a_{a}}^{2} \tilde{r}_{a}(\zeta) \phi_{a}(\cdot + t_{1}x_{a}) f(t_{4} y_{a} + t_{3}x_{a}).$$ 

We let $G : \mathbb{R}_{x}^{3} \oplus \mathbb{R}_{x}^{3} \to \mathbb{C}$ be given by

$$G(x_{a}, y_{a}) = f_{\| s \| a_{a}}^{2} f_{\| s \| a_{a}}^{2} f_{\| s \| a_{a}}^{2} \tilde{r}_{a}(\zeta) \phi_{a}(\cdot + t_{1}x_{a}) G(x_{a}, \cdot).$$ 

Clearly,

$$F(x_{a}, y_{a}) = f_{\| s \| a_{a}}^{2} f_{\| s \| a_{a}}^{2} \tilde{r}_{a}(\zeta) f_{\| s \| a_{a}}^{2} \phi_{a}(\cdot + t_{1}x_{a}) G(x_{a}, \cdot)$$

and by Condition A there exists $C < \infty$ (depending on $m_{i}$ and $\phi_{\beta}$) such that

$$\sup_{x_{a}, y_{a} \in \mathbb{R}^{3}} \left( \int_{y_{a}} \overline{G(x_{a}, y_{a})} \right)^{\frac{2}{3}} \leq C \| f \|_{a_{1}, a_{2}}^{2}. \quad (3.28)$$
The right-hand side of
\[ \| g_x \|_{K_a}^2 = \int dy \int dx g(x) f(x, y) \bar{r}_a(z) f(x, y) G(x, y) \]
\[ + \int dx'' g(x) f(x, y) \bar{r}_a(z) f(x, y) G(x, y) \]
is now estimated using Fubini's theorem and Cauchy-Schwarz' inequality as follows:
\[ \| g_x \|_{K_a}^2 \leq [K(z)]^2 \int dx' \int dx'' |g(x')g(x'')|^2 \left( \int dy |G(x', y)|^2 \right)^{1/2} \left( \int dy |G(x'', y)|^2 \right)^{1/2} \]
where
\[ K(z) = \sup_{x \in R^3} \| f(z) \bar{r}_a(z) f(z) + x \|_{\mathcal{S}(a)} \].

We now use (3.28) and find the estimate
\[ \| r_a(z) < \phi_a | V_\alpha | \phi_\beta \|_{\mathcal{S}(a)}^2 \leq [K(z)]^2 \| g_a \|_{L^1(R^3)}^2 \| f \|_{\mathcal{S}(a)}^2 \]
Hence \( r_a(z) < \phi_a | V_\alpha | \phi_\beta \in \mathcal{S}(a, -s, a, -s) \) by Lemma 3.8. Moreover, since by the same Lemma \( K(z) \to 0 \) for \( \zeta \to \infty \) in \( S_a \),
\[ \lambda_a + e^{-2i\varphi_{a}^{+}} \}
we obtain
\[ \| r_a(z) < \phi_a | V_\alpha | \phi_\beta \|_{\mathcal{S}(a, -s, a, -s)} \to 0 \]
for \( \zeta \to \infty \) in \( S_a \). The Lemma is proved.

4. UNIFORM ESTIMATES

**Lemma 4.1.** Let I be a closed interval contained in \([-a, a]\). Assume that for every \( \varphi \in I \), V(\varphi) has a decomposition
\[ V(\varphi) = V_1(\varphi) + V_2(\varphi) \]
such that
\[ V_1(\varphi) \in L^p \cap L^1 \]
\[ V_2(\varphi) \in L^\infty \]
and
\[ V_1(\varphi) \] is a continuous map from I into \( L^p \) and from I into \( L^1 \) while
\[ V_2(\varphi) \] is a continuous map from I into \( L^\infty \).

Assume moreover that for every \( \varphi_0 \in I \) there exists \( \delta(\varphi_0) \) and \( F_{\varphi_0} \in L^p \cap L^1 \) so that
\[ |V_1(\varphi)(x)| \leq |F_{\varphi_0}(x)|, \quad |\varphi - \varphi_0| \leq \delta(\varphi_0), \quad x \in R^3. \]
Then for every $\delta > 0$ there exists $C_4(\delta)$ such that for all $\varphi \in I$

$$V(\varphi) = V_1^0(\varphi) + V_2^0(\varphi)$$

where

$$V_1^0(\varphi) \in L^p \cap L^1_s, \quad \| V_1^0(\varphi) \|_{L^p} \| V_1^0(\varphi) \|_{L^1_s} < \delta$$

and

$$V_2^0(\varphi) \in L^\infty, \quad \| V_2^0(\varphi) \|_{L^p} < C_4(\delta).$$

**Proof.** — Let $\varphi_0 \in I$. Then by the proof of Lemma 1.1 there exists $N$ such that

$$\| (1 - \chi_N) V_1(\varphi_0) \|_{L^p} < \delta/2, \quad \| (1 - \chi_N) V_1(\varphi_0) \|_{L^1_s} < \delta/2$$

where

$$\chi_N(x) = \chi(x|F_{\varphi_0}(x)| \leq N)|x| \leq N)$$

By the continuity of $V_1(\varphi)$ and $V_2(\varphi)$ there exists $\eta > 0$ such that for

$$| \varphi - \varphi_0 | \leq \eta
\| V_1(\varphi) - V_1(\varphi_0) \|_{L^p} < \delta/2, \quad \| V_1(\varphi) - V_1(\varphi_0) \|_{L^1_s} < \delta/2.$$

Then for $| \varphi - \varphi_0 | \leq \eta$

$$\| (1 - \chi_N) V_1(\varphi) \|_{L^p} \leq \| (1 - \chi_N)(V_1(\varphi) - V_1(\varphi_0)) \|_{L^p} + \| (1 - \chi_N)V_1(\varphi_0) \|_{L^p}
\leq \| V_1(\varphi) - V_1(\varphi_0) \|_{L^p} + \| (1 - \chi_N)V_1(\varphi_0) \|_{L^p}
< \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and similarly with $L^p$ replaced by $L^1_s$. Moreover, for $| \varphi - \varphi_0 | \leq \eta \min \{ \eta, \delta(\varphi_0) \}$

$$\| \chi_N V_1(\varphi) + V_2(\varphi) \|_{L^\infty} \leq \| \chi_N V_1(\varphi) \|_{L^\infty} + \| V_2(\varphi) \|_{L^\infty}
\leq (1 + N)\eta N + \max_{| \varphi - \varphi_0 | \leq \eta} \| V_2(\varphi) \|.$$}

A compactness argument concludes the proof.

**Theorem 4.2.** — Let $I$ be a closed interval contained in $(0, a)$ and assume that for some $p > \frac{3}{2}, s > 1$ all $\alpha$ and $\varphi \in I$ the functions $V_\alpha(\varphi)$ have decompositions as in Lemma 4.1. Assume moreover, that $A$ is satisfied uniformly for $\varphi \in I$. Then for every $\varepsilon > 0$ the set of resonances $\bigcup_{\varphi \in I} \mathcal{R}_{\varphi, -\varepsilon}$ is bounded.

**Proof.** — The estimates of $\| A^2(\varphi, \zeta) \|$ given in Lemmas 3.3-3.6 can be obtained uniformly for $\varphi \in I$ by Lemma 4.1, the assumption that $A$ holds uniformly and the fact that $\varepsilon_1 = \varepsilon \sin 2\varphi \geq D_1 > 0$ for $\varphi \in I$. We need only replace $K$ of Lemma 3.16 (used in the proof of Lemmas 3.3 and 3.5) by a constant $K_1$ valid for all $\varphi \in I$, $\| V_{\alpha}^0 \|_{L^\infty} \leq C_\alpha(\delta)$ (Lemma 4.1), $K$ and $C$ by $k_1$ and $C_1$ (Condition A(i)) and $\varepsilon_1$ by $D_1$ in the various estimates. We illustrate this in the case of Lemma 3.16. We first of all note that (3.12) holds uniformly for $\varphi \in I$. This follows from a uniform version of Lemma 2.5.
obtained by estimating the first term in (2.1) by $C \max \{ \| V \|_{L^p}, \| V \|_{L^q} \}$
and by using in (2.3), (2.4) the trivial estimate $\| V_{\delta}^\pm \|_{L^1_{-\epsilon}} \leq \| V_{\delta}^\pm \|_{L^1}$
together with the uniform estimates of Lemma 4.1. In (2.5) we use
Lemma 4.1. It is easy to see that for every $R > 0, q(\varphi, \zeta)$ is jointly continuous
in $\varphi$ and $\zeta$ for $\varphi \in I$ and $\zeta \in S_{-\epsilon}$, $| \zeta | \leq R$. It follows that $\| q(\varphi, \zeta) \| < C$
for $\varphi \in I, \zeta \in S_{-\epsilon}$. Then (3.13) holds for $\varphi \in I$ with $C_1$ replaced by $C_{1,1}$.
By continuity (3.14) holds for $\varphi \in I$ with $C_2$ replaced by $C_{2,1}$. Also the
integrand in (3.15) is continuous in $\varphi, \zeta$ and $\zeta'$ for $\varphi \in I, \zeta \in D$ and
$| \zeta' - \lambda_\varphi | = \frac{3\delta}{2}$. This implies (3.16) for $\varphi \in I$ with $C_3$ replaced $C_{3,1}$, and
we obtain (3.17) uniformly for $\varphi \in I$, from which Lemma 3.16 follows
with $K$ replaced by $K_1$.

We finally discuss the case when the $V_\alpha$ are $\tilde{S}_a$-dilation-analytic, $a < \frac{\pi}{2}$.

We note the following simple fact.

REMARK 4.3. — Assume that $V$ is $\tilde{S}_a$-dilation-analytic and that
$V(\varphi)$ has a decomposition as in Lemma 4.1 on $[-a, a]$. Then $V(\varphi)$ is continuous
on $[-a, a]$ with values in $C(H^1, H^{-1})$.

The estimates of $A^2(\zeta)$ are valid for the operator $H(e^{i\varphi})$ as well as for
$0 < \varphi < a$. The interval of Theorem 4.2 can then be chosen such that
$I \subset (0, a]$. The constant $C(\delta)$ of Theorem 4.2 can be chosen accordingly,
$\delta \sin 2\varphi \geq C_1 > 0$ for $\varphi \in I$, and we obtain the following result.

THEOREM 4.4. — Assume that for all $\alpha$, $V_\alpha$ has a decomposition as in
Lemma 4.1 on $[-a, a]$. Assume moreover that Condition A holds uni-
formly on $[-a, a]$. Then for every $\varepsilon > 0$ and every closed interval $I \subset (0, a]$
the set of resonances $\bigcup_{\varphi \in I} R_{\varphi, -\varepsilon}$ is bounded.

REMARK 4.5. — The condition of Lemma 4.1 is satisfied, and $V$ is
$\tilde{S}_a$-dilation-analytic, if $V$ satisfies the following condition, expressed in
polar coordinates.

There exists an $L^\alpha(S^2)$-valued function $\tilde{V}(z)$, continuous on $\tilde{S}_a$ and ana-
lytic in $S_a$, such that for some $p > \frac{3}{2}$
$$
\int_0^1 \sup_{-a \leq \varphi \leq a} \{ \| \tilde{V}(e^{i\varphi}) \|_{L^p(S^2)} \} r^2 dr < \infty ,
$$
for some $t > 1$
$$
\sup_{-a \leq \varphi \leq a} \sup_{1 < r < \infty} \{ \| V(e^{i\varphi}) \|_{L^\infty(S^2)} r^t \} < \infty
$$
and for $r \in \mathbb{R}^+$
$$V(r, \cdot) = \tilde{V}(r) .$$

Annales de l'Institut Henri Poincaré - Physique théorique
In this case $V(\varphi)$, $e^{i\varphi} \in \overline{S}_u$, is given by

$$V(\varphi)(r, \cdot) = \tilde{V}(e^{i\varphi}r), \quad r \in \mathbb{R}^+.$$ 

Moreover

$$V_1(\varphi) \in L^p \cap L^1_\sigma \quad \text{and} \quad V_2(\varphi) \in L^s_\infty,$$

where

$$V_1(\varphi)(r, \cdot) = \tilde{V}(e^{i\varphi}r)\chi_{(0,1)}(r) \quad r \in \mathbb{R}^+,$$

$$V_2(\varphi)(r, \cdot) = \tilde{V}(e^{i\varphi}r)\chi_{(1,\infty)}(r) \quad r \in \mathbb{R}^+, \quad \text{and} \quad 1 < s < t.$$ 

REMARK 4.6. — The statement of Theorem 4.2 can be expressed as follows. For any two consecutive negative thresholds $\lambda_1$ and $\lambda_2$, consider the sheet $\mathcal{F}_{\lambda_1, \lambda_2}$ of the Riemann surface of dilation-analytic continuation attached to $(\lambda_1, \lambda_2)$ and given by $\mathcal{F}_{\lambda_1, \lambda_2} = \bigcup \bigcup \{ \lambda \prec \lambda_1 \prec \lambda_2 \ 0 < \varphi < a \}$. Then the set of resonances on $\mathcal{F}_{\lambda_1, \lambda_2}$ is bounded on the subset given by $0 < \delta \leq \varphi \leq a - \delta$ for any $\delta > 0$. Similarly for Theorem 4.4 for $0 < \delta \leq \varphi \leq a$ if $a < \frac{\pi}{2}$.

REMARK 4.7. — Using the well-known fact that $r_0(\zeta)$ has continuous boundary values on $\mathbb{R}^+$ in $(L^2_\sigma, L^{2-} \sigma)$, it is easy to show that $A(\zeta)$ has continuous boundary values $A_\pm(\zeta)$ in $(\mathcal{H}_{-\delta}, \mathcal{H}_{-\delta})$ on all non-zero cuts. The estimates of Lemmas 3.3-3.6 are then valid for $A^2(\zeta)$ including the boundary-values in each strip. Singular points of $A_\pm(\zeta)$ are identical with resonances on the respective sides of the cut [4]. Thus Theorem 3.1 extends to include resonances on the cut.

REMARK 4.8. — Replacing $L^\infty_\sigma$ by $L^\infty_\sigma$ for some $\sigma > 0$, we can prove $\| A^2(\zeta) \|_{(\mathcal{H}, \mathcal{F})} \to 0$ for $\zeta \to \infty$, keeping $\text{dist}(\zeta, \sigma(H)) \geq c > 0$. This yields an improvement of the results of [3] in the three-body case allowing $r^{-2+\varepsilon}$-singularities instead of the $r^{-1+\varepsilon}$-singularities of [3].

5. BOUNDEDNESS OF RESONANCES ALONG THE ZERO-CUT

In this section we shall prove that if the pair potentials decrease roughly speaking faster than $r^{-2}$ as $r \to \infty$, then the set of resonances is bounded also along $e^{-2i\varphi}\mathbb{R}^+$. Finally we establish the uniform estimates up to $\varphi = 0$.

DEFINITION. — We define $S_\pm$ and $\mathcal{R}_{\varphi \pm}$ by

$$S_{+} = \{ \zeta = s + e^{-2i\varphi}t \ | \ s > 0, t \in \mathbb{R} \}, \quad \mathcal{R}_{\varphi \pm} = \mathcal{R}_\varphi \cap S_\pm.$$ 

Vol. 43, n° 4-1985.
THEOREM 5.1. — Assume that $V_{x} = V_{x}(\varphi) \in M_{p}^{p}$ for some $p > \frac{3}{2}$, $s > 2$, $|\varphi| \leq a < \frac{\pi}{2}$, and all $x$. Suppose that every two-body eigenfunction $\phi_{x}$ associated with a negative eigenvalue satisfies $A$. Assume furthermore that for all $x$, $\mathcal{N}(1 + B_{x}r_{0}(0)A_{x}) = \{0\}$. Then $R_{\varphi}^{\pm}$ is bounded.

Assume moreover that the number of two-body resonances in $S_{+}$ and the number of positive eigenvalues of $h_{x}$ are finite for every $x$, that all two-body resonances in $S_{+}$ are simple and that every two-body eigenfunction associated with a positive eigenvalue satisfies $A$. Assume that there are no two-body resonances on $e^{-2i\varphi}R^{+}$. Then $R_{\varphi}^{\pm}$ is bounded.

Proof. — This follows in the same way as Theorem 3.1 from the Lemmas 3.3-3.6 with $S_{\pm}$ replaced by $S_{x}$. Lemmas 3.3 and 3.5 are consequences of Lemmas 5.4 and 5.5, proved below.

REMARK 5.2. — $V_{x} \in M_{p}^{p}$ ($p, s$ as above) implies $V_{x} \in L^{p} \cap L^{q}$ for $\frac{3}{s} < q < \frac{3}{2}$. It is well known, that under this assumption the number of negative eigenvalues of each $h_{x}$ is finite (cf. [12], p. 86). Under the slightly stronger assumption that $(1 + |x|)^{2}V_{x} \in L^{p} + L^{\infty}$ it is proved in [7], Prop. (3.5) that the number of positive eigenvalues is finite. Generically the number of resonances (and positive energy bound states) of each two-body problem is finite. This follows from the existence and compactness of $|V_{x}|^{1/2}r_{0}(0)V_{x}^{1/2} = \lim_{\zeta \to 0} |V_{x}|^{1/2}r_{0}(\zeta)V_{x}^{1/2}$ in $(L^{2}(\mathbb{R}^{3}),$ as noted by A. Jensen [9].

LEMMA 5.3. — (Iorio-O’Carroll [8], Ginibre-Moulin [7]).

(1) $B_{R_{0}(\zeta)}A_{\beta}$ is bounded and uniformly Hölder-continuous in $\mathcal{B}(\mathcal{H})$ for $\zeta \in \mathcal{C}$, including the boundary values on $e^{-2i\varphi}R^{+}$.

(2) $f_{s_{x}}^{1/2}R_{0}(\zeta)A_{\beta}$ is bounded and uniformly Hölder-continuous in $\mathcal{B}(\mathcal{H})$ for $\zeta \in \mathcal{C}$, where $f_{s_{x}}(x_{a}, y_{a}) = f_{s}(y_{a})$.

Proof. — The proof is given in [7] for the two-body problem; as indicated there (1) is proved in the same way, and this also holds for (2).

LEMMA 5.4.

(1) $\|B_{x}R_{0}(\zeta)A_{\beta}\|_{\mathcal{B}(\mathcal{H})} \to 0$ for $|\zeta| \to \infty$

(2) $\|f_{s_{x}}^{1/2}R_{0}(\zeta)A_{\beta}\|_{\mathcal{B}(\mathcal{H})} \to 0$ for $|\zeta| \to \infty$

Proof. — Let $\varepsilon > 0$ be given. By Lemma 5.3 we can choose $\delta > 0$ such that for $0 \leq s \leq \delta$ and $t \in \mathbb{R}$

$$\|B_{x}R_{0}((t - is)e^{-2i\varphi})A_{\beta} - B_{x}R_{0}((t - i\delta)e^{-2i\varphi})A_{\beta}\| < \frac{\varepsilon}{2}$$
By Lemma 3.15 we can then choose $K = K(\delta) > 0$ so that
\[ \| B_2 R_0 (t - iu) e^{-2i\sigma} A_\beta \| < \frac{\varepsilon}{2} \quad \text{for} \quad u \geq \delta \quad \text{and} \quad |t| \geq K , \]
hence
\[ \| B_2 R_0 (\zeta) A_\beta \| < \varepsilon \quad \text{for} \quad \zeta \in S_- , \quad |\zeta| > K , \]
so
\[ \| B_2 R_0 (\zeta) A_\beta \| \to 0 \quad \text{for} \quad |\zeta| \to \infty \quad \text{in} \quad S_- . \]
Similarly we prove this in $S_+$, and (2) is proved in the same way.

**Lemma 5.5.**
\[ \| B_2 R_0^0 (\zeta) A_\beta \| \to 0 \quad \text{for} \quad |\zeta| \to \infty \]
and
\[ B_2 R_0^0 (\zeta) A_\beta = B_2 R_0 (\zeta) (1 - |\phi_\beta \rangle \langle \phi_\beta |) A_\beta - B_2 R_0 (\zeta) A_\beta \cdot B_2 R_0^0 (\zeta) A_\beta . \]

**Proof.** — This follows from Lemma 5.4 and Lemma 3.16, which can be proved also for $\varepsilon = 0$. Here we use the fact that $q(\zeta)$ has continuous boundary values in $\mathfrak{B}(h)$ on $e^{-2i\sigma}\mathbb{R}^+$ and that these boundary values have no singular points on $e^{-2i\sigma}\mathbb{R}^+$. There are no zero-energy resonances by assumption. The fact that there are no singular points on $e^{-2i\sigma}\mathbb{R}^+$ is proved in [2] (note in the case of $S_+$ that by assumption $e^{-2i\sigma}\mathbb{R}^+$ does not contain two-body resonances).

We now proceed to discuss the extension of the uniform estimates of Section 4 to include the zero channel as well as $\varphi$ near 0.

**Theorem 5.6.** — Assume that $V_\varphi = V_\varphi (\varphi) \in M_\varepsilon^p$ for some $p > \frac{3}{2}$, $s > 2$, $|\varphi| \leq a$ and all $\alpha$, and that the conditions of Lemma 4.1 are satisfied for $\varphi \in I = [0, a]$. Suppose that every two-body eigenfunction $\phi_\alpha$ associated with a negative eigenvalue satisfies $A$ uniformly for $\varphi \in I$. Assume furthermore that for all $\alpha$, $h_\alpha$ has no zero-energy resonance for any $\varphi$ (or, equivalently, for one $\varphi \in I$) and no positive eigenvalues. Then $\bigcup_{\varphi \in I} \mathcal{R}_{\varphi-}$ is bounded.

Assume moreover that there exists $b \in (0, a]$ such that there are no two-body resonances for $\varphi \in [0, b]$ (This holds generically). Then $\bigcup_{\varphi \in [0, b]} \mathcal{R}_{\varphi+}$ is bounded.

**Proof.** — We sketch the proof in the case of $\mathcal{R}_{\varphi-}$, the proof for $\mathcal{R}_{\varphi+}$ is similar. The proof follows very closely that of Theorem 4.2, replacing $S_{-\varepsilon}$ by $S_-$ as in Theorem 5.1. A main point to be elaborated further is the existence and joint continuity of $(1 + q(\varphi, \zeta))^{-1}$ in $\varphi$ and $\zeta$, uniformly for $\varphi \in I$ and $\zeta \in S_-$ (= $S_{-}(\varphi)$), where $q(\varphi, \zeta)$ is one of the operators $B_\varphi (\varphi) r_\alpha (\varphi, \zeta) A_\varphi (\varphi)$. As noted in the proof of Theorem 5.1 the continuous boundary values of $q(\varphi, \zeta)$ on $e^{-2i\sigma}\mathbb{R}^+$ have no singular points. By assum-
tion, \(1 + q(\varphi, 0)\) is non-singular. Finally the boundary values of \(q(0, \zeta)\) on \(\mathbb{R}^+\) have no singular points by the assumption that \(h_a\) has no positive eigenvalues. This implies that \((1 + q(\varphi, \zeta))^{-1}\) is continuous in \(\varphi\) and \(\zeta\) for \(\varphi \in I, \zeta \in S_-\). As in the proof of Theorem 4.2 this is used to show that \(B_a(\varphi)R_{\alpha}(\varphi, \zeta)A_{\rho}(\varphi)\) is bounded for \(\varphi \in I, \zeta \in S_-\).

Another central point of the proof is to establish

\[
\|B_a(\varphi)R_{\alpha}(\varphi, \zeta)A_{\rho}(\varphi)\| \to 0 \quad \text{for} \quad |\zeta| \to \infty
\]

uniformly for \(\varphi \in I\). This follows as in the proof of Theorem 5.1, using the estimates

\[
\|V_a(\varphi)\|_{L^p} < C, \quad \|V_a(\varphi)\|_{L^q} < C \quad \text{for} \quad \varphi \in I,
\]

where \(\frac{3}{s} < q < \frac{3}{2}\), leading to the uniform Hölder-continuity in \(\zeta\) of \(B_a(\varphi)R_{\alpha}(\varphi, \zeta)A_{\rho}(\varphi)\) for \(\varphi \in I\) and \(\zeta \in S_-\).

A slight adaptation of the remaining part of the proof of Lemma 4.2 suffices to conclude the proof of the fact that \(\|A^2(\varphi, \zeta)\| \to 0\) for \(|\zeta| \to \infty\), uniformly for \(\varphi \in I\), from which the Theorem follows.

ACKNOWLEDGMENT

We would like to thank Professor N. H. Kuiper for the hospitality of the I. H. E. S., where the last phase of the work was carried out.

REFERENCES


[14] E. Skibsted, Resonances of Schrödinger operators with potentials. \( V(r) = \gamma r^\beta e^{-\zeta r} \), \( \beta > -2, \zeta > 0 \) and \( \alpha > 1 \), to appear in *J. Math. Anal. Appl.*

(Manuscrit reçu le 3 mai 1985)