ANNALES DE L'I. H. P., SECTION A

B. LINET

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Annales de l'I. H. P., section A, tome 45, nº 3 (1986), p. 249-256

<http://www.numdam.org/item?id=AIHPA_1986__45_3_249_0>

Ann. Inst. Henri Poincaré,
Vol. 45, n° 3, 1986, p. 249-256. *Physique théorique*

On the wave equations in the spacetime of a cosmic string

by

B. LINET
Unité Associée au CNRS n° 769
Université Pierre-et-Marie-Curie, Institut Henri Poincaré
11, rue P. et M. Curie, 75231 Paris Cedex 05, France

ABSTRACT. — We determine the Green function for the Helmholtz equation in the spacetime describing a static, cylindrically symmetric cosmic string. We can describe the diffraction of an incident plane wave: besides the waves of the geometric optics there exists the diffracted wave which is a cylindrical wave. We also give the explicit expression for the massive static scalar or vector field due to a point source at rest. We deduce from it an attractive interaction which is short range between the scalar particle and this cosmic string. For a vector field we obtain a repulsive interaction and when the mass of the field vanishes we get the electrostatic case.

RÉSUMÉ. — Nous déterminons la fonction de Green de l'équation d'Helmholtz dans l'espace-temps décrivant une corde cosmique statique et à symétrie cylindrique. Nous pouvons décrire la diffraction d'une onde plane incidente : à côté des ondes de l'optique géométrique il existe une onde diffractée qui est une onde cylindrique. Nous donnons aussi l'expression explicite du champ massif statique, scalaire ou vectoriel, dû à une source ponctuelle au repos. Nous en déduisons une interaction attractive, à courte portée, entre la particule scalaire et cette corde cosmique. Pour un champ vectoriel, nous obtenons une interaction répulsive qui donne le cas électrostatique lorsque la masse du champ s'annule.

Annales de l'Institut Henri Poincaré - Physique théorique - 0246-0211
Vol. 45/86/03/249/ 8 /$ 2,80/© Gauthier-Villars

1. INTRODUCTION

Cosmic strings could be produced at a phase transition in the early universe and have survived to the present day [1] [2]. In order to study some distinctive gravitational effects of a cosmic string, we consider the simple model which is static and cylindrically symmetric [3] [4] [5]. The spacetime describing such a cosmic string has the metric [5]

$$ds^2 = dt^2 - d\rho^2 - dz^2 - B^2\rho^2 d\varphi^2 \qquad \text{with} \qquad 0 < B \leqslant 1 \tag{1}$$

in a coordinate system (t, ρ, z, φ) with $\rho \geqslant 0$ and $0 \leqslant \varphi \leqslant 2\pi$. Metric (1) induces on the axis $\rho = 0$ a singular line source [6] of the Einstein equations having the following energy-momentum tensor

$$T_t^t = T_z^z = \left[\frac{1 - B}{4G}\right]\frac{\delta(\rho)}{(-\tilde{g})^{1/2}} \qquad \text{and} \qquad T_\rho^\rho = T_\varphi^\varphi = 0 \tag{2}$$

where $\tilde{g}$ is the determinant of the induced metric on the two-surface $t=\text{const.}$ and $z=\text{const.}$ Form (2) is characteristic of a cosmic string with the linear mass density μ given by

$$\mu = \frac{1 - B}{4G} \qquad \text{with} \qquad 0 \leqslant \mu < \frac{1}{4G}. \tag{3}$$

On the cosmological level, $G\mu \sim 10^{-6}$ (we have chosen units in which $c=1$).

The spacetime described by metric (1) is locally flat but of course it is not globally flat. Consequently, cosmic string (1) acts as a gravitational lens [3] [7] [8] and induces a repulsive force on an electric charge at rest [9]. The purpose of this paper is to give other effects induced specifically by a cosmic string (1) in determining the Green function for the Helmholtz equation in this spacetime.

The plan of the present work is as follows. In section 2, we indicate how the solution of the wave equations is reduced to finding the solution of the usual wave equations in a wedge of the Minkowski spacetime. Two cases are considered: the diffraction of an incident plane wave in section 3 and the case of a massive static scalar or vector field due to a point source at rest in section 4. We add in section 5 some concluding remarks.

2. THE WAVE EQUATION

We consider the question of finding a solution Φ of the wave equation in background (1) corresponding to a time harmonic source placed at the point $\rho = \rho_0$, $z = z_0$ and $\varphi = \varphi_0$, having the source strength λ. We can put

$$\Phi = \mathrm{U}e^{ikt} \tag{4}$$

then quantity U satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}+\frac{1}{\mathrm{B}^2\rho^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\mathrm{U}=-4\pi\lambda\frac{\delta(\rho-\rho_0)\delta(z-z_0)\delta(\varphi-\varphi_0)}{\mathrm{B}\rho}. \tag{5}$$

The solution to equation (5), denoted $\mathrm{G}(\rho, z, \varphi)$, having only outgoing waves at infinity, i. e.

$$\mathrm{G} \propto \frac{e^{-ik(\rho^2+z^2)^{1/2}}}{(\rho^2+z^2)^{1/2}} \qquad \text{as} \qquad (\rho^2+z^2)^{1/2} \to \infty \tag{6}$$

will be the Green function for the Helmholtz equation.

With the coordinate transformation

$$\theta = \mathrm{B}\varphi \tag{7}$$

equation (5) reduces to the usual Helmholtz equation in the subset of the Minkowski spacetime covered by the coordinate system (t, ρ, z, θ) with $0 \leqslant \theta \leqslant 2\pi\mathrm{B}$. We get

$$(\Delta + k^2)\mathrm{U} = -4\pi\lambda\frac{\delta(\rho-\rho_0)\delta(z-z_0)\delta(\theta-\theta_0)}{\rho} \tag{8}$$

where we have $\theta_0 = \mathrm{B}\varphi_0$. However the values of U and that of its derivatives on the hypersurfaces $\theta = 0$ and $\theta = 2\pi\mathrm{B}$ must coincide. But it is known that a solution $\mathrm{G_B}(\rho, z, \theta)$ to equation (8) is determined by the following conditions

$$\left.\begin{aligned} \mathrm{G_B}(\rho, z, 0) &= \mathrm{G_B}(\rho, z, 2\pi\mathrm{B}) \\ \frac{\partial \mathrm{G_B}}{\partial\theta}(\rho, z, 0) &= \frac{\partial \mathrm{G_B}}{\partial\theta}(\rho, z, 2\pi\mathrm{B}) \end{aligned}\right\} \tag{9}$$

and the radiation condition in the far field. Such a solution has been the subject of extensive studies, for a review [*10*] [*11*]. It is obvious that

$$\mathrm{G}(\rho, z, \varphi) = \mathrm{G_B}(\rho, z, \mathrm{B}\varphi) \tag{10}$$

and our problem is solved.

We now recall some results which will be needed for our present work. By formula (10), we deduce that the Green function G can be expressed as the sum of two functions

$$\mathrm{G} = \mathrm{U}^* + \mathrm{U_B}. \tag{11}$$

The first term in (11) has the following expression

$$\mathrm{U}^* = \sum_n \lambda\frac{e^{-ik[\rho^2+\rho_0^2+(z-z_0)^2-2\rho\rho_0\cos\mathrm{B}(\varphi-\varphi_0+2\pi n)]^{1/2}}}{[\rho^2+\rho_0^2+(z-z_0)^2-2\rho\rho_0\cos\mathrm{B}(\varphi-\varphi_0+2\pi \mathrm{n})]^{1/2}} \tag{12}$$

where the limits of the summation in (12) for a given value of φ $(0 \leqslant \varphi \leqslant 2\pi)$ are the positive or negative integer verifying

$$\varphi_0 - \frac{\pi}{B} \leqslant \varphi + 2\pi n \leqslant \varphi_0 + \frac{\pi}{B}. \tag{13}$$

When the value of φ ensures equality in inequality (13) for a certain integer n, the corresponding term for this integer must be halved. The second term in (11) has the integral expression

$$U_B = \frac{\lambda}{2\pi B} \int_0^\infty \frac{e^{-ikR}}{R} \left[\frac{\sin\left(\varphi - \varphi_0 - \frac{\pi}{B}\right)}{\cosh\frac{x}{B} - \cos\left(\varphi - \varphi_0 - \frac{\pi}{B}\right)} - \frac{\sin\left(\varphi - \varphi_0 + \frac{\pi}{B}\right)}{\cosh\frac{x}{B} - \cos\left(\varphi - \varphi_0 + \frac{\pi}{B}\right)} \right] dx \tag{14}$$

with $R = [\rho^2 + \rho_0^2 + (z - z_0)^2 + 2\rho\rho_0 \cosh x]^{1/2}$. It may be of interest to remark that the functions U^* and U_B on the boundaries defined by equalities (13) are both discontinuous, whereas their sum is completely regular. We point out that formulas (12) and (14) are also valid for the complex value $k = -im$.

We now turn to a more detailed study of two cases: the diffraction of an incident plane wave and the massive static scalar or vector field due to a point source at rest.

3. DIFFRACTION OF A PLANE WAVE

We consider the problem of the diffraction of an incident plane wave by a cosmic string described by metric (1). The solution for a plane wave coming from the direction φ_0 is obtained as the limiting case of formulas (12) and (14) for $\rho_0 \to \infty$. For the sake of simplicity, we assume that $B > \frac{1}{2}$ which is physically justified. Without loss of generality, we may choose $\varphi_0 = \pi$. The corresponding diffraction problem in a wedge of the Minkowski spacetime is such that $\theta_0 = \pi B$ and $\frac{\partial G_B}{\partial \theta}(\rho, z, 0) = \frac{\partial G_B}{\partial \theta}(\rho, z, 2\pi B) = 0$, which is an usual boundary condition in such a problem.

In formula (11), U^* represents the geometrical optics contribution. We can state explicitly formula (12). Inequalities (13) define the following regions:

For $2\pi - \pi \dfrac{1-B}{B} < \varphi < \pi \dfrac{1-B}{B}$, we have $n = 0$ and there is only the incident plane wave

$$U^* = \lambda e^{ik\rho\cos(B\varphi - B\pi)}. \tag{15}$$

For $0 < \varphi < \pi \dfrac{1-B}{B}$, we have $n = 1$ and there are the incident plane wave and the deflected wave

$$U^* = \lambda e^{ik\rho\cos(B\varphi - B\pi)} + \lambda e^{ik\rho\cos(B\varphi + B\pi)}. \tag{16}$$

For $2\pi - \pi \dfrac{1-B}{B} < \varphi < 2\pi$, we have $n = -1$ and there are the incident plane wave and the deflected wave

$$U^* = \lambda e^{ik\rho\cos(B\varphi - B\pi)} + \lambda e^{ik\rho\cos(B\varphi - 3B\pi)}. \tag{17}$$

From expressions (15) (16) and (17), we obtain immediately the eikonal for incident plane wave and deflected wave. We can then construct light rays in the whole spacetime and of course we find the effect of lens [3] [7] [8].

On the other hand in formula (11), U_B represents the diffraction term which may be written in the integral form

$$U_B = \frac{\lambda}{2\pi B} \int_0^\infty e^{-ik\rho\cosh x} \left[\frac{\sin\left(\varphi + \frac{\pi}{B}\right)}{\cosh\frac{x}{B} + \cos\left(\varphi + \frac{\pi}{B}\right)} - \frac{\sin\left(\varphi - \frac{\pi}{B}\right)}{\cosh\frac{x}{B} + \cos\left(\varphi - \frac{\pi}{B}\right)} \right] dx \tag{18}$$

We can rewrite (18) in the following form

$$U_B = I_B(\pi - B\varphi + B\rho) + I_B(\pi + B\varphi - B\pi) + I_B(\pi - B\varphi - B\pi) + I_B(\pi + B\varphi + B\pi)$$

where

$$I_B(\delta) \doteq - \frac{\sin\frac{\delta}{2B}}{4\pi B} \int_0^\infty \frac{e^{-ik\rho\cosh x}}{\cos\frac{x}{2B} - \cos\frac{\delta}{2B}} dx . \tag{19}$$

In Oberhettinger [12], one finds how integral (19) can be expressed in the form of an asymptotic series for large ρ. One can merely interpret result (18) as a cylindrical wave going out from the cosmic string whose amplitude depends on φ. Of course it is clear from (18) that U_B vanishes for $B = 1$. The order of magnitude of the cylindrical wave is $\lambda G\mu$.

4. MASSIVE STATIC FIELDS

We now consider a massive scalar field of inverse Compton wavelength m in background metric (1). The field due to a point source with source strength λ located at the point $\rho = \rho_0$, $z = z_0$ and $\varphi = \varphi_0$ satisfies equation (5) in which we have put $k = -im$. In this case, Garnir [*10*] has given an integral form of the Green function G_B. We obtain thereby

$$G = \frac{\lambda}{\pi B(2\rho\rho_0)^{1/2}} \int_\eta^\infty \frac{\cos\,[m(2\rho\rho_0)^{1/2}(\cosh\zeta - \cosh\eta)^{1/2}]}{(\cosh\zeta - \cosh\eta)^{1/2}} \times \frac{\sinh\dfrac{\zeta}{B}}{\cosh\dfrac{\zeta}{B} - \cos(\varphi - \varphi_0)}\, d\zeta \tag{20}$$

where η is defined by $\cosh\eta = \dfrac{\rho^2 + \rho_0^2 + (z - z_0)^2}{2\rho\rho_0}$ $(\eta \geqslant 0)$.

For a static case in metric (1), it should be noted that the only nonvanishing component A^t of a massive vector field satisfies the same equation. In the case where $m = 0$, formula (20) reduces to the expression of the electrostatic potential that we have previously obtained [*9*] ignoring this known expression for G_B.

However form (20) is not convenient to determine the self-force acting on the point source. In the neighbourhood of the point source, we can use expression (11) in which we expressed immediately U*

$$G = \lambda \frac{e^{-m[\rho^2 + \rho_0^2 + (z-z_0)^2 - 2\rho\rho_0\cos B(\varphi - \varphi_0)]^{1/2}}}{[\rho^2 + \rho_0^2 + (z - z_0)^2 - 2\rho\rho_0 \cos B(\varphi - \varphi_0)]^{1/2}} + U_B \tag{21}$$

where U_B is given by (14) with $k = -im$. The first term in (21) is the usual static field due to a point source in the Minkowski spacetime. The second term in (21) is regular at the position of the point source and is a solution of the homogeneous equation (5). Therefore the field U_B may be considered as an « external » field which exerts a force on the point source. The energy of the particule in the « external » field U_B is given by

$$W = \varepsilon \frac{\lambda}{2} U_B(\rho_0, z_0, \varphi_0) \tag{22}$$

where $\varepsilon = -1$ for a scalar field and $\varepsilon = 1$ for a vector field. This difference arises from the form of the interaction term in the Lagrangian for a scalar field [13]. We obtain finally

$$W = -\varepsilon \frac{\lambda^2 \sin\frac{\pi}{B}}{2\sqrt{2\pi B\rho_0}} \int_0^\infty \frac{e^{-\sqrt{2}m\rho_0(1+\cosh x)^{1/2}}dx}{(1+\cosh x)^{1/2}\left(\cosh\frac{x}{B} - \cos\frac{\pi}{B}\right)}. \tag{23}$$

Hence from (23), the exerted force is given by $f^\rho = -\frac{\partial W}{\partial \rho_0}$ and $f^z = f^\varphi = 0$. In our present case $\frac{1}{2} < B < 1$, it is easy to see from (23) that there exists an attractive interaction between the scalar particle and this cosmic string. On the contrary, there is a repulsive interaction in the case of the vector field. The limit of integral (23) where $\rho_0 \gg \frac{1}{m}$ is obtained by the usual asymptotic expansion [14]. We find

$$W \sim -\varepsilon \frac{\lambda^2 \sin\frac{\pi}{B}}{4\sqrt{\pi}B\left(1-\cos\frac{\pi}{B}\right)} \times \frac{e^{-2m\rho_0}}{\rho_0(\rho_0 m)^{1/2}} \quad \text{as} \quad \rho_0 m \to \infty. \tag{24}$$

We see from (24) that the interaction is short range with a range $\frac{1}{2m}$.

The physically interesting limit of integral (23) is the one $B \to 1$, i. e. $\mu \to 0$ following formula (3). We find

$$W \sim \varepsilon \frac{\sqrt{2}\lambda^2 G\mu}{\rho_0} \int_0^\infty \frac{e^{-\sqrt{2}m\rho_0(1+\cosh x)^{1/2}}}{(1+\cosh x)^{3/2}} dx \quad \text{as} \quad \mu \to 0 \tag{25}$$

which can be integrated in terms of the third repeated integral of the modified Bessel function K_0 [15]

$$W \sim e \frac{\lambda^2 G\mu}{\rho_0} Ki_3(2m\rho_0) \quad \text{as} \quad \mu \to 0. \tag{26}$$

In the limit where $m = 0$, the energy of the particle (26) reduces to

$$W \sim \varepsilon \frac{\pi\lambda^2 G\mu}{4\rho_0} \quad \text{as} \quad \mu \to 0. \tag{27}$$

When $\varepsilon = 1$, we get the electrostatic case where λ is replaced by q to denote electric charge.

5. CONCLUSION

The expression of the Green function for the Helmholtz equation in the spacetime of a static, cylindrically symmetric cosmic string has been deduced from the theory of the Helmholtz equation is Minkowski spacetime subject to boundary conditions on a wedge formed from two semi-infinite planes.

The diffraction of an incident plane wave should be reexamined within an astrophysical context. However the amplitude of the cylindrical wave going out from the cosmic string is very small and thus this diffracted wave will be probably unobservable.

In the case of a massive static scalar or vector field due to a point source, the self-force has been determined. The origin of the interaction between the point source and this cosmic string is purely topological since the spacetime is locally flat. In electrostatics, in the limit where the linear mass density of the cosmic string goes to zero, the exact expression of the self-force confirms our numerical approximation given in a previous work [9].

REFERENCES

[1] T. W. B. KIBBLE, Topology of cosmic domains and strings, *J. Phys.*, t. **A 9**, 1976, p. 1387.

[2] A. VILENKIN, Cosmic strings and domain walls, *Phys. Rep.*, t. **121**, 1985, p. 263.

[3] J. R. GOTT III, Gravitational lensing effects of vacuum strings: exact solutions, *Astrophys. J.*, t. **288**, 1985, p. 422.

[4] W. A. HISCOCK, Exact gravitational field of a string, *Phys. Rev.*, t. **D 31**, 1985, p. 3288.

[5] B. LINET, The static metrics with cylindrical symmetry describing a model of cosmic strings, *Gen. Rel. Grav.*, t. **17**, 1985, p. 1109.

[6] A. H. TAUB, Spacetimes with distribution valued curvature tensors, *J. Math. Phys.*, t. **21**, 1980, p. 1423.

[7] A. VILENKIN, Cosmic strings as gravitational lenses, *Astrophys. J. Lett.*, t. **282**, 1984, p. 51.

[8] C. J. HOGAN and R. NARAYAN, Gravitational lensing by cosmic strings, *M. N. R. A. S.*, t. **211**, 1984, p. 575.

[9] B. LINET, Force on a charge in the spacetime of a cosmic string, *Phys. Rev.*, t. D 33, 1986, p. 1833.

[10] H. G. GARNIR, Fonction de Green de l'opérateur métaharmonique pour les problèmes de Dirichlet et de Neumann posés dans un angle ou un dièdre, *Bull. Soc. R. Sc. Liège*, 1952, p. 119 and 207.

[11] F. OBERHETTINGER, On the diffraction and reflection of waves and pulses by wedges and corners, *J. Res. N. B. S.*, t. **61**, 1958, p. 343.

[12] F. OBERHETTINGER, On asymptotic series for functions occurring in the theory of diffraction of waves by wedges, *J. Math. Phys.*, t. **34**, 1956, p. 245.

[13] J. L. ANDERSON, *Principles of Relativity Physics*, Academic Press, 1967, p. 288.

[14] A. EDERLY, *Asymptotic Expansions*, Dover, 1956, p. 36.

[15] M. ABRAMOWITZ and I. A. STEGUN, (Eds.), *Handbook of Mathematical Functions*, Dover, 1968, p. 482.

(Manuscrit reçu le 25 février 1986)