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E. BALSLEV
A. GROSSMANN
T. PAUL

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A characterisation of dilation-analytic operators

by

E. BALSLEV (*), **A. GROSSMANN** **and T. PAUL**

Centre de Physique théorique (**), Section II, C. N. R. S., Luminy, Case 907, 13288 Marseille, Cedex 9, France

ABSTRACT. — Dilation analytic vectors and operators are characterized in a new representation of quantum mechanical states through functions analytic on the upper half-plane. In this space H_0-bounded operators are integral operators and criteria for dilation analyticity are given in terms of analytic continuation outside of the half-plane for functions and for kernels. A sufficient condition is given for an integral operator in momentum space to be dilation-analytic.

RÉSUMÉ. — On caractérise les vecteurs et les opérateurs analytiques par dilatation dans une nouvelle représentation des états de la Mécanique Quantique par des fonctions analytiques dans le demi-plan supérieur. Dans cet espace, les opérateurs H_0-bornés sont des opérateurs intégraux, et on donne des critères d'analyticité par dilatation en termes de prolongement analytiques de fonctions et de noyaux en dehors de ce demi-plan. On donne une condition suffisante d'analyticité par dilatation pour un opérateur intégral dans l'espace des impulsions.

(*) Matematisk Institut, Aarhus Universitet.
(**) Laboratoire propre du C. N. R. S.

0. INTRODUCTION

The concept of dilation analyticity has proved fruitful in the study of Schrödinger operators, in particular for the many-body problem, cf. [1] [2].

Dilation analytic potentials are defined in a purely operator theoric manner by requiring that the operator-valued function $V(\rho)=U(\rho)VU(\rho)^{-1}$ have an analytic extension in ρ to a sector, where $\{ U(\rho) \mid \rho > 0 \}$ is the dilation group on the representation space $\mathscr{H}$.

The problem remains of characterizing in a given representation the class of dilation-analytic potentials. In the usual position and momentum representation this is a very difficult (impossible) problem unless some further restriction is made on the class of operators.

Thus, in position space the class of multiplicative, dilation-analytic operators has been characterized by a simple analyticity condition on the potentials (3).

In momentum space, bounded integral operators, whose kernels are analytic in both variables, have been treated (4).

For a general investigation of this problem, the representation of quantum mechanics on a space $\mathscr{H}_\alpha$ of analytic functions on a half-plane developed in (5), (6) seems particularly well adapted. If we write the momentum space of quantum mechanics as $L^2(\mathbb{R}^+, h; p^{N-1}dp)$ (with for example $h = L^2(S^{n-1})$), the space we use is a space $\mathscr{H}_\alpha^h = \mathscr{H}_\alpha \otimes h$ of h-valued analytic functions on the half-plane, square integrable with respect to a two-dimensional measure $dv_\alpha(z)$ defined below. The correspondance between the two spaces is explicitly given by associating to each function $\psi \in L^2(\mathbb{R}^+, h; p^{N-1}dp)$ the function f on $\mathbb{C}^+$ given by

$$f(z) = \frac{1}{\sqrt{\pi\Gamma\left(\beta+\frac{N}{2}-1\right)}} \int_0^\infty p^\beta e^{iz\frac{p^2}{2}} \psi(p) p^{N-1} dp \tag{0.1}$$

This representation has the advantage first of all, that analyticity is already built into the theory, providing a natural basis of an analytic continuation theory such as the dilation-analytic theory.

Secondly, a large class of operators are represented by an integral kernel, which makes it possible to treat all dilation-analytic operators in an unified way. In particular every H_0-bounded operator V is represented by an integral operator on $\mathscr{H}_\alpha^h$ that is:

$$V(f)(z) = \int_{\mathbb{C}^+} v(z, \bar{z}') f(z') d\mu_\alpha(z') \tag{0.2}$$

with $v(z, \bar{z}')$ a $\mathscr{B}(h)$-valued analytic function in z and $\bar{z}'$ (here $\mathscr{B}(h)$ is the set of bounded operators on h and the sense of convergence of (0.2) will be given below).

And thirdly the space has a reproducing kernel, which is a powerful technical tool for constructing simple proofs.

The main aim of the present paper is to analyse the class of dilation-analytic operators in this representation. As a first step we characterize dilation-analytic vectors in the space $\mathscr{H}_\alpha^h$ as functions having an analytic continuation to a certain angular region and satisfying certain estimates (Theorem 1). This permits to recover in an easy way (Proposition 5) the characterization of dilation-analytic vectors given in [*3*].

Our main result is theorem 2, which gives a characterization of dilation-analytic operators in terms of certain analytic continuation properties of the integral kernel. This cannot be expected to give a complete characterization of dilation-analytic, H_0-bounded or H_0-compact operators, since no characterization of integral kernels defining bounded or compact operators exist.

However, various sufficient conditions for boundedness or compactness of an integral operator may be combined with theorem 1 to provide sufficient conditions on an operator on $\mathscr{H}_\alpha^h$ to be dilation-analytic and H_0-bounded or H_0-compact. Theorem 3 gives one such sufficient condition for an integral operator in momentum space to be S_a-dilation analytic.

1. HILBERT SPACES

1.1. The space $\mathscr{H}_\alpha$.

Let $\alpha > -1$ be a real number which will be kept fixed throughout this paper. We denote by $\mathscr{H}_\alpha$ the Hilbert space of analytic functions discussed in the introduction, and defined as follows:

Denote by $\mathbb{C}^+$ the open upper half-plane:

$$\mathbb{C}^+ = \{ z \mid x + iy, y > 0 \}. \tag{1.1}$$

Denote by $\mathscr{H}_\alpha$ the set of all complex-valued functions f which are analytic on $\mathbb{C}^+$ and such that

$$(f, f)_\alpha = \iint | f(z) |^2 y^\alpha dx dy = \int | f(z) |^2 d\mu_\alpha(z) < \infty . \tag{1.2}$$

With the scalar product

$$(f, g)_\alpha = \iint \bar{f}(z) g(z) y^\alpha dx dy . \tag{1.3}$$

$\mathcal{H}_\alpha$ is a Hilbert space; moreover $\mathcal{H}_\alpha$ has a reproducing kernel, which is here defined as follows:

For $z \in \mathbb{C}^+$ and $w \in \mathbb{C}^+$, consider the function

$$\rho_z(w) = \frac{\alpha + 1}{4\pi} \left(\frac{w - \bar{z}}{2i} \right)^{-\alpha - 2} \tag{1.4}$$

where $\bar{z}$ is the complex conjugate of z, and where the (possibly non-integer) power on the r. h. s. is defined as usual, with a possible cut along the negative real axis. Then *i*), for every fixed $z \in \mathbb{C}^+$ the function $\rho_z(\cdot)$ belongs to $\mathcal{H}_\alpha$ and, *ii*) for every $f \in \mathcal{H}_\alpha$ we have

$$(\rho_z, f) = f(z); \tag{1.5}$$

the evaluation functional in $\mathcal{H}_\alpha$ is given by a scalar product in $\mathcal{H}_\alpha$. This is the reproducing kernel property.

The norm of ρ_z in $\mathcal{H}_\alpha$ is

$$\| \rho_z \| = \sqrt{\frac{\alpha + 1}{4\pi}} (\operatorname{Im} z)^{-\frac{\alpha}{2} - 1}. \tag{1.6}$$

1.2. The space $\mathcal{H}_\alpha^h$.

As we have seen in the introduction the space $\mathcal{H}_\alpha$ is used to describe the « radial » behaviour of a one-body hamiltonian in n dimensions. In order to describe the full hamiltonian (which does not need to be spherically symmetric), we use, as is customary [*3*], a space of functions on $\mathbb{C}^+$ with values in the Hilbert space $L^2(S^{n-1}) = h$ of square integrable functions on the unit sphere in $\mathbb{R}^n$. The precise nature of this auxiliary space is not important here, and this leads to the following definition:

Let h be a separable Hilbert space, with inner product $(\ , \ )_h$ and norm $| \cdot |_h$. Denote by $\mathcal{H}_\alpha^h$ the set of all h-valued functions f, defined and analytic on $\mathbb{C}^+$, and such that

$$(f, f)_{\alpha,h} = \iint | f(z) |_h^2 \, y^\alpha dx dy < \infty \tag{1.7}$$

Then $\mathcal{H}_\alpha^h$ is a Hilbert space. In the special case where $h = \mathbb{C}$, we have $\mathcal{H}_\alpha^{\mathbb{C}} = \mathcal{H}_\alpha$, the space considered above. In general, $\mathcal{H}_\alpha^h$ does not have a reproducing kernel.

However, if we fix an arbitrary vector $\chi \in h$ and an arbitrary $z \in \mathbb{C}^+$ we may consider in $\mathcal{H}_\alpha^h$ the vector ρ_z^χ defined by

$$\rho_z^\chi(z') = \rho_z(z')\chi \tag{1.8}$$

where $\rho_z(z')$ is given by (1.4), and is the reproducing kernel for $\mathcal{H}_\alpha^{\mathbb{C}} = \mathcal{H}_\alpha$. Then for every $f \in \mathcal{H}_\alpha^h$ we have

$$(\rho_z^\chi, f)_{\alpha,h} = \int (\rho_z^\chi(z'), f(z'))_h d\mu_\alpha(z') = \int \overline{\rho}_z(z')(\chi, f(z'))d\mu_\alpha(z') \,. \quad (1.9)$$

Now $(\chi, f(z'))_h$ is a complex-valued function in $\mathcal{H}_\alpha$; by the reproducing property of $\overline{\rho}_z(z')$, we get

$$(\rho_z^\chi, f)_{\alpha,h} = (\chi, f(z))_h \quad (1.10)$$

We shall now prove

PROPOSITION. — If f belongs to $\mathcal{H}_\alpha^h$, then, for every $z \in \mathbb{C}^+$, one has:

$$| f(z) |_h \leqslant \sqrt{\frac{\alpha+1}{4\pi}} (\mathrm{Im}\, z)^{-1-\frac{\alpha}{2}} \| f \|_{\alpha,h} \,. \quad (1.11)$$

Proof. — We consider the vector $\rho_z^{f(z)}$, i. e. we choose, in (1.8), $\chi = f(z)$ (the value of z is kept fixed).

Then $\rho_z^{f(z)}(z') = \rho_z(z')f(z)$. By (1.10) we have, (with $\chi = f(z)$)

$$(f(z), f(z))_h = (\rho_z^{f(z)}, f)_{\alpha,h} \,. \quad (1.12)$$

By Schwarz's inequality, we have

$$| f(z) |_h^2 \leqslant \| f \|_{\alpha,h} \| \rho_z^{f(z)} \|_{\alpha,h} \,. \quad (1.13)$$

Now

$$\| \rho_z^{f(z)} \|_{\alpha,h}^2 = \int | \rho_z(z') |^2 | f(z) |_h^2 d\mu_\alpha(z') = | f(z) |_h^2 \| \rho_z \|_\alpha^2 \,.$$

Consequently, by (1.13)

$$| f(z) |_h^2 \leqslant \| f \|_{\alpha,h} | f(z) |_h \| \rho(z) \|_\alpha = \| f \|_{\alpha,h} | f(z) |_h \sqrt{\frac{\alpha+1}{4\pi}} (\mathrm{Im}\, z)^{-\frac{\alpha}{2}-1} ;$$

if $| f(z) |_h \neq 0$, division by $| f(z) |_h$ gives (1.11). If $| f(z) |_h = 0$, (1.11) still holds.

Dilations:

In $\mathcal{H}_\alpha^h$, there is a natural unitary representation of the one-parameter dilation group:

$$(\mathrm{U}(k)f)(z) = k^{1+\frac{\alpha}{2}} f(kz) \qquad (k > 0, \quad z \in \mathbb{C}^+) \,. \quad (1.14)$$

We shall be concerned with complexifications of this representation, and start by defining natural domains for the complexified variable. Let a be such that $0 \leqslant a < \frac{1}{2}\pi$. We denote by S_a the set $\mathrm{S}_a = \{ k \in \mathbb{C}^+ \mid -a < \arg(k) < a \}$.

2. A CHARACTERIZATION OF DILATION-ANALYTIC VECTORS

We shall say that a vector $f \in \mathscr{H}_\alpha^h$ is S_a-dilation analytic if the family of vectors

$$f^k = U(k)f \qquad (k > 0) \tag{2.1}$$

in $\mathscr{H}_\alpha^h$ is the restriction, to the positive k-axis, of an analytic $\mathscr{H}_\alpha^h$-valued function f^k defined for $k \in S_a$.

This is the standard definition of dilation analyticity, (see [1] [2] [3]), transported to the space $\mathscr{H}_\alpha^h$. It is stated in terms of the whole family f^k; so it is natural to ask for a criterion involving only f and allowing us to decide whether f is S_a-dilation analytic.

An advantage of the representation that we are using is that such a criterion can be readily obtained from a consideration of f as an analytic h-valued function.

The purpose of this section is to derive this criterion, which says, essentially, that f is dilation-analytic if the function f has a suitably bounded analytic continuation from the upper half-plane into a suitable « cut pie » extending into the lower half-plane.

We begin with the definition of the cut pie:

For $0 \leqslant a < \frac{1}{2}\pi$, denote by τ_a the set

$$\tau_a = \{ z \in \mathbb{C} \mid -a < \arg z < \pi + a \} = e^{-ia}(\mathbb{C}^+)^{1+\frac{2a}{\pi}} . \tag{2.2}$$

We now define $\mathscr{D}(a, \alpha)$ as the set of all f in $\mathscr{H}_\alpha^h$ which are such that

i) The h-valued function $z \to f(z)$ $(z \in \mathbb{C}^+)$ is the restriction to $\mathbb{C}^+$ of an h-valued function defined and analytic in τ_a.

ii) For every φ such that $-a<\varphi<a$, the h-valued function $z \to f(e^{-i\varphi}z)$ is square integrable over $\mathbb{C}^+$ with respect to the measure $y^\alpha dxdy$ (i. e. belongs to $\mathscr{H}_\alpha^h$):

$$N_\varphi = \iint | f(e^{-i\varphi}z) |_h^2 \, y^\alpha dxdy < \infty . \tag{2.3}$$

iii) The numbers N_φ are uniformly bounded in any closed interval contained in the open interval $(-a, a)$: for every $\delta > 0$, there exists a $C_\delta > 0$ such that $N_\varphi \leqslant C_\delta$ for every φ in $[-a+\delta, a-\delta]$.

The space of dilation-analytic vectors can now be characterized as follows.

THEOREM 1. — A vector f is S_a dilation-analytic, if and only if it belongs to $\mathscr{D}(a, \alpha)$.

Proof.

a) Assume that f is S_a-analytic. In order to prove that f is in $\mathscr{D}(a, \alpha)$ we show first:

i) The h-valued function $f(z)$ has an analytic continuation to τ_a.

By the assumption on f, there exists an $\mathscr{H}_\alpha^h$-valued analytic family f^k $(k \in S_a)$, such that, for $k > 0$,

$$f^k(z) = k^{1+\frac{\alpha}{2}} f(kz).$$

Choose an arbitrary $\chi \in h$, and consider the inner product, in $\mathscr{H}_\alpha^h$, of f^k $(k \in S_a)$ with the vector ρ_z^χ defined in (1.8). We have by (1.10),

$$(\rho_z^\chi, f^k)_{\alpha,h} = (\chi, f^k(z))_h. \tag{2.4}$$

Denoting temporarily the r. h. s. of (2.4) by $F(z, k)$, we see that $F(z, k)$ is a complex-valued function separately, and hence jointly, analytic in $\mathbb{C}^+ \times S$. Furthermore, for $k > 0$, we have

$$F(z, k) = k^{1+\alpha/2}(\chi, f(kz)). \tag{2.5}$$

Introducing the variable $kz = u$, we can write (2.5) as

$$(\chi, f(u))_h = \left(\frac{u}{z}\right)^{-1-\frac{\alpha}{2}} F\left(z, \frac{u}{z}\right).$$

The analyticity properties of F then show that $(\chi, f(u))_h$ is analytic in the pie $u \in \tau_a$.

Since χ was arbitrary the analyticity assertion about $f(z)$ is proved.

ii) For every φ such that $-a < \varphi < a$, the h-valued function $z \to f(e^{-i\varphi}z)$ belongs to $\mathscr{H}_\alpha^h$.

Indeed, by *i*), we have

$$f(e^{-i\varphi}z) = k^{-1-\frac{\alpha}{2}} f^k(z) \qquad \text{with} \qquad k = e^{-i\varphi},$$

and the function $z \to f^k$ belongs to $\mathscr{H}_\alpha^h$ by the assumption of S_a-dilation analyticity.

iii) The numbers $N_\varphi = \int | f(e^{-i\varphi}z) |^2 d\mu_\alpha(z)$ are uniformly bounded on any closed sub-interval of $(-a, a)$.

This in an immediate consequence of *ii*) and of the assumption of S_a-dilation analyticity.

We have now proved that $f \in S_a$ implies $f \in \mathscr{D}(a, \alpha)$.

b) Let us now assume that $f \in \mathscr{D}(a, \alpha)$. By the condition *i*) in the definition of $\mathscr{D}(a, \alpha)$, we may define, for every $k \in S_a$, the h-valued function

$$\overset{\circ}{f}{}^k : z \to k^{1+\frac{\alpha}{2}} f(kz) \tag{2.4}$$

and this function is analytic in $z \in \mathbb{C}^+$. Condition *ii*) says that $\overset{\circ}{f}{}^k \in \mathscr{H}_\alpha^h$ for every $k \in S_a$. So we have defined a $\mathscr{H}_\alpha^h$-valued function in S_a. This function coincides with (2.1) for $k > 0$. Our proof will be complete if we show that $\overset{\circ}{f}{}^k$ is an analytic vector family. This will be achieved if we display, *a*) : a dense family of vectors in $\mathscr{H}_h^\alpha$ such that, for every ψ in this family, the scalar product $(\psi, \overset{\circ}{f}{}^k)$ is analytic in $k \in S_a$, and if *b*) : we show that the norms $\| \overset{\circ}{f}{}^k \|_{\alpha,h}$ are bounded uniformly in compacts in S_a.

a) : Consider again the vectors $\rho_z^\chi \in \mathscr{H}_\alpha^h$ defined in (1.8). It is clear from (1.10) that the linear span of these vectors (and even of appropriate subsets of such vectors) is dense in $\mathscr{H}_\alpha^h$; indeed, the orthogonality of $f \in \mathscr{H}_\alpha^h$ and of $\rho_z^h \in \mathscr{H}_\alpha^h$ means $(\chi, f(z))_h = 0$. The inner product of ρ_z^χ and of $\overset{\circ}{f}{}^k$ is

$$(\rho_z^\chi, \overset{\circ}{f}{}^k)_{\alpha,h} = (\chi, \overset{\circ}{f}{}^k(z))_h .$$

By the definition (2.4) of $\check{f}^k$ and the analyticity assumption this function is analytic in $k \in S_a$. Finally the boundedness of the norms $\| \overset{\circ}{f}{}^k \|_{\alpha,h}$ in compacts of S_a follows from the condition *iii*) in the definition of $\mathscr{D}(a, \alpha)$.

3. A CHARACTERIZATION OF DILATION-ANALYTIC OPERATORS

The preceding arguments show that one can define a family $U(k)$ of operators from $\mathscr{D}(a, \alpha)$ into $\mathscr{H}_\alpha^h$ by the equation

$$(U(k)f)(z) = k^{1+\frac{\alpha}{2}} f(kz) \qquad (k \in S_a, \quad z \in \mathbb{C}^+, \quad f \in \mathscr{D}(a, \alpha)) . \quad (3.1)$$

Moreover, if we define in $\mathscr{D}(a, \alpha)$ a topology with the help of the family of (semi) norms N_φ (see sec. 2) then

i) $\mathscr{D}(a, \alpha)$ is complete in this topology and

ii) for every $k \in S_a$, $U(K)$ is continuous from $\mathscr{D}^h(a, \alpha)$ into $\mathscr{H}_\alpha^h$.

Let H_0^h be a selfadjoint operator in $\mathscr{H}_\alpha^h = \mathscr{H}_\alpha \otimes h$; assume that H_0^h is of the form $H_0 \otimes \mathbb{1}_h$ where H_0 acts in $\mathscr{H}_\alpha$ (In the example of the appendix, this will mean that H_0^h is « radial ».)

Assume furthermore that the domain of H_0 contains all the functions ρ_z^χ ($z \in \mathbb{C}^+$, $\chi \in h$).

Let V be symmetric and H_0^h-bounded i. e. the domain of V contains the domain of H_0^h and V is bounded as an operator from the domain of H_0^h with the graph norm into $\mathscr{H}_\alpha^h$. Since $V \subseteq V^*$, V is closable.

By the assumption on V, the vector $V\rho_z^\chi = V(\rho_z \otimes \chi)$ is in $\mathscr{H}_\alpha^h$ for every $z \in \mathbb{C}^+$, $\chi \in h$. For every $z' \in \mathbb{C}^+$, consider the vector in h defined as $(V\rho_z^\chi)(z')$.

Define $v(z, \bar{z}')\chi$ for $\chi \in h$ and $z, z' \in \mathbb{C}^+$ by:

$$v(z, \bar{z}')_\chi = (\mathrm{V}\rho_{z'}^\chi)(z). \tag{3.2}$$

Proposition 2. — For fixed z, z' the correspondence $\chi \to v(z, \bar{z})\chi$ is a bounded linear operator in h, and $v(z, \bar{z}') = v^*(z', \bar{z})$.

Proof. — Let $\chi_n \to \chi$ in h. Then $\rho_{z'}^{\chi_n} \to \rho_{z'}^{\chi}$ in $\mathcal{H}_\alpha^h$ and

$$\mathrm{H}_0^h \rho_{z'}^{\chi_n} = (\mathrm{H}_0\rho_{z'}) \otimes \chi_n \to (\mathrm{H}_0\rho_{z'}) \otimes \chi = \mathrm{H}_0^\chi \rho_{z'}^\chi \qquad \text{in } \mathcal{H}_\alpha^h.$$

Since V is H_0^h-bounded, this implies

$$\mathrm{V}\rho_{z'}^{\chi_n} \to \mathrm{V}\rho_{z'}^{\chi} \qquad \text{in } \mathcal{H}_\alpha^h.$$

By Proposition 1 it follows that

$$v(z, \bar{z}')\chi_n \to v(z, z')\chi \qquad \text{in } h$$

hence $v(z, \bar{z}') \in \mathcal{B}(h)$.

Using (1.10) we obtain for χ, $\eta \in h$

$$\begin{aligned}(v(z, \bar{z}')\chi, \eta)_h &= (\mathrm{V}\rho_{z'}^\chi(z), \eta)_h = (\mathrm{V}\rho_{z'}^\chi, \rho_z^\eta)_{\alpha,h} \\ &= (\rho_{z'}^\chi, \mathrm{V}\rho_z^\eta)_{\alpha,h} = (\chi, \mathrm{V}\rho_z^h(z'))_h = (\chi, v(z', \bar{z})\eta)_h.\end{aligned} \tag{3.3}$$

So $v(z, \bar{z}')^* = v(z', \bar{z})$, and the proposition is proved.

Proposition 3. — Let V be a symmetric operator having all vectors ρ_z^h in its domain. Then, for every f in $\mathcal{D}(\mathrm{V})$, and every $\chi \in h$, one has

$$(\chi, (\mathrm{V}f)(z))_h = \int (\chi, v(z, \bar{z}')f(z'))_h d\mu(z'),$$

where the integral on the r. h. s. is absolutely convergent.

Proof. — By (1.10), we have

$$(\chi, (\mathrm{V}f)(z)) = (\rho_z^\chi, \mathrm{V}f)_{\alpha,h} = (\mathrm{V}\rho_z^\chi, f)_{\alpha,h} = \int ((\mathrm{V}\rho_z^\chi)(z'), f(z'))_h d\mu_\alpha(z')$$

where the integral is absolutely convergent.

Now $(\mathrm{V}\rho_z^\chi)(z') = v(z', \bar{z})\chi$, by (3.2), so

$$(\chi, (\mathrm{V}f)(z))_h = \int (v(z', \bar{z})\chi, f(z'))_h d\mu_\alpha(z') = \int (\chi, v(z, \bar{z}')f(z'))_h d\mu_\alpha(z')$$

where we have used Proposition 2 in the last step and the proof is complete.

Definition 1. — V is S_a-dilation analytic if the map $k \to \mathrm{U}(k)\mathrm{V}\mathrm{U}(k)^{-1}$ from $\mathbb{R}^+$ into $\mathcal{B}(\mathcal{D}(\mathrm{H}_0), \mathcal{H}_\alpha^h)$ (where $\mathcal{D}(\mathrm{H}_0)$ is equipped with the graph norm) has an analytic extension V^k to S_a.

DEFINITION 2. — Define F_a as the subset of $\mathbb{C}^2$, defined as follows:

$$F_a = \bigcup_{-a<\varphi<a} (e^{i\varphi}\mathbb{C}^+) \times (e^{i\varphi}\mathbb{C}^-)$$

where $\mathbb{C}^-$ is the lower open half-plane.

F_a can also be written as

$F_a = \{(z, w) \in \mathbb{C}^2 \mid z \in \tau_a \text{ and } \operatorname{Max}(-\pi - a, \arg z - 2\pi) < \arg \overline{w} < \operatorname{Min}(\arg z, a)\}$

THEOREM 2. — Suppose that H_0 and V are as above. Then V is S_a-dilation analytic if and only if the three conditions below hold:

i) The family $v(z, w)$ has a $\mathscr{B}(h)$-valued analytic continuation from $\mathbb{C}^+ \times \mathbb{C}^-$ to F_a.

ii) For each $k \in S_a$, the operator V^k defined by the kernel

$$v^k(z, \overline{z}') = k^{\alpha+2} v(kz, k\overline{z}')$$

is H_0-bounded.

iii) The operator norms

$$\| (H_0^h - i)^{-1} V^{\exp(-i\varphi)} \|$$

in the space $\mathscr{H}_\alpha^h$, are uniformly bounded in any closed interval $-a + \delta \leqslant \varphi \leqslant a + \delta$.

Proof.

a) Assume that V is S_a-dilation analytic; for $k > 0$, the operator

$$V^k = U(k) V U(k)^{-1}$$

is H_0^h-bounded. By Proposition 2, for every $z \in \mathbb{C}^+$ and $z' \in \mathbb{C}^+$ a bounded operator in h is defined by

$$v^k(z, \overline{z}')\chi = V^k \rho_{z'}^{\chi}(z)$$

for every $\chi \in h$. One has by (1.10)

$$(\chi, v^k(z, \overline{z}')\chi')_h = (\rho_z^\chi, U(k) V U(k)^{-1} \rho_{z'}^{\chi'})_{\alpha,h} = k^{\alpha+2} (\rho_{kz}^\chi, V \rho_{kz'}^{\chi'})_{\alpha,h}. \quad (3.4)$$

Consequently, by (3.3)

$$v^k(z, \overline{z}') = k^{\alpha+2} v(kz, k\overline{z}'). \quad (3.5)$$

By the assumption that V is S_a-dilation analytic, the matrix element $(\rho_z^\chi, V^k \rho_{z'}^{\chi'})_{\alpha,h}$ has, for fixed z, z' in $\mathbb{C}^+$, an analytic continuation into $k \in S_a$. This matrix element is $k^{\alpha+2}(\chi, v(kz, k\overline{z}')\chi')$ by (3.4) and (3.5). Furthermore, for any fixed $k \in S_a$, by (3.4), this is analytic in $z \in \mathbb{C}^+$, $z' \in \mathbb{C}^+$. Consequently $(\chi, v^k(z, \overline{z}')\chi')_h$ is jointly analytic in $(k, z, \overline{z}') \in S_a \times \mathbb{C}^+ \times \mathbb{C}^-$. Using the identity (3.5) valid for $k > 0$, one can now conclude that $(\chi', v(z, \overline{z}')\chi)$ has an analytic continuation to $(z, \overline{z}') \in F_a$; hence $v(z, \overline{z}')$ has an analytic continuation to this domain as, a $\mathscr{B}(h)$ valued function.

The conditions *ii*) and *iii*) follow from the definition (analyticity and continuity).

b) Suppose now that there is given an operator V which is H_0-bounded; assume that the corresponding family $v(z, \bar{z}')$ of operators in $\mathscr{B}(h)$ satisfies the conditions *i*), *ii*) and *iii*). Define $v^k(z, \bar{z}')$ by

$$v^k(z, \bar{z}') = k^{\alpha+2}v(kz, k\bar{z}'). \tag{3.4}$$

Then define V^k by the kernel $v^k(z, \bar{z}')$ in the sense of (3.2). By (3.3) we have $(\rho_{z'}^{\chi'}, V^k\rho_z^{\chi})_{\alpha,h} = (\chi', v^k(z, \bar{z}')\chi)_h$. By analyticity for fixed z and z', of $v^k(z, \bar{z}')$ in $k \in S_a$ one obtains analyticity of $(\rho_{z'}^{\chi'}, V^k\rho_z^k)$. From the density of the linear span of $\{ \rho_z^{\chi} \mid z \in \mathbb{C}^+, \chi \in h \}$ and *iii*) we get the analyticity of V^k. It follows from (3.4) that $U(k)VU(k)^{-1} = V^k$ for $k > 0$.

The theorem is proved.

4. EXAMPLES AND APPLICATIONS

In this section we first recover the result of (3) on the characterisation of dilation analytic vectors in a very simple way and then give a dilation analyticity criterion for an integral operator in momentum space.

a) **Analytic vectors.**

We prove the following proposition of [*3*] (note a difference of notation: because of the transformation (1.14), a vector which is S_a dilation analytic in our terminology is $S_{a/2}$ dilation analytic in that of [*3*]).

PROPOSITION 5. — A vector φ in $L^2(R^+, h; p^{N-1}dp)$ is S_a dilation analytic if and only if $\varphi(p)$ is equal almost everywhere to a function $\psi(p)$ satisfying the following conditions.

i) $\psi(p)$ is the restriction to $\mathbb{R}^+$ of a function $\psi(pe^{i\varphi})$ analytic from $S_{a/2}$ into h.

ii) for every $\varepsilon > 0$

$$\sup_{\varphi \in \left[-\frac{a}{2}+\varepsilon, \frac{a}{2}-\varepsilon\right]} \int_0^\infty | \psi(pe^{i\varphi}) |_h^2 p^{N-1}dp < +\infty .$$

The dilation analytic extension φ^τ of the vector φ is given by

$$\varphi^\tau(p) = \tau^{-N/4}\varphi(\tau^{-1/2}p). \tag{4.0}$$

Proof. — We first prove the following Lemma:

LEMMA. — If f belongs to $\mathscr{D}_\mu^\alpha$ for some $\mu > 0$, $\alpha > 0$ then the integral

$$\mathrm{I}(p) = \int_{\mathbb{C}^+} p^\beta e^{-i\bar{z}p^2/2} f(z)(\mathrm{Im}\, z)^\alpha d^2 z \tag{4.1}$$

with $\beta = \alpha + 2 - \frac{\mathrm{N}}{2}$ is absolutly convergent for $p > 0$.

Proof. — Writing $\mathrm{I}(p)$ in polar coordinates, setting $z = \rho e^{i\theta}$, we get

$$\mathrm{I}(p) = \int_0^\pi \int_0^\infty p^\beta e^{-\frac{p^2}{2}(\sin\theta + i\cos\theta)\rho} f(\rho e^{i\theta})(\rho\sin\theta)^\alpha \rho d\rho d\theta\,. \tag{4.2}$$

By the estimate (1.11):

$$|\, f(\rho e^{i\theta})\,|_h \leqslant \sqrt{\frac{\alpha+1}{4\pi}}(\rho\sin\theta)^{-1-\alpha/2}\,\|\, f\,\|_{\alpha,h}\,,$$

the integral over ρ is absolutely convergent for each $\theta \in\,]0, \pi[$. We just need to control it for $\theta \to 0$ and $\theta \to \pi$. Since f belongs to $\mathscr{D}_\mu^a$, $f^{e^{i\mu'}}$ belongs to $\mathscr{H}_\alpha$ for each $|\,\mu'\,| < \mu$. Then the estimate (1.11) applied to $f^{e^{i\mu'}}$ gives

$$|\, f(z)\,|_h \leqslant \sqrt{\frac{\alpha+1}{4\pi}}(\rho\sin(\theta-\mu'))^{-1-\alpha/2}\,\|\, f^{e^{i\mu'}}\,\|_{\alpha,h} \tag{4.3}$$

and making the change of variable $\rho' = \rho\sin\theta$ we get

$$\left|\int_0^\infty p^\beta e^{-\frac{p^2}{2}\rho(\sin\theta + i\cos\theta)} f(\rho e^{i\theta})\rho^{\alpha+1} d\rho\right|$$
$$\leqslant \int_0^\infty p^\beta e^{-\rho'\frac{p^2}{2}} \sqrt{\frac{\alpha+1}{4\pi}}(\sin(\theta-\mu'))^{-1-\alpha/2}\,\|\, f^{e^{i\mu'}}\,\|_{\alpha,h}\rho'^{\frac{\alpha}{2}} d\rho' (\sin\theta)^{-1-\alpha/2}$$

which ensures the convergence of $\mathrm{I}(p)$ for $\theta \to 0$. The same argument for $\mu' > 0$ gives the result for $\theta \to \pi$ and the Lemma is proved. We now prove the proposition. Let us suppose φ $\mathrm{S}_{\alpha/2}$-dilation analytic. Then, by the inversion formula (A.2′) and theorem 1, $\varphi(p)$ is equal almost everywhere to the function:

$$\psi(p) = \frac{1}{\sqrt{\pi\Gamma\left(\beta + \frac{\mathrm{N}}{2} - 1\right)}} \int p^\beta e^{-i\bar{z}\frac{p^2}{2}} f(z)(\mathrm{Im}\, z)^\alpha d^2 z \tag{4.4}$$

with f belonging to $\mathscr{D}(a, \alpha)$ and $\beta = \alpha + 2 - \frac{\mathrm{N}}{2}$.

Moreover, for $\tau > 0$, we have from (A.4)

$$\tau^{-N/4}\psi(\tau^{-1/2}p) = \frac{1}{\sqrt{\pi\Gamma\left(\beta+\frac{N}{2}-1\right)}} \int p^{\beta} e^{-i\bar{z}\frac{p^2}{2}} f^{\tau}(z)(\operatorname{Im} z)^{\alpha} d^2 z . \quad (4.5)$$

From the analyticity of the family of vectors f^{τ} with respect to τ in S_a and the fact that for fixed τ in S_a, f^{τ} belongs to $\mathscr{D}(\mu, \alpha)$ with $\mu < (a - \operatorname{Arg} \tau)$ we get, by use of Lemma 6, the analyticity of ψ in $S_{a/2}$ and the fact that the dilation-analytic extension φ^{τ} is given by (4.0). Property *ii*) of Proposition 5 is a direct consequence of the definition of $\mathscr{D}(a, \alpha)$.

If now ψ satisfies *i*) and *ii*) of Proposition 5 it is very easy to see that the function f defined by

$$f(z) = \frac{1}{\sqrt{\pi\Gamma\left(\beta+\frac{N}{2}-1\right)}} \int p^{\beta} e^{iz\frac{p^2}{1}} \psi(p) p^{N-1} dp$$

is in $\mathscr{D}(a, \alpha)$ and hence that ψ is $S_{a/2}$-dilation analytic.

b) Integral operators in momentum space.

As far as operators are concerned, it is of course not possible to characterize the kernels of bounded integral operators. Based on theorem 2 and a well-known criterion for an integral operator to be bounded we shall give a sufficient condition for an integral operator in momentum space to be dilation-analytic.

LEMMA 7. — Let an integral operator K in the sense of section 3, in $L^2(\mathbb{C}^+, h, d\mu_\alpha)$ given by the kernel $K(z, \bar{z}')$ which is a $\mathscr{B}(h)$-valued analytic function on $\mathbb{C}^+ \times \mathbb{C}^-$.

Let us suppose that

$$\sup_{z \in \mathbb{C}^+} \int_{\mathbb{C}^+} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z')^{\alpha} d^2 z' = C_1^2 < +\infty \quad (4.6)$$

and

$$\sup_{z' \in \mathbb{C}^+} \int_{\mathbb{C}^+} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z)^{\alpha} d^2 z = C_1^2 < +\infty .$$

Then K is a bounded operator in $L^2(\mathbb{C}^+, h, d\mu_\alpha)$ and $\| K \| \leqslant C_1$.

Proof. — We refer to Weidman [7] theorem 6.24 and corollary. The proof given there for scalar-valued kernels is easily adapted to $\mathscr{B}(h)$-valued kernels on replacing the modulus of kernel function and scalar function by the norms in $\mathscr{B}(h)$ and h respectively.

LEMMA 8. — Let K be as above. Let

$$\kappa_n = \left\{ \xi \in \mathbb{C}^+ / - n < \operatorname{Re} \xi < n, \frac{1}{n} < \operatorname{Im} \xi < n \right\} \tag{4.7}$$

$$\kappa'_n = \mathbb{C}^+ \backslash \kappa_n$$

$$\sup_{z \in K_n} \int_{K'_n} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z)^\alpha d^2 z = C_n \tag{4.8}$$

$$\sup_{z' \in K'_n} \int_{K_n} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z)^\alpha d^2 z = C'_n \tag{4.8'}$$

$$\sup_{z \in K'_n} \int_{K_n} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z')^\alpha d^2 z' = D_n \tag{4.9}$$

$$\sup_{z \in K'_n} \int_{K_n} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z')^\alpha d^2 z' = D'_n \tag{4.9'}$$

$$\sup_{z \in K'_n} \int_{K'_n} \| K(z, \bar{z}') \|_{\mathscr{B}(h)} (\operatorname{Im} z')^\alpha d^2 z' = E_n \tag{4.10}$$

$$\sup_{z' \in K'_n} \int_{K'_n} \| K(z, z') \|_{\mathscr{B}(h)} (\operatorname{Im} z)^\alpha d^2 z = E'_n . \tag{4.10'}$$

Assume that, for $n \to \infty$

$$C_n, C'_n \to 0 ; \qquad D_n, D'_n \to 0 ; \quad E_n, E'_n \to 0 . \tag{4.11}$$

Then the operator K is compact on $L^2(\mathbb{C}^+, h ; d\mu_\alpha)$.

Proof. — We write K as

$$K = K_{1n} + K_{2n} + K_{3n} + K_{4n}$$

where $K_{1n} = \chi_{K_n} K \chi_{K_n}$, $K_{2n} = \chi_{K_n} K \chi_{K'_n}$, $K_{3n} = \chi_{K'_n} K \chi_{K_n}$

and $K_{4n} = \chi_{K'_n} K \chi_{K'_n}$; $\chi_A(z) = 1$ if $z \in A$ and $= 0$ if $z \notin A$.

The operator K_{1n} is a Hilbert-Schmidt operator and therefore compact. By Lemma 7 it follows that for $n \to \infty$,

$$\| K_{in} \| \to 0 \qquad \text{for} \quad i = 2, 3, 4$$

and hence K is compact.

THEOREM 3. — Let U be an operator given, in momentum representation, using polar coordinates (p, u), by

$$(Uf)(p) = \int_{\mathbb{R}^+} U(p, p') f(p') p'^{n-1} dp' \tag{4.2}$$

where $U(p, p') \in \mathscr{B}(h), \qquad h = L^2(S^{n-1})$.

Let $V(z, \bar{z}')$ and $K(z, \bar{z}')$ be the operators in $\mathscr{B}(h)$ given for $z, z' \in \mathbb{C}^+$ by:

$$V(z, \bar{z}') = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (pp')^\beta e^{iz\frac{p^2}{2} - i\bar{z}'\frac{p'^2}{2}} u(p, p') p^{n-1} p'^{n-1} dp dp' \tag{4.13}$$

and

$$K(z, \bar{z}') = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (pp')^\beta e^{iz\frac{p^2}{2}} e^{-i\bar{z}'\frac{p'^2}{2}} (p^2 - i)^{-1} u(p, p') p^{n-1} p'^{n-1} dp dp' \tag{4.14}$$

Assume that $K(z, \bar{z}')$ satisfies the conditions of lemma 8 and that $V(z, \bar{z}')$ has a $\mathscr{B}(h)$-valued analytic extension to the domain F_a.

Moreover assume that

$$\sup_{z \in \mathbb{C}^+} \int | K(e^{i\varphi}z, e^{i\varphi}\bar{z}') |_{\mathscr{B}(h)} d^2z' < C(\varphi) \tag{4.15}$$

and

$$\sup_{z' \in \mathbb{C}^+} \int | K(e^{i\varphi}z, e^{i\varphi}\bar{z}') |_{\mathscr{B}(h)} d^2z < C'(\varphi) \tag{4.16}$$

with $C(\varphi)$ and $C'(\varphi)$ uniformly bounded on every compact of $]-a, a[$.

Then U is S_a-dilation analytic.

Proof. — The operator $R_0(i)U$ is clearly represented in $\mathscr{H}_\alpha^h$ by the kernel $K(z, \bar{z}')$ of (4.14). By Lemma 8, U is H_0-compact. The analyticity projection of $V(z, \bar{z}')$ and the conditions (4.15) and (4.16) together with Lemma 7 imply that the operator defined by the kernel $V(\bar{z}, \bar{z}')$ is S_a-dilation analytic.

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