YVONNE CHOQUET BRUHAT

Global existence theorems for hyperbolic harmonic maps


<http://www.numdam.org/item?id=AIHPA_1987__46_1_97_0>
Global existence theorems for hyperbolic harmonic maps

by

Yvonne CHOQUET BRUHAT
Institut de Mécanique, Université Paris VI

INTRODUCTION

Harmonic maps between properly riemannian manifolds have a long mathematical history, with global results of great interest in geometry and physics (cf. [1] [2] [16]). The harmonic maps with source a riemannian manifold of hyperbolic (lorentzian) signature have been studied more recently, however such maps appear in numerous problems of physics, from the harmonic gauge of General Relativity to the non-linear $\sigma$ models and many others, as pointed out in [15] [17] and [14].

The natural problem for such « hyperbolic harmonic maps » is the Cauchy problem, that is the determination of the map from its value, and the value of its first derivative, on a space like submanifold of the source. A local existence theorem of a solution of the problem for harmonic maps from $M^{n+1}$, Minkowski space time of arbitrary dimension, into various compact riemannian manifold has been proved by Ginibre and Velo [10], using I. Segal’s theory of non linear semi-group [12]. They also prove for such problems a global existence theorem when $n = 1$, by using energy estimates. A global existence from $M^2$ into a complete riemannian has been proved, by a different method, for smooth data by Gu Chao Hao [11].

In this article we prove a local existence theorem for harmonic maps from a globally hyperbolic manifold $(M, g)$ into a riemannian manifold $(N, h)$ both arbitrary except for some regularity conditions. The proof uses the embedding of $(N, h)$ in an euclidean space $(\mathbb{R}^q, e)$, like Ginibre and Velo [10]. Another proof, which used only the standard local existence and uniqueness results for hyperbolic equations had been indicated in [3], which treated harmonic gauges in General Relativity.
We prove in § 5 a global existence theorem in the case $n = 1$, using the second order equation satisfied by the differential of an harmonic mapping and, like Ginibre and Velo, the resulting a priori estimates.

In § 6 we prove a global existence theorem for $(M, g) = M^{n+1}$, with $n$ odd, arbitrary if the Cauchy data are sufficiently near from those of a constant map. We use the method of conformal transformation as in [6] [7] and [8]. The theorem is valid for $n = 1$, because of conformal invariance, and for $n = 3$ because the operator satisfies an analogue of the condition indicated as sufficient by Christodoulou in [8] (which treats scalar systems).

It results from counter examples constructed by Shatah [9] that this last theorem cannot be true for arbitrary, large, data.

1. DÉFINITIONS

Let $(M, g)$ and $(N, h)$ be two smooth riemannian manifolds of arbitrary signature and dimensions. Let $f : M \to N$ be a smooth map. The differential of $f$ at $x \in M$ is a linear map

$$\nabla f(x) : T_x M \to T_{f(x)} N$$

it is therefore an element of $T^*_x M \otimes T_{f(x)} N$. The differential itself, $\nabla f$ is a mapping $x \to \nabla f(x)$, that is a section of the vector bundle with base $M$ and fiber at $x$ the vector space $T^*_x M \otimes T_{f(x)} N$. This vector bundle—one forms on $M$ with values at $x$ in $T_{f(x)} N$—is denoted $T^* M \otimes f^{-1} N$. The vector bundle with base $M$ and fiber $T_{f(x)} N$ at $x$ is denoted $f^{-1} TN$. If $(x^a)$ and $(y^a)$ are respectively local coordinates in $M$ and $N$, and $f$ is represented in these coordinates by

$$y^a = f^a(x^\gamma)$$

the derivative $f$ is represented by

$$(x^\gamma) \mapsto \left( \frac{\partial f^a}{\partial x^\gamma}(x^\gamma) \right).$$

The metrics $g$ on $M$ and $h$ on $N$ endow the fiber at $x$ of the vector bundle $E = T^* M \otimes f^{-1} TN$ with a scalar product $G(x) = g^a(x) \otimes h(f(x))$, where $g^a$ is the contravariant tensor canonically associated with $g$. In coordinates, if $u$ and $v$ are two sections of $E$ :

$$G(x)(u, v) = g^{ab}(x) h_{ab}(f(x)) u^a(x) v^b(x).$$

The vector bundle $E = T^* M \otimes f^{-1} TN$ is endowed with a linear connection $\nabla$, mapping sections of $E$ into sections of $T^* M \otimes E$, by the usual rules: if $s$ is a section of $f^{-1} TN$ and $t$ a section of $T^* M$ we have:

$$\nabla_t (t \otimes s) = s \nabla_t + t \otimes f^* \nabla_s$$

Annales de l'Institut Henri Poincaré - Physique théorique
with $\partial V$ and $\partial H$ the riemannian covariant derivatives in the metrics $g$ and $h$ respectively. In local coordinates if $(x^a) \mapsto (u^a(x^a))$ is a section of $E$, we have:

(2) $\nabla_a u^b(x^a) = \partial_a u^b(x^a) + \frac{\partial f^b}{\partial x^a} \Gamma^b_{c \rho}(f^c(x^a)) u^c(x^a) - \Gamma^a_{\rho \beta}(x^\lambda) u^\lambda(x^\lambda)$

where $\Gamma^\alpha_{\beta \gamma}$ and $\Gamma^\lambda_{\rho \beta}$ denote respectively the riemannian connexions of $g$ and $h$. The mapping $f$ is called harmonic if

(3) $\text{tr}_g \nabla^2 f = 0$

that is, in local coordinates

$$g^{ab} \partial_a \partial_b f^c \equiv g^{ab}(\partial_a \partial_b f^c - \Gamma^c_{ab} \partial^a f^b + \Gamma^a_{bc} \partial_b \partial_c f^b) = 0.$$ 

If $f$ satisfies (3) it is a critical point of the functional

$$f \mapsto E(f) = \int_M G(\nabla f, \nabla f) d\mu(g) = \int_M g^{ab}(x^a) h_{ab}(f^a(x^a)) \partial_a f^a \partial_b f^b d\mu(g).$$

2. HYPERBOLIC HARMONIC MAPS. ENERGY INTEGRAL

When the metrics $g$ and $h$ are properly riemannian the integral (4) is called the energy of the mapping $f$. When $g$ is of hyperbolic signature we will define another integral as the energy of $f$, like for usual wave equations.

We define the stress energy tensor of the map $f$ as the covariant 2-tensor on $M$ given by

$$T = (h \circ f)(\nabla f, \nabla f) - \frac{1}{2} g(g^a \otimes h \circ f)(\nabla f \otimes \nabla f)$$

that is

(2.1) $T_{x^\beta} = (h_{ab} \circ f) \partial_a f^a \partial_b f^b - \frac{1}{2} g_{ab} g^{c\mu}(h_{ab} \circ f) \partial_c f^a \partial_\mu f^b$

We have on $M$

(2.2) $\nabla_a T_{x^\beta} \equiv (h_{ab} \circ f) \partial_\beta f^a \nabla^a \partial_b f^b$

that is

$$\nabla \cdot T = (h \circ f)(\nabla f, \text{tr}_g \nabla^2 f).$$

Thus $\nabla \cdot T = 0$ if $f$ is a harmonic map.

We suppose that $(M, g)$ is a hyperbolic manifold with $M = S \times \mathbb{R}$, $S_t \equiv S \times \{ t \}$ space-like, we denote by $n$ their unit time like normal. Let $X$ be a time like vector field. We define the energy density of $f$ relative to $S_t$ and $X$ by:

$$e(f) = T(X, n) = X^2 h^\rho \partial_\rho T_{x^\beta}$$

we have

(2.3) $e(f) = \frac{1}{2} g^{ab}(h_{ab} \circ f) \partial_a f^a \partial_b f^b = \frac{1}{2} g^a \otimes (h \circ f)(\nabla f, \nabla f)$

where $\gamma^g$ is the quadratic form

$$\gamma^g = n^\beta X^\beta + n^\alpha X^\alpha - g^\alpha_\beta X^\alpha n_\beta.$$

It is well known that this form is positive definite if $g$ of hyperbolic signature $(+, -, \ldots)$ with $X$ and $n$ time like.

We deduce from

$$V_\beta T^\beta = 0$$

when $f$ is a harmonic map that

$$(2.4) \quad \nabla_\beta (T_\beta T^\beta) = \frac{1}{2} T^\beta (\nabla_\beta X_\beta + \nabla_\beta X_\alpha).$$

By integration of $2.4$ on $S \times [0, t]$ we obtain the following:

**Proposition.** — Let $f$ be a smooth map such that $\nabla f |_{S_t}$ has a compact support for $0 \leq \tau \leq t$ then, if $f$ is harmonic it satisfies the identity:

$$(2.5) \quad \int_{S_t} \nabla f d\mu_t = \int_{S_0} \nabla f d\mu_0 + \frac{1}{2} \int_0^t \int_{S_t} N(T.LX) d\mu_t$$

$d\mu_t$ denotes the volume element of the metric $\bar{g}_t$, induced on $S_t$ by $g$, $N$ is the lapse function, $N = g(X, n)$, that is $N = (g^{00})^{-1/2}$ if $X$ is the tangent vector to the curves $\{x\} \times \mathbb{R}$ and the coordinates are adapted to the product $S \times \mathbb{R}$; the volume element of $(M, g)$ is

$$d\mu(g) = N d\mu_0 dx^0.$$

**Definition 1.** — The manifold $(M, g)$ is said regularly hyperbolic if

1) $M$ and $g$ are smooth and $M = S \times \mathbb{R}$, the metrics $\bar{g}_t$ induced on $S_t = S \times \{t\}$ are (properly) riemannian (1), and uniformly equivalent to the metric $\bar{g}_0$ which is complete.

2) The tangent vector $X$ to the lines $\{x\} \times \mathbb{R}$ is time like, and there exists strictly positive numbers $a$ and $b$ such that

$$\inf_M g(X, X) \geq a \geq 0 \quad \text{and} \quad \sup_M N \leq b$$

we then have also, since $N = g(X, n) \geq (g(X, X))^{1/2}$

$$\inf_M N \geq a^{1/2} \quad \text{and} \quad \sup_M g(X, X) \leq b^2.$$

If $(M, g)$ is regularly hyperbolic the metric $\gamma$ on $M = S \times \mathbb{R}$ defined by $2.3$ is uniformly equivalent to the metric $\Gamma = (dx^0)^2 - \bar{g}_0$.

We shall then add to the definition of regular hyperbolicity the following.

3) The riemann curvature of $g$, together with as many of its covariant derivatives as is relevant, is bounded in $\Gamma$-norm.

---

(1) These metrics are negative definite: $-\bar{g}_t$ is positive definite.
DEFINITION 2. — A (properly) riemannian manifold is said to be regular if it is smooth, has a non zero injectivity radius (thus is complete), and has a bounded riemannian curvature, as well as its covariant derivatives up to the relevant order.

Tensor products of the metrics $\Gamma$ and $h$ give scalar products and norms in the fiber at $x \in M$ of vector bundles $(\otimes T_x M)^p (\otimes f^{-1} TN)^q$ or their duals. We denote this norm by $| |$. We have, for instance,

$$| \nabla f |^2 = \Gamma^{ab} (h_{ab} \circ f) \partial_a f^c \partial_b f^d.$$  

If $s$ and $u$ are two sections of such vector bundles we have, at a point $x \in M$

$$| s \otimes u | = | s | | u |, \quad | s . u | \leq | s | | u |$$

if $s . u$ is some contracted tensor product.

Therefore, in particular

$$| LX . T | \leq | LX | | T |.$$  

It results from the definition that $| g |$ and $| g^\# |$ are uniformly bounded if $(M, g)$ is regularly hyperbolic (cf. [5]). Thus, due to the expression of $T$, there exists a constant $C$ such that

(2.6)  

$$| T | \leq C e(f)$$

and also, if $| X |$ is uniformly bounded on $M$, a constant still denoted $C$ such that

$$| LX . T | \leq C e(f).$$

From the equality (2.5) results then the inequality ($C_0$ and $C$ positive constants)

(2.7)  

$$y(t) \leq C_0 y(0) + C \int_0^t y(\tau) d \tau$$

with

$$y(t) = \int_{S_t} | \nabla f |^2 d \mu_0.$$  

If $y$ is a continuous function of $t$ we deduce from (2.7), by the Gromwall lemma

(2.8)  

$$y(t) \leq K(t) y(0)$$

with $K(t)$ the continuous function of $t$

$$K(t) = C_0 e^{Ct}.$$  

3. SECOND ORDER EQUATION FOR $f$

PROPOSITION. — Every smooth harmonic map $f : (M, g) \to (N, h)$ satisfies the equation

$$(\nabla . \nabla) (\nabla f) - \text{Ricc} (g) \nabla f + \text{tr}_g (f^* \text{Riem} (h) . \nabla f) = 0.$$
that is, in local coordinates

\[(3.1) \quad \nabla^a \nabla_a \partial_a f^a - R^a_{\beta \mu} \partial_\beta f^a + R_{\sigma \mu} \partial_a f^a \partial_\sigma \partial_\mu f^b = 0.\]

The proof, independent of signature, is straightforward and given in [1].

We now consider the case \( g \) hyperbolic and \( h \) properly riemannian. We set \( (h_{ab} \) stands always for \( h_{ab} \circ f \))

\[(3.2) \quad T_{ab}^{(1)} = e^{\lambda \mu} h_{ab} \left\{ \nabla_a \partial_\lambda f^a \nabla_\beta \partial_\mu f^b - \frac{1}{2} g_{ab} g_{\rho \sigma} \nabla_\rho \partial_\lambda f^a \nabla_\sigma \partial_\mu f^b \right\} \]

We have identically, after use of the Ricci identity

\[(3.3) \quad \nabla_a T_{a \beta}^{(1)} = e^{\lambda \mu} h_{ab} \nabla_a \partial_\lambda f^a \nabla_\beta \partial_\mu f^b + e^{\lambda \mu} h_{ab} \nabla_a \partial_\lambda f^a (- R^b_{\beta \mu} \partial_\beta f^b + \partial_\sigma \partial_\beta f^b \partial_\mu f^c R_{cd} \epsilon) + h_{ab} \left\{ \nabla_a \partial_\lambda f^a \nabla_\beta \partial_\mu f^b - \frac{1}{2} g_{ab} g_{\rho \sigma} \nabla_\rho \partial_\lambda f^a \nabla_\sigma \partial_\mu f^b \right\} \]

Using (3.1) we see that, for a harmonic map, \( \nabla_a T_{a \beta}^{(1)} \) is a polynomial \( Q(f, \nabla f, \nabla^2 f) \) of degree 2 in \( \nabla^2 f \), with coefficients of degree 1 or 3 [respectively 0] in \( \nabla f \) for the terms of degree 1 [respectively 2] in \( \nabla^2 f \). The mapping \( f \) itself appears through \( h \circ f \) and \( \text{Riem}(h) \circ f \).

On the other hand we have:

\[ e_{1,f} = T_{a \beta}^{(1)} X_a \mu^\beta = \frac{1}{2} \epsilon^{\alpha \beta \gamma} \mu h_{ab} \nabla_a \partial_\lambda f^a \nabla_\beta \partial_\mu f^b = 0. \]

Integrating the identity:

\[ \nabla_a (X_a \mu^{a \beta} T_{a \beta}^{(1)}) = X_{\beta} \nabla_{\beta} T_{a \beta}^{(1)} + \frac{1}{2} T_{a \beta}^{(1)} (\nabla_a X_\beta + \nabla_\beta X_a) \]

with \( T_{a \beta}^{(1)} \) and \( \nabla_a T_{a \beta}^{(1)} \) given by (3.2) and (3.3) gives for a harmonic map, with compact support in space, using (3.1), an equality:

\[(3.4) \quad \int_{S_t} Q_1(f, \nabla f, \nabla^2 f) \, d\mu(g_\tau) = \int_{S_0} Q_1(f, \nabla f, \nabla^2 f) \, d\mu(g_0) + \int_0^t \int_{S_\tau} Q_1(f, \nabla f, \nabla^2 f) \, d\mu(g_\tau) \, d\tau \]

with \( Q_1 \) of the type

\[ Q_1(f, \nabla f, \nabla^2 f) = \Sigma k \nabla^2 f \cdot \{ \text{Riem} \cdot \nabla f + \text{Riem} \circ f, (\text{Riem} \circ f)^3 \} + \nabla^2 f \]

with \( k \) polynomial in \( g^\#, h, X \) and \( \nabla X \).

If \( (M, g) \) is regularly hyperbolic, and \( (N, h) \) regularly riemannian we deduce from (3.4) an inequality, as in § 2; with \( C_0, C_1, C_2, C_3 \) positive constants:

\[(3.5) \quad y_1(t) \leq C_0 y_1(0) + C_1 \int_0^t y_1(\tau) d\tau + \int_0^t \left( C_2 | \nabla^2 f | + | \nabla f | + C_3 | \nabla^2 f | + \nabla f |^3 \right) d\mu_0 d\tau \]

Annales de l'Institut Henri Poincaré - Physique théorique
where, by definition
\[ y_1(t) = \int_{s_1} |\nabla^2 f|^2 d\mu_0 \]
and
\[ |\nabla^2 f|^2 = e^{\lambda u} e^{a^b (h_{ab} \circ f)} \nabla_a \partial_x f^a \nabla_b \partial_x f^b. \]

4. LOCAL EXISTENCE

Let \( N \) be a submanifold of the riemannian manifold \((Q, q)\) and \( h \) be the metric induced on \( N \) by \( q \). We shall suppose that \( N \) is defined by \( p \) equations
\[ N : \Phi^I(z) = 0, \quad z \in Q, \quad I = 1, \ldots, p \]
where \( \phi = (\Phi^I : Q \to \mathbb{R}^p) \) is a smooth map of rank \( p \) at each point of \( N \). The matrix \( m = (m^{IJ}) \) given by
\[ m = q(\nabla \phi, \nabla \phi), \quad \text{i.e.,} \quad m^{IJ} = (q^{AB} \partial_A \phi^I \partial_B \phi^J) \circ f \]
\((x^A \text{coordinates in } Q)\) is then positive definite on \( M \) when \( f \) takes its values in \( N \). We denote by \( m^{-1} = m_{IJ} \) the inverse matrix.

**Lemma 1.** — A necessary and sufficient condition for the mapping \( f : M \to N \subset Q \) to be a harmonic map from \((M, g)\) into \((N, h)\) is that, as a mapping \( M \to Q \) it satisfies the equations which read in local coordinates \( x^a \) in \( M \) and \( x^A \) in \( Q \):
\[ \hat{\nabla}^A \nabla_x f^A + \lambda_4 (q^{AB} \partial_B \Phi^I) \circ f = 0 \]
where \((\hat{\nabla}^A \nabla_x f^A)\) is the tension field of the map \( f : (M, g) \to (Q, q) \) and
\[ \lambda_4 = m_{IJ} g^{\#} \partial_x f^A \partial_\phi f^B (\hat{\nabla}^A \partial_B \phi^I) \circ f \]
together with the conditions
\[ \phi \circ f = 0. \]

**Proof** (cf. a particular case in Ginibre and Velo [10]).
A mapping \( f : M \to N \) defines a mapping \( F : M \to Q \) by
\[ F = i \circ f \]
where \( i \) denotes the embedding (identity map) \( N \to Q \).

The integral constructed with \( F : M \to Q \)
\[ \mathbb{E}(F) = \int_M (g^\# \otimes q)(\nabla F, \nabla F) d\mu(g) \]
is equal to the integral (1.4) constructed with \( f \) since
\[ \nabla F = \nabla_i f^a, \quad \text{i.e.,} \quad \partial_a F^A = \partial_a f^A \]
and \( h \) is the metric induced by \( i \) on \( N \); that is \( h_{ab} = q_{AB} \partial_a i^A \partial_b i^B \), and
\[
E(f) = E(F).
\]
Thus a critical point of \( E(f) \) is a critical point of \( E(F) \) with the constraint \( \phi(F) = 0 \), that is a solution of equations of the form:
\[
\hat{\nabla}^a \nabla_a F^A + \lambda_i q^{AB} \partial_B \Phi^I = 0
\]
where the \( \lambda_i \) (Lagrange multipliers) are determined by derivating twice the conditions
\[
\Phi^I \circ F = 0
\]
with \( \Phi^I \circ F \) considered as a mapping \( M \to Q \to \mathbb{R}^p \), and contracting with \( g \):
\[
\nabla^a \nabla_a (\Phi^I \circ f) = \partial_A \Phi^I \hat{\nabla}^a \partial_2 F^A + g^{ab} \partial_b \partial_a F^B \partial_B \partial_2 \Phi^I = 0
\]
comparing (4.5) and (4.6) gives
\[
\lambda_i q^{AB} \partial_B \Phi^I = g^{ab} \partial_b \partial_a F^B \partial_B \partial_2 \Phi^I
\]
(4.7) is equivalent to (4.2) \( F = i \circ f = f \), since \( i \) is the identity mapping \( N \to N \subset Q \).

**Lemma 2.** — The equations (4.1) satisfied by a mapping \( M \to Q \), with \( \lambda_i \) given by (4.2), imply that the mapping \( \phi \circ f : M \to \mathbb{R}^p \) satisfies the homogeneous wave equation on \( M \)
\[
\nabla^a \nabla_a (\phi \circ f) = 0.
\]

**Proof.** — (4.6) implied by (4.1) and (4.2).

**Definition 3.** — A submanifold \( N \) of \( \mathbb{R}^q \) given by \( N = \{ y \in \mathbb{R}^q, \phi(y) = 0 \} \) with \( \phi \) a smooth map \( \mathbb{R}^q \to \mathbb{R}^p \) is said to be regularly defined by \( \phi \) if there exists \( a > 0 \) and \( \varepsilon > 0 \) such that
\[
\inf_{y \in N_\varepsilon} \det m^I(y) \geq a > 0, \quad N_\varepsilon = \{ y \in \mathbb{R}^q, d(y, N) < \varepsilon \}
\]
denotes the euclidean distance. The definition means that \( \phi \) is uniformly of rank \( p \) in some uniform neighbourhood of \( N \).

**Definition 4.** — We denote by \( H^q(S) \) the Sobolev space of \( \mathbb{R}^q \)-valued functions on the regularly riemannian manifold \( (S, g_0) \), closure of \( C^\infty \), \( \mathbb{R}^q \) valued functions with compact support on \( S \) in the norm:
\[
\| \phi \|_{H^q}^2 = \int_S \sum_{k=0}^q |D^k \phi|^2 d\mu_0
\]
where \( D \) is the covariant derivative in the metric \( \bar{g}_0 \) for each scalar valued

\[Annales de l’Institut Henri Poincaré - Physique théorique\]
map $\varphi^A : S \to \mathbb{R}$, and $|\varphi|$ is the $g_0$ and $e$ norm of the set $D^k \varphi = (D^k \varphi^A)$, for instance

$$\left| D^2 \varphi \right|_e^2 = e_{AB} \delta_0 ^{\delta_0 D^2 _{ij} \delta_0 D^2 \varphi^A D^2 _{jk} \varphi^B, \quad e_{AB} = \delta_{AB}. $$

**Theorem (local existence).** — Let $(M, g)$, $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension $n + 1$ (definition 1).

Let $(N, h)$ be a regular riemannian manifold, regularly defined by a mapping $\varphi : \mathbb{R}^q \to \mathbb{R}^p$. Let $\varphi, \psi$ be mappings $S \to \mathbb{R}^q$, $\varphi \in H_d(S), \psi \in H_{s-1}(S), \quad s > \frac{n}{2} + 1$ such that $\Phi \circ \varphi = 0, (\nabla \varphi, \varphi) \cdot \psi = 0$.

Then there exists $l > 0$ and on $S \times (-l, l)$ a harmonic map $f : (M, g) \to (N, h)$, with $h$ the metric induced on $N$ by the euclidean metric $e$ of $\mathbb{R}^q$, such that

$$f |_{S_0} = \varphi, \quad \partial_0 f |_{S_0} = \psi.$$ 

**Proof.** — We apply lemma 1 with $(Q, q) = (\mathbb{R}^q, e)$. Equations (4.1) reads then:

$$\nabla^2 \nabla \varphi^A + (\nabla \varphi^T \nabla \varphi) \Delta^{-1}(f) g^{\alpha \beta} \partial_\alpha \varphi^A \partial_\beta f^B (\delta_{AB} \Phi)(f) = 0;$$

they are a system of $q$, numerical, quasi-linear, quasi-diagonal second order hyperbolic equations on $M$, with smooth coefficients if $d(f, N) < e$. The local existence theorem, on $S \times (-l, l)$ is a standard result, since $d(\varphi, N) = 0$. The solution $f : S \times (-l, l) \to \mathbb{R}^q$ satisfies $\varphi \circ f = 0$ because $\varphi \circ f$ satisfies the homogeneous wave equation (4.8) with zero Cauchy data:

$$\varphi \circ f |_{S_0} = \varphi | = 0, \quad \partial_0 (\varphi \circ f) |_{S_0} = (\nabla \varphi \circ f \cdot \partial_0 f |_{S_0} = (\nabla \varphi \circ f \cdot \psi = 0.$$

**Remark 1.** — The local existence theorem for a numerical hyperbolic system gives that the interval of existence depends continuously on the $H^1_0 \times H^1_{s-1}$ norm, $s_0$ smallest integer such that $s_0 > \frac{n}{2} + 1$, of the Cauchy data, and tends to infinity when these norms tend to zero. If $N$ is a submanifold of $\mathbb{R}^q$, we can always, by translation, take the origin of $\mathbb{R}^q$ at some arbitrary given point $y_0$ of $N$. The $H_{d}(S_0)$ norm of a map $\varphi : S_0 \to \mathbb{R}^q$ is by definition

$$\| \varphi \|_s = \left\{ \sum_{k=0}^s \int_S |\nabla^k \varphi |^2 d\mu_0 \right\}^{1/2}.$$
with

$$|φ|^2 = \sum_{\lambda=1}^{q} |φ^\lambda|^2.$$  

Small $H_2$ norm for $φ$ means then nearness of $φ$ from the constant map $M \to y_0$.

**REMARK 2.** — Every riemannian manifold $(N, h)$ can be isometrically embedded in a space $(\mathbb{R}^q, e)$—we have supposed moreover that $N$ is given by equations $Φ^l = 0$. We could have proceeded without this hypothesis, either inspired by techniques used by Eells and Sampson in the elliptic case, either by using atlases on $M$ and $N$, together with local existence and uniqueness theorems (cf. an indication of such a proof in [3]).

### 5. GLOBAL EXISTENCE WHEN $n = 1$

The global existence of a solution of the Cauchy problem for harmonic maps from 2-dimensional Minkowski space time $M^2$ into a complete riemannian manifold $(N, h)$ has been proved by Gu Chao Hao [11] for smooth initial data. It has been proved by Ginibre and Velo [10] from the two dimensional Minkowski space into various compact riemannian manifolds for $H_2 \times H_1$ Cauchy data. This result can be generalized:

**Theorem.** — Let $(M, g)$, $M = S \times \mathbb{R}$, be a regularly hyperbolic manifold of dimension 2, and $(N, h)$ be a regular riemannian manifold, regularly defined as a submanifold of $(\mathbb{R}^q, e)$ by mapping $φ : \mathbb{R}^q \to \mathbb{R}^p$, $N = \{ y \in \mathbb{R}^q, $ \phi(y) = 0 \}$. Let

$$φ \in H_s(S), \quad ϕ \in H_{s-1}(S). \quad s \geq 2$$

be given maps $φ : S \to \mathbb{R}^q, ϕ : S \to \mathbb{R}^q$, such that $ψ \circ ϕ = 0, (Vψ \circ ϕ). ϕ = 0$. Then there exists on $M$ a harmonic map $f : (M, g) \to (N, h)$ taking on $S_0 = S \times \{ 0 \}$ these Cauchy data.

**Proof.** — In the case $n = 1$ the local existence theorem is valid with $s \geq 2$. The solution $f : S \times (-l, l) \to N \subset \mathbb{R}^q$ admits a restriction on each $S_t = S \times \{ t \}, |t| < l$ which is a mapping $f_t : S_t \to N \subset \mathbb{R}^q$ which belongs to $H_2(S)$ (definition 4). The derivative $∂_0 f$ admits a restriction $(∂_0 f)_t : S_t \to f^{-1}TN$ by $x \to T_{f(t,0)}N \subset \mathbb{R}^q, (∂_0 f)_t \in H_1(S)$. The energy inequality (2.9) implies, when $f = i \circ f$ is considered as a mapping into $N$:

$$y(t) \leq K(t)y(0)$$  

**Annales de l’Institut Henri Poincaré - Physique théorique**
with

$$y(t) = \int_S |\nabla f|^2 d\mu_0$$

with, due to the definition of $\Gamma$:

$$|\nabla f|^2_h = (h_{ab} f^a \nabla^b f^a + \tilde{g}_{ij} \partial_i f^a \partial_j f^b)$$

(5.2)

but we have, since $f = i \circ f$ and $h = i * e$

$$|\nabla f|^2 = |\nabla f|^2 - e_{AB}(\tilde{c}^A \partial_0 f^A \partial_0 f^B - \tilde{g}_{ij} \partial_i f^A \partial_j f^B).$$

(5.3)

The inequality (2.9) implies therefore the non-blow up of the $L^2(S, g_0)$ norm of $Df_t$ and of $(\partial_0 f)_t$, as mappings $S \to \mathbb{R}^q$, and thus also of $f_t$ : the norms $\| f_t \|_{W^{1,2}(S)}$ (and thus $\| f_t \|_{C^0(S)}$) are bounded by continuous functions of $t$ which extend to $t = + \infty$. To prove the non blow up of the second derivatives we look at the identity (3.4). Due to the regularity hypothesis we have, with $C$ a constant

$$\int_{S_t} Q_{t} d\mu_t \leq C \int_{S_t} |\nabla^2 f|^2_h (|\nabla f|^2 + |\nabla f|^4 + |\nabla^2 f|^2) d\mu_0$$

(5.4)

using the fact that $f = i \circ f$ we find inequalities of the form ($C_1$ and $C_2$ positive constants)

$$|\nabla^2 f|^2 \leq |\nabla^2 f|^2 + C_1 |\nabla f|^4$$

$$|\nabla^2 f|^2 \leq |\nabla^2 f|^2 + C_2 |\nabla f|^4$$

(5.5)

(recall that $\nabla$ denotes the covariant derivative of $f$ as a mapping $(M, g) \to (\mathbb{R}^q, e)$ and $\nabla$ as mapping $M \to N$).

We deduce from (5.4), by the Cauchy-Schwartz inequality

$$\int_{S_t} Q_{t} d\mu_t \leq C \int_{S_t} |\nabla^2 f|^2 d\mu_0$$

$$+ \left\{ \int_{S_t} |\nabla^2 f|^2 d\mu_0 \right\}^{1/2} \left\{ \left( \int_{S_t} |\nabla f|^2 d\mu_0 \right)^{1/2} + \left( \int_{S_t} |\nabla f|^6 d\mu_0 \right)^{1/2} \right\}.$$

(5.6)

Considering $f$ as mapping $M \to \mathbb{R}^q$, and setting

$$z(t) = \int_{S_t} |\nabla^2 f|^2 d\mu_0$$

we obtain, using (3.5) (5.5) and (5.6), with $C$ some constant

$$z(t) \leq C_0 y(0) + C \int_0^t \left\{ z(\tau) + \int_{S_\tau} (|\nabla f|^2 + |\nabla f|^4 + |\nabla f|^6) d\mu_0 \right\} d\tau.$$
By the inequality (2.9) we know that
\[ \int_{S^r} |\nabla f|^2 d\mu_0 = \gamma(t) \leq K(t)\gamma(0). \]
We bound the integrals \( \int_{S^r} |\nabla f|^4 d\mu_0 \) and \( \int_{S^r} |\nabla f|^6 d\mu_0 \) by using the following Sobolev inequality, valid in any dimension if \( a = \frac{n(p-1)}{p} \) and \( S \) admits a uniformly locally finite atlas:
\[ \| u \|_{L^p(S)} \leq C \| Du \|_{L^1(S)}^{\frac{1}{1-a}} \| u \|_{L^1(S)}^{\frac{1}{a}} \]
by taking
\[ u = |\nabla f|^2. \]
If \( n = 1, a = \frac{p-1}{p}, p = 1, 2 \) or 3 we have therefore:
\[ \int_{S^r} |\nabla f|^2 p d\mu_0 = (\| u \|_{L^p})^p \leq C \| Du \|_{L^1(S)}^{(p-1)/p} \| u \|_{L^1(S)}^{1/p} \]
we have
\[ \| u \|_{L^1(S)} = \gamma(t) \]
and
\[ \| Du \|_{L^1(S)}^2 \leq C \gamma(t) \zeta(t). \]
We then obtain an integral inequality for \( z(t) \), with coefficients continuous functions of \( t \) extendable for all \( t \), and at most of degree one in \( z(t) \). The non-blow up of \( z(t) \) follows.

6. GLOBAL EXISTENCE WHEN \((M, g) = M^{n+1}, \)
SMALL DATA

Let \( M^{n+1} = (\mathbb{R}^{n+1}, \eta) \) be \( n + 1 \) dimensional Minkowski space time. Let \( \Sigma^{n+1} = (S^n \times \mathbb{R}, g) \) be the Einstein cylinder with its canonical metric. \( M^{n+1} \) is known to be conformal to a subset \( V \) of \( \Sigma^{n+1} \), that is:
(6.1)
\[ g = \Omega^2 \eta \quad \text{on} \quad V = \mathbb{R}^{n+1} \]
the identification of \( V \subset \Sigma^{n+1} \) with \( \mathbb{R}^{n+1} \) being given in canonical polar coordinates respectively \((t, r, \ldots)\) on \( \mathbb{R}^{n+1} \) and \((T, \alpha, \ldots)\) on \( \Sigma^{n+1} \) by (cf. [7])
(6.2)
\[ T = \text{Arctg} (t + r) + \text{Arctg} (t - r) \]
\[ \alpha = \text{Arctg} (t + r) - \text{Arctg} (t - r) \]
\[ V : \alpha - \Pi < T < \Pi - \alpha \]

Annales de l'Institut Henri Poincaré - Physique théorique
We have
\begin{equation}
(6.3) \quad g = dT^2 - d\Omega^2 - \sin^2 \Omega d\sigma^2 = \Omega^2(\Omega^2 dt^2 - dr^2 - r^2 g_{S^{n-1}})
\end{equation}
\[ \Omega = \cos T + \cos \alpha \]
g_{S^{n-1}} metric of the sphere $S^{n-1}$.

We remark that $\Omega$ extends to an analytic function on $\Sigma^{n+1}$, which vanishes on $\partial V$; the submanifold $t = 0$ (i.e. $\mathbb{R}^n \times \{0\}$) is mapped diffeomorphically onto $T = 0$ (i.e. $S^n \times \{0\}$), minus its north pole $\alpha = \Pi$. On $S^n \times \{0\}$ we have ($\alpha = \Pi$ is $r = +\infty$)
\[ \Omega|_{T = 0} = 1 + \cos \alpha = 2(1 + r^2)^{-1}. \]

To a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ corresponds, by the previous diffeomorphism a mapping, still denoted $\varphi$, defined almost everywhere on $S^n$.

**THEOREM.** — Let $(N, h)$ be a riemannian submanifold of $(\mathbb{R}^q, e)$ regularly defined by a smooth map $\phi : \mathbb{R}^q \to \mathbb{R}^p$
\[ N = \{ y \in \mathbb{R}^q, \phi(y) = 0 \}. \]

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ and $\phi : \mathbb{R}^n \to \mathbb{R}^q$ be given mappings such that
\begin{equation}
(6.4) \quad \phi \circ \varphi = 0, \quad (\nabla \phi \circ \varphi) \cdot \phi = 0
\end{equation}
with
\[ (1 + \cos \alpha)\varphi \in H_s(S^n), \quad (1 + \cos \alpha)\phi \in H_{s-1}(S^n), \quad s > \frac{n}{2} + 1 \]
then there exists a harmonic map $f : M^{n+1} \to (N, h)$ such that
\[ f|_{\mathbb{R}^n} = \varphi, \quad \tilde{\gamma}_0 f|_{\mathbb{R}^n} = \phi \]
if the mappings $\varphi$ and $\phi$ are sufficiently near in the relevant norms respectively from a constant map and zero.

**Proof.** — It is inspired from the proofs of [7] and [8].

We do not restrict the generality by supposing that $N$ passes through the origin of $\mathbb{R}^p$, that is $\phi(0) = 0$.

To a mapping $f : \mathbb{R}^{n+1} \to \mathbb{R}^q$ we associate $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}^q$ defined on $V$ by:
\begin{equation}
(6.5) \quad \tilde{f} = \Omega^{(1-n)/2} f.
\end{equation}

We deduce then from (6.1), since the scalar curvature of $\Sigma^{n+1}$ is $n(n - 1)$ that
\begin{equation}
(6.6) \quad \Box_g \tilde{f} - \frac{(n-1)^2}{4} \tilde{f} = \Omega^{-3 + n/2} \Box_g f
\end{equation}
where $\Box_g$ and $\Box_\eta$ are the wave operators in the metrics $g$ and $\eta$ respectively. We see that $f$ satisfies (4.1) (4.2), with $(Q, q) = (\mathbb{R}^d, e)$ that is

\[(6.6 \ a) \quad \Box_\eta f^A + \lambda_1 \big( (\partial_A \phi^1) \circ f \big) = 0, \]
\[(6.7 \ b) \quad \phi(f) = 0 \]

with
\[
\lambda_1 = m_1(f) \eta^{\alpha\beta} \partial_\alpha f^\Lambda \partial_\beta f^B \big[ (\partial_{AB}^2 \phi^1) \circ f \big], \quad m_1 = \sum_{A=1}^q (\partial_A \phi^1 \partial_B \phi^1) \circ f
\]
if and only if
\[(6.8) \quad \Box_g \tilde{f} - \frac{(n-1)^2}{4} \tilde{f} + \Omega^{-(3+n/2)} \lambda_1 \big( (\partial_A \phi^1) \circ \Omega^{(n-1)/2} \tilde{f} \big) = 0
\]
with
\[
\lambda_1 = m_1 \Omega^{2} \eta^{\alpha\beta} \partial_\alpha \Omega^{(n-1)/2} \tilde{f}^A \partial_\beta \Omega^{(n-1)/2} \tilde{f}^B \big[ (\partial_{AB}^2 \phi^1) \circ (\Omega^{(n-1)/2} \tilde{f}) \big]
\]
and
\[
\phi(\Omega^{(n-1)/2} \tilde{f}) = 0.
\]

We have
\[
\partial_x (\Omega^{(n-1)/2} \tilde{f}^A) = \Omega^{(n-1)/2} \partial_x \tilde{f}^A + (n-1)/2 \Omega^{(n-3)/2} \partial_x \Omega \tilde{f}^A
\]
$\forall \Omega$ extends to a bounded function on $\Sigma^{n+1}$, and so does $\Omega^{-1}g(\nabla \Omega, \nabla \Omega)$ (cf. [8]) since
\[
g(\nabla \Omega, \nabla \Omega) = g^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega = \cos^2 \alpha - \cos^2 T = \Omega (\cos \alpha - \cos T)
\]
Therefore the equation (6.8) extends to a semi-linear, semi-diagonal second order system with smooth coefficients for a mapping $\tilde{f}$ from an open set $U$ of $\Sigma^{n+1}$ into $\mathbb{R}^p$ if on the one hand $n$ is odd and if, on the other hand, $\tilde{f}$ is such that
\[
d((\Omega^{(n-1)/2} \tilde{f})(X), N) < \eta \quad \forall X \in U
\]
this last property will be a fortiori satisfied since $0 \in N$ if
\[
\sup_{X \in U} | \Omega^{(n-1)/2}(X) \tilde{f}(X) |_e < \eta
\]
thus if
\[
\sup_{X \in U} | \tilde{f}(X) |_e < \eta.
\]
The existence of an open set $U = S^n \times (-l, l)$ where the equation (6.8) has a solution $\tilde{f}$ taking the Cauchy data:
\[
\tilde{f} \big|_{S^n \times \{0\}} = (1 + \cos \alpha)^{(1-n)/2} \phi
\]
\[
\partial_0 \tilde{f} \big|_{S^n \times \{0\}} = (1 + \cos \alpha)^{-(1+n)/2} \phi
\]
is then a consequence of the local existence theorem, and Sobolev inequalities, if $s > \frac{n}{2} + 1$. The length depends continuously on the norms of the Cauchy data, and we have $l > \Pi$ if these norms are small enough.

Annales de l'Institut Henri Poincaré - Physique théorique
The mapping \( f = \Omega^{(n-1)/2} f \) is defined on \( M^{n+1} \), satisfies (4.1), and also (4.2) (lemma 2).

**Remark.** The hypothesis (6.4) on \( \varphi \) implies that \( \varphi \) tends to the constant map \( \mathbb{R}^n \to 0 \in \mathbb{N} \), when \( r \) tends to infinity.

From the theorem follow decay estimates for \( f \) on \( M^{n+1} \) (i.e. rate of approximating the constant map \( M^{n+1} \to 0 \in \mathbb{N} \)) when \( t \) or \( r \) tend to infinity.

**References**


(Manuscrit reçu le 12 mai 1986)