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## Quasi $*$ -algebras valued quantized fields

by

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**ABSTRACT.** — Point-like fields are considered as elements of the quasi  $*$ -algebra  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ . This approach allows to recover in natural way a necessary and sufficient condition for the existence of a field  $A(x)$  associated to a given Wightman field  $A(f)$ .

**RÉSUMÉ.** — Les champs localisés en un point sont considérés comme éléments de la quasi  $*$ -algèbre  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ . Cette approche permet d'obtenir de façon naturelle une condition nécessaire et suffisante pour l'existence d'un champ  $A(x)$  associé à un champ de Wightman donné  $A(f)$ .

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### 1. INTRODUCTION

One of the most unpleasant features in the mathematical description of quantized fields is that the point-like field  $A(x)$  cannot be described as an operator over some state space. Thus it turns out that a satisfactory mathematical approach to quantum fields must make use of more singular objects. In a previous paper [1] one of us *et al.*, following some idea of Haag [2], proposed a definition of field at a point as a mapping from the Minkowski space-time  $M$  into the weak sequential completion  $\tilde{C}_{\mathcal{D}}$  of the algebra of unbounded operators  $C_{\mathcal{D}}$  ( $= \mathcal{L}^+(\mathcal{D})$ ). This corresponds to the heuristic approach where the field at a point is a limit of observables localized in a shrinking sequence of space-time regions.

$\tilde{C}_{\mathcal{D}}$ -valued fields allowed to give a precise mathematical meaning to relation of the form

$$(1) \quad A(f) = \int d^4x f(x)A(x) \quad f \in \mathcal{S}(\mathbf{M})$$

where  $\mathcal{S}(\mathbf{M})$  is the Schwartz space of fast decreasing  $C^\infty$ -functions on  $\mathbf{M}$ .

In fact, it is proved in [1] that, under suitable assumptions, a Wightman field is presentable in the form (1).

In [3] Fredenhagen and Hertel considered a point-like field as a sesquilinear form on an appropriate pre-Hilbert space  $\mathcal{D}$  which fulfils an high order energy bound.

As proposed by several authors [see reff. from 4 to 8] we consider here the field as an element of  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ , where  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is the set of all linear continuous maps from  $\mathcal{D}$  endowed with a topology finer than the Hilbert-norm into its topological dual  $\mathcal{D}'$ . In particular we consider  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$ , where  $\mathbf{H}$  is the energy operator. In this case  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is a quasi-algebra [9].

Starting from the notion of  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field, for  $\mathcal{D}$  as above, we show first that integrals of the form (1) always converge and that energy-bounds of the kind considered in [3] are consequences of the definition itself.

Following this approach the theorem 4 of [1] can be extended to a more suitable domain and completely inverted. Moreover some propositions proved in [3] are given here under different assumptions which appear to be in some sense, more natural.

## 2. MATHEMATICAL FRAMEWORK

Let us first recall some known facts about sets of operators.

Let  $\mathcal{D}$  be a pre-Hilbert space and  $\mathcal{H}$  its norm-completion. By  $C_{\mathcal{D}}$  ( $= \mathcal{L}^+(\mathcal{D})$ ) we denote the set of all operators in  $\mathcal{D}$  which have an adjoint in  $\mathcal{D}$ .

In  $C_{\mathcal{D}}$  it is possible to introduce the  $\mathcal{D}$ -weak topology defined by the set of seminorms

$$A \rightarrow |(A\phi, \psi)| \quad \phi, \psi \in \mathcal{D}$$

$C_{\mathcal{D}}$  is a topological  $*$ -algebra, in general not complete. We call  $\tilde{C}_{\mathcal{D}}$  its  $\mathcal{D}$ -weak sequential completion.

In recent years some partial algebraic structures, such as partial  $*$ -algebras [10], quasi  $*$ -algebras [9], have been studied by several authors. In this paper a remarkable role is played by quasi  $*$ -algebras.

A quasi  $*$ -algebra is a pair  $(\mathcal{A}, \mathcal{A}_0)$  where  $\mathcal{A}$  is a linear space with involution  $A \rightarrow A^+$  and  $\mathcal{A}_0 \subseteq \mathcal{A}$  is a  $*$ -algebra such that both the left- and

right-products of elements of  $\mathcal{A}$  and elements of  $\mathcal{A}_0$  are defined in  $\mathcal{A}$ . If a topology  $\tau$  on  $\mathcal{A}$  is given such that both the right- and left-multiplications are continuous and  $\mathcal{A}_0$  is dense in  $\mathcal{A}$ ,  $\mathcal{A}$  is said to be a topological quasi \*-algebra with distinguished algebra  $\mathcal{A}_0$ .

We are interested, in particular, to the set of operators  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  which is, for special  $\mathcal{D}$ , a quasi \*-algebra.

Let  $\mathcal{D}$  be a pre-Hilbert space, endowed with a topology  $t$  stronger than the norm-topology,  $\mathcal{D}'$  its topological dual. Thus we get the familiar triplet

$$\mathcal{D} \subseteq \mathcal{H} \subseteq \mathcal{D}'$$

which is called « rigged Hilbert space ». We will consider on  $\mathcal{D}'$  the strong dual topology  $t'$ .

Following Lassner [9], we denote by  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  the set of all continuous operators from  $\mathcal{D}[t]$  into  $\mathcal{D}'[t']$ . The equation  $(A^+ \phi, \psi) = \overline{(A\psi, \phi)}$ ,  $\phi, \psi \in \mathcal{D}$  defines an involution in  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ , which becomes so a \*-invariant linear space.

If  $T$  is any self-adjoint operator in  $\mathcal{H}$ , the set  $\mathcal{D} = \mathcal{D}^\infty(T) = \bigcap_{n>0} D(T^n)$  endowed with the  $t_T$ -topology defined by the set of seminorms

$$\phi \rightarrow \|T^n \phi\| \quad n \in \mathbb{N}$$

provides a very simple example of the situation discussed above. In this case  $\mathcal{D}[t_T]$  is a reflexive Fréchet space (and hence barrelled). For this  $\mathcal{D}$ ,  $C_{\mathcal{D}} \subseteq \mathcal{L}(\mathcal{D}, \mathcal{D}')$  and the latter is a quasi \*-algebra. If we endow  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  with the  $\tau_{\mathcal{D}}$ -topology defined by the seminorms

$$A \rightarrow \sup_{\phi, \psi \in \mathcal{M}} |(A\phi, \psi)|, \quad \mathcal{M} \text{ bounded in } \mathcal{D}[t_T]$$

then  $C_{\mathcal{D}}$  is dense in  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  and thus  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is a topological quasi \*-algebra [9].

In general, to each element  $A \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  it corresponds a sesquilinear form on  $\mathcal{D}$  which is, as is readily checked, separately continuous with respect to  $t$ . If  $B \in C_{\mathcal{D}}$  then the product  $AB$  is always defined in the sense of composition of maps, whereas the product  $BA$  can be defined in the sense of forms. Nevertheless both  $AB$  and  $BA$  are not, in general, elements of  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ . For reflexive  $\mathcal{D}$ , however, we get  $AB, BA \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  and  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is a quasi \*-algebra with distinguished algebra  $C_{\mathcal{D}}$ . If, moreover,  $\mathcal{D}$  is a Fréchet space (e. g.  $\mathcal{D} = \mathcal{D}^\infty(T)$ ) the correspondence between  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  and separately continuous sesquilinear forms is an isomorphism of linear spaces, because all separately continuous sesquilinear forms are continuous.

From now on, we will consider the case where  $\mathcal{D} = \mathcal{D}^\infty(T)$  for some self-adjoint operator  $T$  in  $\mathcal{H}$  and prove some propositions about  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ .

**PROPOSITION 1.** — Let  $\mathcal{D} = \mathcal{D}^\infty(T)[t_T]$ . Then  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is sequentially

complete with respect to the  $\mathcal{D}$ -weak topology defined by the set of seminorms

$$A \rightarrow |(A\phi, \psi)| \quad \phi, \psi \in \mathcal{D}.$$

*Proof.* — First notice that  $\mathcal{D}'$  is  $\sigma(\mathcal{D}', \mathcal{D})$ -quasi complete since it is the dual of a barrelled space ([11] 23 n. 1 (3)).

Let  $A_n$  be a weak Cauchy sequence in  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  i. e.

$$((A_n - A_m)\phi, \psi) \rightarrow 0 \quad \forall \phi, \psi \in \mathcal{D}.$$

Then the sequence  $\{A_n\phi\}$  is  $\sigma(\mathcal{D}', \mathcal{D})$ -Cauchy in  $\mathcal{D}'$ , therefore there exists an element  $\Phi \in \mathcal{D}'$  such that

$$(A_n\phi, \psi) \rightarrow (\Phi, \psi) \quad \forall \psi \in \mathcal{D}.$$

Put  $\Phi = A\phi$ ; in this way we define an operator which maps  $\mathcal{D}$  into  $\mathcal{D}'$ . Taking into account that the operation of taking adjoints is continuous, we can also define the operator  $A^+$ . Therefore  $A$  is continuous from  $\mathcal{D}$  with the  $\sigma(\mathcal{D}, \mathcal{D}')$ -topology into  $\mathcal{D}'$  with the  $\sigma(\mathcal{D}', \mathcal{D})$  or equivalently with respect to the Mackey topologies  $\tau(\mathcal{D}, \mathcal{D}')$  and  $\tau(\mathcal{D}', \mathcal{D})$ . Since  $\mathcal{D}$  is metrizable the  $\tau(\mathcal{D}, \mathcal{D}')$ -topology coincides with the topology  $t_T$  ([11], § 21 n. 5 (3)); on the other hand, for the reflexivity of  $\mathcal{D}$ , the topology  $\tau(\mathcal{D}', \mathcal{D})$  coincides with  $t'_T$  ([11] § 23 n. 3 (1)).

**PROPOSITION 2.** — In the hypothesis of the previous proposition,  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$  is isomorphic to  $\tilde{C}_{\mathcal{D}}$ .

*Proof.* — Since  $\mathcal{D} = \mathcal{D}^\infty(T)$ ,  $C_{\mathcal{D}} \subseteq \mathcal{L}(\mathcal{D}, \mathcal{D}')$  thus it is enough to prove that  $C_{\mathcal{D}}$  is dense in  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ .

Let  $T = \int_0^\infty \lambda dE(\lambda)$  be the spectral decomposition of  $T$ ; put  $P_n = E(n+1) - E(n)$   $n = 0, 1, \dots$ ; we get thus a decomposition of  $\mathcal{H}$  in mutually orthogonal subspaces  $\mathcal{H}_n$ . Each of the  $\mathcal{H}_n$ 's is contained in  $\mathcal{D}$  because if  $f \in \mathcal{H}$  the vector  $P_n f$  is analytic for  $T$ .

On the other hand each  $P_n$  can be extended by continuity to the whole of  $\mathcal{D}'$  and we get an operator  $P_n : \mathcal{D}' \rightarrow \mathcal{H}_n$ .  $\forall k, l \in \mathbb{N}$  the operator  $P_k A P_l$ , with  $A \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  is a bounded operator of  $C_{\mathcal{D}}$  (see [12]). Let

$$A_{nm} = \sum_{k=0}^n \sum_{l=0}^m P_k A P_l. \text{ We will show that } \forall A \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$$
 the sequence  $A_{nm}$ ,

defined in this way, converges to A.

$$\begin{aligned}
 |(A_{nm} - A)\phi, \psi| &= \left| \left( \left( \sum_{k=0}^n \sum_{l=0}^m P_k A P_l - A \right) \phi, \psi \right) \right| \leq \\
 &\leq \left| \left( \sum_{l=0}^m A P_l \phi, \sum_{k=0}^n P_k \psi \right) - \left( A \phi, \sum_{k=0}^n P_k \psi \right) \right| + \left| \left( A \phi, \sum_{k=0}^n P_k \psi \right) - (A \phi, \psi) \right| = \\
 &= \left| \left( \sum_{l=0}^m P_l \phi, \sum_{k=0}^n A^+ P_k \psi \right) - \left( \phi, \sum_{k=0}^n A^+ P_k \psi \right) \right| + \left| \left( A \phi, \sum_{k=0}^n P_k \psi \right) - (A \phi, \psi) \right| < \\
 &< \varepsilon.
 \end{aligned}$$

**PROPOSITION 3.** — Let  $\mathcal{D} = \mathcal{D}^\infty(T)$  with  $T \geq 1$ . If A is a sesquilinear form in  $\mathcal{D}$ , then  $A \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  if, and only if, there exists a natural number k such that  $T^{-k}AT^{-k}$  is defined as a bounded operator in  $\mathcal{D}$ .

*Proof.* — The operator  $T^{-k}AT^{-k}$  is defined since  $T^{-1}\mathcal{D} = \mathcal{D}$ . Now it is enough to prove that  $T^{-k}AT^{-k}$  is bounded if, and only if A is a  $t_T$ -continuous sesquilinear form. But

$$|(T^{-k}AT^{-k}\phi, \psi)| \leq \|\phi\| \|\psi\|$$

is clearly equivalent to

$$|(A\phi, \psi)| \leq \|T^k\phi\| \|T^k\psi\|$$

**PROPOSITION 4.** — Let  $\mathcal{D} = \mathcal{D}^\infty(T)$  with  $T \geq 1$  and  $A \in C_{\mathcal{D}}$ . Then there exists a natural number k such that  $AT^{-k}$  is defined as a bounded operator in  $\mathcal{D}$ .

*Proof.* — Let  $A \in C_{\mathcal{D}}$ ; then A is  $\sigma(\mathcal{D}, \mathcal{D}')$ -continuous; then its graph  $G_A$  is weakly closed in  $\mathcal{D} \times \mathcal{D}$ . It follows that  $G_A$  is also closed in  $\mathcal{D} \times \mathcal{D}$  with respect to the product topology induced on  $\mathcal{D} \times \mathcal{D}$  by the  $t_T$ -topology of  $\mathcal{D}$ . Therefore, by the closed graph theorem, A is  $t_T$ -continuous. Then  $\forall r \in \mathbb{N}$  there exists  $n \in \mathbb{N}$  such that

$$\|T^r A \phi\| \leq \|T^n \phi\| \quad \forall \phi \in \mathcal{D}.$$

This is in particular true for  $r = 0$ ; taking  $\phi = T^{-n}\psi$  we get

$$\|AT^{-n}\psi\| \leq \|\psi\| \cdot \quad \forall \psi \in \mathcal{D}.$$

### 3. $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -VALUED FIELDS

DEFINITION 5. — An  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field is a mapping from the Minkowski space-time  $M$  into  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$

$$A : x \in M \rightarrow A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$$

which satisfies the following axioms:

1) *Translation invariance*: There exists in  $\mathcal{H}$  a strongly continuous representation  $U$  of the group of the translations in  $M$  such that  $\forall a \in M$   $U(a)\mathcal{D} \subset \mathcal{D}$  and

$$U(a)A(x)U(-a) = A(x+a)$$

(where the product is intended in the sense discussed in § 2).

2) *Existence (and uniqueness) of a translation invariant vacuum*: There exists a unique vector  $\Omega \in \mathcal{D}$  such that  $\forall a \in M$

$$U(a)\Omega = \Omega$$

( $\Omega$  is unique up to a constant phase vector).

3) *Spectral condition*: The eigenvalues of the energy-momentum operator (of the theory)  $P^n$  lie in or on the plus cone.

We call Wightman field what is in general understood with these words (see, for instance, [13]) and suppose for this field the following axioms to be verified:

- W1) Translation invariance
- W2) Existence of a translation invariant vacuum
- W3) Cyclicity of the vacuum vector.

From now on we choose  $\mathcal{D} = \mathcal{D}^\infty(H)$  where  $H = P^0$  is the energy operator and we consider in  $\mathcal{D}$  the  $t_H$ -topology.

As consequence of Proposition 2, in this case, the two approaches with  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued fields and  $\tilde{C}_{\mathcal{D}}$ -valued fields may be regarded as equivalent.

PROPOSITION 6. — Let  $x \rightarrow A(x)$  be an  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field with  $\mathcal{D} = \mathcal{D}^\infty(H)$ , let  $R = (1 + H)^{-1}$ ; then there exists a natural number  $k$  such that  $R^k A(x) R^k$  is defined as a bounded operator in  $\mathcal{D}$ . ( $H$ -bound condition).

The proof follows immediately from Proposition 3 taking into account that for the spectral condition  $H$  is a self-adjoint positive operator in  $\mathcal{H}$ .

PROPOSITION 7. — Let  $x \rightarrow A(x)$  be an  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field with  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$  then the integral

$$(A(f)\phi, \psi) = \int d^4x f(x)(A(x)\phi, \psi)$$

converges for all  $\phi, \psi \in \mathcal{D}$  and defines for each  $f \in \mathcal{S}(\mathbf{M})$  an operator of  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ .

*Proof.* — Since  $A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  for each  $\phi \in \mathcal{D}$  the form  $(A(x)\phi, \psi)$  is an (anti)-linear continuous form on  $\mathcal{D}$ . Thus there exists an integer  $n$  such that

$$|A(x)\phi, \psi| \leq \|H^n \psi\|$$

where  $n$  may depend on  $x \in \mathbf{M}$  and on  $\phi \in \mathcal{D}$ . But

$$|(A(x)\phi, \psi)| = |(A(0)U(-x)\phi, U(-x)\psi)| \leq \|H^n U(-x)\psi\| = \|H^n \psi\|$$

therefore  $n$  does not depend on  $x$ .

$$\begin{aligned} |(A(f)\phi, \psi)| &= \left| \int d^4x f(x)(A(x)\phi, \psi) \right| \leq \int d^4x |f(x)| |(A(x)\phi, \psi)| \leq \\ &\leq \|H^n \psi\| \int d^4x |f(x)| < \infty. \end{aligned}$$

Thus the integral exists in the sense of the weak convergence. The above inequality also shows that  $A(f)\phi \in \mathcal{D}'$ . We will now prove that  $A(f) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  i. e. that  $A(f)$  maps continuously  $\mathcal{D}[t_{\mathbf{H}}]$  into  $\mathcal{D}'[t_{\mathbf{H}}]$ . Now, since  $A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  it is continuous from  $\mathcal{D}$  into  $\mathcal{D}'$ ; hence for each bounded set  $\mathcal{M}$  in  $\mathcal{D}$  there exists  $k > 0$  and  $n \in \mathbb{N}$  such that

$$\sup_{\psi \in \mathcal{M}} |(A(x)\phi, \psi)| \leq k \|H^n \phi\|$$

then

$$\begin{aligned} \sup_{\psi \in \mathcal{M}} |(A(f)\phi, \psi)| &\leq \sup_{\psi \in \mathcal{M}} \int d^4x |f(x)| |(A(0)U(-x)\phi, U(-x)\psi)| \leq \\ &\leq \int d^4x |f(x)| \sup_{\psi \in \mathcal{M}} |(A(0)U(-x)\phi, U(-x)\psi)| \leq k \|H^n \phi\| \int d^4x |f(x)|. \end{aligned}$$

Thus  $A(f)$  is also continuous.

At this stage of our discussion we know that the « smeared » field associated with an  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field, with  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$ , is also  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued. But what is usually required is that  $A(f) \in C_{\mathcal{D}} \forall f \in \mathcal{S}(\mathbf{M})$ .

In [3] Fredenhagen and Hertel proposed the idea of selecting point-like fields  $A(x)$  in a class  $F$  satisfying the requirement that  $R^k A(0) R^k$ , with  $R = (1 + H)^{-1}$ , be bounded for some natural  $k$  and they proved that in this case the  $A(f)$ 's are operators of  $C_{\mathcal{D}}$ . In our approach, because of Pro-

position 6, if  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$ , this H-bound is a natural consequence of the definition itself. Therefore, in this case all  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued fields are elements of the class F.

We will prove here that the H-bound condition is not only sufficient but also necessary. Actually in [3] a sort of necessary condition was proved but under stronger assumptions ([3], relation 2.6).

**PROPOSITION 8.** — Let  $\mathcal{S}(\mathbf{M}) \ni f \rightarrow \mathbf{A}(f) \in \mathbf{C}_{\mathcal{D}}$  with  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$  a Wightman field; then there exists a natural  $k$  (independent of  $f$ ) such that  $\mathbf{A}(f)\mathbf{R}^k$  is a bounded operator  $\forall f \in \mathcal{S}(\mathbf{M})$ .

*Proof.* — We proceed by absurd, showing that if  $\forall k > 0$  there exists an  $f_k \in \mathcal{S}(\mathbf{M})$  such that  $\mathbf{A}(f_k)\mathbf{R}^k$  is unbounded whereas  $\mathbf{A}(f_k)\mathbf{R}^{k+1}$  is bounded, then it is possible to find a sequence  $\{a_k\}$  of real numbers and a sequence  $\{\varphi_k, \psi_k\}$  of pairs of vectors of  $\mathcal{D}$  with  $\|\varphi_k\| = \|\psi_k\| = 1$  such that the series

$$\sum_{k=1}^{\infty} a_k f_k$$

converges to an element  $f \in \mathcal{S}(\mathbf{M})$  and such that  $\forall n \in \mathbb{N}$

$$\left| \sum_{k=1}^n (a_k \mathbf{A}(f_k) \mathbf{R}^n \varphi_n, \psi_n) \right| > 2^n$$

and

$$|(a_{n+r} \mathbf{A}(f_{n+r}) \mathbf{R}^n \varphi_n, \psi_n)| \leq 2^{-r}.$$

If such a sequence exists, taking into account that the map

$$f \in \mathcal{S}(\mathbf{M}) \mapsto (\mathbf{A}(f)\varphi, \psi)$$

is,  $\forall \varphi, \psi \in \mathcal{D}$ , a tempored distribution, one has

$$|(\mathbf{A}(f)\mathbf{R}^n \varphi_n, \psi_n)| \geq \left| \sum_{k=1}^n a_k (\mathbf{A}(f_k) \mathbf{R}^n \varphi_n, \psi_n) - \sum_{r=1}^{\infty} (a_{n+r} \mathbf{A}(f_{n+r}) \mathbf{R}^n \varphi_n, \psi_n) \right| > 2^n - 1.$$

Let  $n_0$  be the smallest number such that  $\mathbf{A}(f)\mathbf{R}^{n_0}$  is bounded (Prop. 4); taking into account that  $\|\mathbf{R}\| \leq 1$  we get for  $n > n_0$

$$\begin{aligned} \|\mathbf{A}(f)\mathbf{R}^{n_0}\| &= \|\mathbf{R}^{n_0} \mathbf{A}^+(f)\| \geq \|\mathbf{R}^n \mathbf{A}^+(f)\| \geq |(\varphi_n, \mathbf{R}^n \mathbf{A}^+(f)\psi_n)| = \\ &= |(\mathbf{A}(f)\mathbf{R}^n \varphi_n, \psi_n)| \geq 2^n - 1 \end{aligned}$$

which is a contradiction.

Let us now prove the existence of the sequences  $\{a_k\}$  and  $\{\varphi_k, \psi_k\}$  as described above.

We denote with  $\| \cdot \|_k$  the  $k$ -th seminorm defining the topology of  $\mathcal{S}(M)$  and which may be supposed to increase with  $k$ .

To begin with, take  $n = 1$ .

The functional  $(A(f_1)R\varphi, \psi)$  is unbounded then  $\forall M > 0$  there exist two vectors  $\varphi_1, \psi_1$  with  $\|\varphi_1\| = \|\psi_1\| = 1$  such that  $|(A(f_1)R\varphi_1, \psi_1)| > M$ . We suppose  $M > 2 \|f_1\|_1 e$ . We choose a number  $a_1^1$  such that

$$2M^{-1} \leq a_1^1 \leq (e \|f_1\|_1)^{-1}$$

and a sequence  $\{a_{1+r}^1\}_{r \geq 1}$  such that

$$a_{1+r}^1 \leq \min \{ (\|f_{1+r}\|_{1+r} \exp(r+1))^{-1}, 2^{-r} / |(A(f_{1+r})R\varphi_1, \psi_1)| \}$$

then we get

$$|(a_1^1 A(f_1)R\varphi_1, \psi_1)| > 2$$

and

$$|(a_{1+r}^1 A(f_{1+r})R\varphi_1, \psi_1)| \leq 2^{-r}.$$

We put  $a_1 = a_1^1$ .

Following an analogous procedure, for each  $n$ , we put

$$C_n = \left\| \sum_{k=1}^{n-1} a_k A(f_k) R^n \right\|$$

and choose a number  $M > \max_{j < n} \{ (2^n + C_n) \|f_n\|_n \exp(n), (2^n + C_n) / a_n^j \}$ .

The functional  $(A(f_n)R^n\varphi, \psi)$  is unbounded; then there exist two vectors  $\varphi_n, \psi_n \in \mathcal{D}$  with  $\|\varphi_n\| = \|\psi_n\| = 1$  such that  $|(A(f_n)R^n\varphi_n, \psi_n)| > M$ ; thus we can choose a number  $a_n^n$  with

$$(2^n + C_n)M^{-1} \leq a_n^n \leq (\|f_n\|_n \exp(n))^{-1}$$

and a sequence  $a_{n+r}^n, r \geq 1$  with

$$a_{n+r}^n \leq \min \{ (\|f_{n+r}\|_{n+r} \exp(n+r))^{-1}, 2^{-r} / |(A(f_{n+r})R^n\varphi_n, \psi_n)| \}$$

then we get

$$\begin{aligned} & \left| \left( \sum_{k=1}^{n-1} a_k A(f_k) R^n \varphi_n, \psi_n \right) + (a_n^n A(f_n) R^n \varphi_n, \psi_n) \right| \geq \\ & \quad - \left| \left( \sum_{k=1}^{n-1} a_k A(f_k) R^n \varphi_n, \psi_n \right) \right| + a_n^n |(A(f_n)R^n\varphi_n, \psi_n)| > 2^n \end{aligned}$$

and

$$|(a_{n+r}^n A(f_{n+r})R^n\varphi_n, \psi_n)| \leq 2^{-r}.$$

We put  $a_n = \min \{ a_n^1, a_n^2, \dots, a_n^n \}$ .

The sequence  $\sum_{k=1}^n a_k f_k$ , for the above choice of the  $a_k$ 's is a Cauchy sequence in  $\mathcal{S}(\mathbf{M})$  and then it converges to an  $f \in \mathcal{S}(\mathbf{M})$ ; the other requirements are, also fulfilled and thus the theorem is proved.

**PROPOSITION 9.** — Let  $\mathcal{D} = \mathcal{D}^\infty(\mathbf{H})$  and  $f \rightarrow A(f)$  a hermitean (scalar) Wightman field satisfying the axioms  $W_1, W_2, W_3$ , then in order that there exists an  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued field  $x \in \mathbf{M} \rightarrow A(x) \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$  such that

$$\forall f \in \mathcal{S}(\mathbf{M}) \quad (A(f)\phi, \psi) = \int d^4x f(x) (A(x)\phi, \psi) \quad \forall \phi, \psi \in \mathcal{D}$$

it is necessary and sufficient that the sesquilinear form  $R^k A(0) R^k$  is bounded for some  $k$ , where  $A(0) = U(-x) A(x) U(x)$ .

*Proof.* — The sufficient part follows from the proof given in [3] taking into account that  $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ -valued fields belong to the class F.

We prove now the necessity.

Let us call  $\mathcal{D}_0$  the linear manifold of  $\mathcal{D}$  which is obtained by applying to the vacuum  $\Omega$  the algebra generated by the set of operators  $\{A(f) | f \in \mathcal{S}(\mathbf{M})\}$ . Then, by Theorem 4 of [1] one finds a sesquilinear form  $A(x)$  such that  $\forall f \in \mathcal{S}(\mathbf{M})$

$$(A(f)\phi, \psi) = \int d^4x f(x) (A(x)\phi, \psi) \quad \forall \phi, \psi \in \mathcal{D}_0.$$

Now, because of Proposition 8 there exists a natural number  $k$  (independent of  $f$ ) such that  $R^k A(f) R^k$  is bounded  $\forall f \in \mathcal{S}(\mathbf{M})$  and taking into account that in the proof of Theorem 4 of [1]  $A(x)$  is obtained as a limit of  $(A(f_n)\phi, \psi)$  where  $f_n \rightarrow \delta_x$  in the topology of  $\mathcal{S}'(\mathbf{M})$  we get, for  $x = 0$  and  $\phi, \psi \in \mathcal{D}_0$

$$|(A(0)\phi, \psi)| = \lim_{n \rightarrow x} |(A(f_n)\phi, \psi)| \leq \|R^{-k}\phi\| \|R^{-k}\psi\|.$$

Therefore the functional  $R^k A(0) R^k$  is bounded on  $\mathcal{D}_0$ , then it can be extended to a bounded sesquilinear form all over  $\mathcal{H}$ . Let us call  $B(0)$  the so obtained form. Let us consider the form  $\hat{A}(0) = R^{-k} B(0) R^{-k}$ ; this is clearly a sesquilinear form on  $\mathcal{D}$  which extends  $A(0)$  and such that  $R^k \hat{A}(0) R^k$  is bounded. For simplicity of notations we denote still  $\hat{A}(0)$  by  $A(0)$ . The sesquilinear form  $A(0)$  satisfies therefore the requirements of Proposition 3 and thus the statement is proved.

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