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ERRATUM

The quantum stability problem for time-periodic perturbations of the harmonic oscillator

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In the above mentioned paper there is some error pertaining to the fact that the « forbidden set » I that one has to remove from B (lemma 2.2) should be:

$$\mathrm{I} = \{ \Omega \in \mathrm{B}_i \ \operatorname*{Inf}_i | (b - \mathrm{T}_k b)_i | < \gamma | k |^{-\sigma}, \text{ some } k \in \mathbb{Z}^2 \backslash \{ 0 \} \}$$

with Inf instead of Sup, and some instead of $\forall$. But the above I has not a finite Lebesgue measure.

The way out is to modify the paper as follows:

DEFINITION 2.3. — Given a closed Borel set $\mathrm{B} \subset \mathbb{R}$ and any non-negative numbers r, σ, let $\mathrm{M}_{r,\sigma}(\mathrm{B})$ denote the set of functions A from B to the space of bounded operators in $l^2(\mathbb{Z}^2)$, represented for any $\Omega \in \mathrm{B}$ by an infinite matrix $(\mathrm{A}_{ij}(\Omega))_{i,j\in\mathbb{Z}^2}$ such that

$$\operatorname*{Sup}_{\Omega,\Omega'\in \mathrm{B}} \operatorname*{Sup}_{i\in\mathbb{Z}^2} \sum_{j\in\mathbb{Z}^2} e^{r|i-j|} \langle i + j \rangle^{\sigma} \left(| \mathrm{A}_{ij}(\Omega) | + \left| \frac{\mathrm{A}_{ij}(\Omega) - \mathrm{A}_{ij}(\Omega')}{\Omega - \Omega'} \right| + i \leftrightarrow j \right) \equiv \| \mathrm{A} \|_{r,\sigma,\mathrm{B}} < \infty .$$

REMARK 2.1. — One sees easily (Schur's lemma), that any $\mathrm{A} \in \mathrm{M}_{0,0}(\mathrm{B})$ is such that $\mathrm{A}(\Omega)$ is a bounded operator in $l^2(\mathbb{Z}^2)$ and this is true *a fortiori* for $\mathrm{M}_{r,\sigma}(\mathrm{B})$, $r, \sigma > 0$.

THEOREM 2.1. — Given any $r > 0$, $1 < \sigma < 2$, $\mathrm{B} \subset \mathbb{R}$, let $\mathrm{P} \in \mathrm{M}_{r,4\sigma}(\mathrm{B})$ be such that

$$\| \mathrm{P} \|_{r,4\sigma,\mathrm{B}} \leqslant d(\sigma)\gamma^2 \rho^{10\sigma+1}$$

for some $\gamma > 0$, $\rho : 0 < \rho < r$ and $d(\sigma)$ only depending on σ with $\rho \leqslant \sigma e$.

Let D be the diagonal matrix whose sequence $d_i(\Omega)$ of diagonal elements is

$$d_i(\Omega) = i_1\Omega + i_2 \qquad i = (i_1, i_2) \in \mathbb{Z}^2 .$$

Then there exists a closed Borel set $B' \subset B$ s. t.

$$| B \backslash B' | < \gamma \qquad \text{(Lebesgue measure)}$$

and an invertible $V \in M_{r-\rho,0}(B')$ satisfying

$$\| V - 1 \|_{r-\rho,0,B} \quad \text{and} \quad \| V^{-1} - 1 \|_{r-\rho,0,B} \leqslant \frac{\| P \|_{r,4\sigma,B}}{d(\sigma)\gamma^2\rho^{10\sigma+1}}$$

such that

$$V^{-1}(D + P)V = \Delta$$

where Δ is a diagonal matrix, whose sequence δ of diagonal elements satisfies

$$\| d - \delta \|_{\mathscr{M}_{B'}} \leqslant \frac{3}{2} \| P \|_{r,4\sigma,B} .$$

LEMMA 2.2. — Let $a \in \mathscr{M}_B$ be such that $\| a \|_{\mathscr{M}_B} \leqslant 1/4$, and let b be the sequence

$$i \in \mathbb{Z}^2 \ \rightarrow \ b_i(\Omega) = i_1\Omega + i_2 + a_i(\Omega) .$$

Then for any $\sigma > 1$, there exists a positive constant $C(\sigma)$ such that if:

$$I \equiv \{ \Omega \in B : \exists i \text{ and } j, i \neq j : | b_i(\Omega) - b_j(\Omega) | < \gamma | i-j |^{-\sigma} \langle i+j \rangle^{-2\sigma} / C(\sigma) \}$$

the Lebesgue measure of I satisfies $| I | \leqslant \gamma$.

Proof. — Given η any positive number $< 1/2$, $i, k \in \mathbb{Z}^2$ we define,

$$I_{ik}(\eta) = \{ \Omega \in B : | b_{\frac{i+k}{2}}(\Omega) - b_{\frac{i-k}{2}}(\Omega) | < \eta \} \qquad k \neq 0 .$$

It is clear that the first component k_1 of k has to be non-zero, in order that $I_{ik}(\eta)$ be non-empty. But if Ω_1 and Ω_2 both belong to $I_{ik}(\eta)$ we have:

$$| \Omega_1 - \Omega_2 | < \frac{2\eta}{| k_1 | - 2 \| a \|_{\mathscr{M}_B}} \leqslant \frac{2\eta}{| k_1 | - 1/2}$$

which implies that the Lebesgue measure of $I_{ik}(\eta)$ is smaller than $2\eta / | k_1 | - 1/2$. Now it is clear that

$$I \subset \bigcup_{\substack{i \in \mathbb{Z}^2 \backslash \{0\} \\ k \in \mathbb{Z}^2 \backslash \{0\}}} I_{ik}(\gamma | k |^{-\sigma} \langle i \rangle^{-2\sigma} C(\sigma)^{-1})$$

and therefore

$$| I | \leqslant \sum_{\substack{i \in \mathbb{Z}^2 \\ k \in \mathbb{Z}^2 \backslash \{0\}}} \frac{2\gamma \langle i \rangle^{-2\sigma} C(\sigma)^{-1}}{| k |^\sigma (| k_1 | - 1/2)} \leqslant \frac{4\gamma h(\sigma)}{C(\sigma)} \sum_1^\infty \frac{\text{Log } 2N}{N^\sigma} \leqslant \gamma$$

where

$$h(\sigma) \equiv \sum_{i \in \mathbb{Z}^2} \langle i \rangle^{-2\sigma}$$

and

$$C(\sigma) \equiv 4h(\sigma) \Big/ \sum_{1}^{\infty} N^{-\sigma} \operatorname{Log} 2N$$

LEMMA 2.3. — Given ρ, σ, $r > 0$, $A \in M_{r,\sigma}(B)$ and $C \in M_{r+\rho,0}(B)$ with $\sigma \leqslant 2$ and $\rho \leqslant \sigma e$, we have AC and $CA \in M_{r,\sigma}(B)$ with

$$\| AC \|_{r,\sigma,B} \quad \text{and} \quad \| CA \|_{r,\sigma,B} \leqslant 2^{2+\sigma/2} \left(\frac{\sigma}{e\rho} \right)^{\sigma} \| A \|_{r,\sigma,B} \| C \|_{r+\rho,0,B} .$$

Proof. — Let $D = AC$. Then $D_{ij} = \sum_{k \in \mathbb{Z}^2} A_{ik} C_{kj}$. But

$$\begin{aligned} \operatorname*{Sup}_i \sum_j e^{r|i-j|} \langle i + j \rangle^{\sigma} | D_{ij} | &\leqslant 2^{\sigma/2} \operatorname*{Sup}_i \sum_{j,k} e^{r|i-k| + r|k-j|} (\langle i + k \rangle^{\sigma} + \langle j - k \rangle^{\sigma}) | A_{ik} | | C_{kj} | \\ &\leqslant 2^{\sigma/2} \operatorname*{Sup}_i \sum_k e^{r|i-k|} \langle i + k \rangle^{\sigma} f_{\sigma}(\rho) \operatorname*{Sup}_k \sum_j e^{(r+\rho)|j-k|} | C_{kj} | \end{aligned}$$

where $\quad f_{\sigma}(\rho) = \operatorname*{Sup}_x e^{-\rho x} (1 + x^2)^{\sigma/2} \leqslant 1 + \left(\frac{\sigma}{e\rho} \right)^{\sigma} < 2 \left(\frac{\sigma}{e\rho} \right)^{\sigma}$.

Proceeding similarly with the other terms of the norm, we get the result.

LEMMA 2.5. — Let D be a diagonal matrix whose diagonal sequence d satisfies for $\Omega \in B$:

$$| d_i(\Omega) - d_j(\Omega) | > \gamma | i - j |^{-\sigma} \langle i + j \rangle^{-2\sigma} C(\sigma)^{-1} \qquad i \neq j .$$

Then given any $P \in M_{r,4\sigma}(B)$ with $\operatorname{diag} P = 0$ and given any $\rho : 0 < \rho < r$, there exists a unique $W \in M_{r-\rho,0}(B)$ with $\operatorname{diag} W = 0$ solution of

$$[D, W] + P = 0 .$$

Furthermore

$$\| W \|_{r-\rho,0,B} \leqslant 2C(\sigma)^2 \gamma^{-2} \left(\frac{\sigma}{e\rho} \right)^{2\sigma+1} \| P \|_{r,4\sigma,B} .$$

The proof of lemma 2.5 is immediate, using definition 2.3.

Now the proof of Theorem 2.1 proceeds along exactly the same lines as in the paper, B_n being redefined according to lemma 2.2, and noting that the power law decay in $\langle i+j \rangle$ propagates from P_n to P_{n+1}, due to lemma 2.3:

$$r_n = r_{n-1} - 2\rho_n$$
$$\| P_n \|_{r_n, 4\sigma, B_n} \leqslant \theta_n^{2^n}$$

the sequence $(\theta_n)_{n\in\mathbb{N}}$ being defined inductively by

$$\theta_{n+1}^{2^{n+1}} \leqslant 2^{6(3\sigma+1)} C(\sigma)^2 \left(\frac{\sigma}{e\rho_n}\right)^{10\sigma+1} \gamma_n^{-2} \theta_n^{2^{n+1}}$$

and (2.17) being modified accordingly.

The rest of the paper works without change, except that $r > 9$ in Theorem 3.1, and in assumption ($\mathscr{C}$) of Section 4 has to be replaced by $r > 25$, and (3.2) by

$$| P_{ij} | \leqslant C\gamma^2 \, | i - j |^{-r} \langle i + j \rangle^{-4\sigma} \qquad (i \neq j).$$