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Quantized fields and operators on a partial inner product space

by

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ABSTRACT. — We investigate the connection between the space OpV of all operators on a partial inner product space V and the weak sequential completion of the $*$-algebra $L^+(V^\#)$ of all operators X such that $V^\# \subset D(X) \cap D(X^*)$ and both X and its adjoint X^* leave $V^\#$ invariant. This connection allows us to describe quantized fields at a point as mappings from the Minkowski space-time into OpV.

RÉSUMÉ. — Nous analysons la relation entre l'espace OpV de tous les opérateurs sur un espace à produit interne partiel V et la complétion séquenciellement faible de l'$*$-algèbre $L^+(V^\#)$ de tous les opérateurs X tels que $V^\# \subset D(X) \cap D(X^*)$ et tels que X et son adjoint X^* laissent $V^\#$ invariant. Cette relation nous permet de décrire des champs quantiques en un point comme des applications de l'espace-temps de Minkowski dans OpV.

1. INTRODUCTION

The fundamental concept of Wightman axiomatics is the concept of quantized field $A(x)$ at a point x, which is usually defined [*1*] as an operator-valued distribution on some space of test functions (x is the four-dimensional coordinate of space-time).

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Let $\mathscr{D}$ be a dense linear manifold of a Hilbert space $\mathscr{H}$ and denote by $L^+(\mathscr{D})$ the $*$-algebra of all operators X such that $\mathscr{D} \subset D(X) \cap D(X^*)$ and both X and its adjoint X^* leave $\mathscr{D}$ invariant. It has been first proposed by Haag [*2*] that a quantized field $A(x)$ at any point x should be described in terms of sesquilinear forms on $\mathscr{D} \times \mathscr{D}$, corresponding to the heuristically defined mapping $(f, g) \mapsto (A(x)f, g)$. This idea has been particularized by Ascoli, Epifanio and Restivo [*3*] in such a way that these sesquilinear forms may be considered as elements of the weak sequential completion $\widetilde{L^+(V^\#)}^w$ of $L^+(V^\#)$.

On the other hand, it is well known that if V is an arbitrary partial inner product (PIP) space [*4*], which is quasi complete in its canonical Mackey topology $\tau(V, V^\#)$, then the space $L^+(V^\#)$ is isomorphic to the $*$-algebra Reg V of all regular operators on V [*5*].

In this note, after a brief recall in Section 2 of basic facts on PIP spaces and operators on them [*4-7*] we investigate in Section 3 the connection between the space OpV of all operators on a PIP space V, and the weak sequential completion of $L^+(V^\#)$. In particular, we show that if V is an arbitrary PIP space, and $\langle V^\#, V \rangle$ is a reflexive dual pair, then OpV is isomorphic to $\widetilde{L^+(V^\#)}^w$, which means that a quantized field at a point may be considered as a mapping from the Minkowski space-time M into OpV. This corresponds to the idea that a field at a point is a limit of observables localized in a shrinking sequence of space-time regions [*8*] i. e. $A(x) = w - \lim_{n \to \infty} A(f_n)$ where $f_n \to \delta_x$ (Dirac delta at the point $x \in M$) in the topology of the dual $\mathscr{S}'(M)$ of the Schwartz space $\mathscr{S}(M)$ of fast decreasing C^∞-functions on M.

At this stage we should mention some related works on the mathematical formulation of point like fields as operators on some PIP space. In [*9*], extending the machinery of Fock space (a symmetric tensor algebra over a Hilbert space), Grossmann defines the unsmeared free field at a point as an operator on some nested Hilbert space [*10*]. Grossmann's approach is summarized in [*4*]. In [*11*], Nelson defines a Euclidian free field as an operator on the PIP space corresponding to the scale built from the Hamiltonian. This fact was already noticed by Antoine and Karwowski [*12*] and extensively used by Fredenhagen and Hertel [*8*].

Consider on $\mathscr{D}$ (a dense linear subspace of $\mathscr{H}$) a topology t finer than the norm-topology and let $\mathscr{D}'[t']$ be the topological dual of $\mathscr{D}$, equipped with the strong dual topology t'. Let $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ be the set of all continuous operators from $\mathscr{D}[t]$ into $\mathscr{D}'[t']$. It has been shown [*13*] that if

$$\mathscr{D} = \mathscr{D}^\infty(T) = \bigcap_{n>0} D(T^n)$$

(where T is any self-adjoint operator in $\mathscr{H}$) and $\mathscr{D}$ is equipped with the t_T-topology defined by the family of seminorms $\phi \to \| T^n \phi \|$, $n \in \mathbb{N}$, then $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is a topological quasi $*$-algebra with distinguished algebra $L^+(V^\#)$ (This result has been generalized in [*14*] to the case of arbitrary domains $\mathscr{D}$).

Recently, in their study of point-like fields, Epifanio and Trapani [*15*] have exploited systematically the quasi $*$-algebra structure of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$. This approach is in fact in the spirit of operators on a PIP space V, since OpV is isomorphic to $\mathscr{L}(V^\#, V)$ [*4*].

In Section IV we introduce the concept of OpV-valued fields and Wightman fields. OpV-valued fields may be used in order to give a precise mathematical meaning to relations of the type

$$A(f) = \int d^4x f(x) A(x), \qquad f \in \mathscr{S}(M).$$

In this Section, using some results of Ref [*15*] we compare our approach to that of Fredenhagen and Hertel [*8*].

2. PIP-SPACES AND OPERATORS ON THEM [*4-7*]

A PIP-space V is a complex vector space with the following structure:

i) $\mathscr{I} = \{ V_r, r \in I \}$ is a collection of vector subspaces of V which covers V and is an involutive lattice with respect to set intersection, vector sum and lattice involution: $V_r \leftrightarrow V_{\bar{r}}$.

Besides elements of $\mathscr{I}$, we consider the extreme spaces:

$$V^\# \equiv \bigcap_{r \in I} V_r \qquad \text{and} \qquad V \equiv \bigcup_{r \in I} V_r$$

ii) A nondegenerate hermitian form $\langle \,.\, | \,.\, \rangle$ (the partial inner product) is defined on $\bigcup_{r \in I} V_r \times V_{\bar{r}}$.

iii) There exists a unique element $0 = \bar{0}$ in I such that $V_0 = V_{\bar{0}} \equiv \mathscr{H}$ is a Hilbert space with respect to $\langle \,.\, | \,.\, \rangle$.

The nondegeneracy assumption $(V^\#)^\perp = \{ 0 \}$ implies that every pair $\langle V_r, V_{\bar{r}} \rangle$, as well as $\langle V^\#, V \rangle$ is a dual pair with respect to the form $\langle \,.\, | \,.\, \rangle$. We may therefore equip each V_r with its Mackey topology $\tau(V_r, V_{\bar{r}})$ and similarly for $V^\#$, V.

An operator A on a PIP space V is a map $D(A) \to V$, where D(A) is the largest union of V_r's such that the restriction of A to any of them is linear and continuous into V.

The set of all operators on V, denoted by OpV is isomorphic to $\mathscr{L}(V^\#, V) = \{$ linear continuous maps $V^\# \to V \}$. Equivalently OpV is isomorphic to $B(V^\#, V^\#) = \{$ separately continuous sesquilinear forms on

$V^{\#} \times V^{\#}$ }. Thus, OpV is a vector space. Moreover, OpV carries an involution $A \leftrightarrow A^{\times}$ (adjoint of A), but it is not an algebra since the multiplication is not always defined. Such sets are called partial $*$-algebras [*16*].

An operator on a PIP-space V is called regular [*5*], if $D(A)=D(A^*)=V$. Equivalently, a regular operator is a linear continuous map of $V^{\#}$ into itself, which maps V into itself continuously. The set of all regular operators on V, denoted by Reg V is a $*$-algebra.

We assume that V is quasi complete in its Mackey topology. Then Reg V is isomorphic to the $*$-algebra $L^+(V^{\#})$ of all closable operators on $\mathscr{H}$ which, together with their (Hilbertian) adjoint leave $V^{\#}$ invariant. Actually almost all PIP-spaces are quasi complete in the $\tau(V, V^{\#})$-topology, the only known exceptions being quite pathological [*17*].

We will endow OpV with the weak topology defined by the family of seminorms

$$A \mapsto |\langle Af, g \rangle|; \qquad f, g \in V^{\#}.$$

On Reg V $\simeq L^+(V^{\#})$ we will consider the weak topology inherited from OpV.

3. OpV AND THE WEAK SEQUENTIAL COMPLETION OF $L^+(V^{\#})$

Following [*3*] we denote by $S_{V^{\#}}$ the space of all sesquilinear forms on $V^{\#} \times V^{\#}$. It has been proved in [*3*] that the space $S_{V^{\#}}$ endowed with the topology of pointwise convergence given by the set of seminorms:

$$F \mapsto |F(f, g)|; \qquad f, g \in V^{\#}$$

is isomorphic to the weak completion of $L^+(V^{\#})$, i. e. in notations of [*3*]

$$S_{V^{\#}} \simeq \widehat{L^+(V^{\#})^w}.$$

On the other hand, it is clear that $S_{V^{\#}}$ contains the space OpV which is isomorphic to the space $B(V^{\#}[\tau], V^{\#}[\tau])$ of all Mackey separately continuous sesquilinear forms on $V^{\#} \times V^{\#}$.

In what follows, we want to answer the following question: given a PIP space V, when is OpV isomorphic to the weak sequential completion $\widetilde{L^+(V^{\#})^w}$ of $L^+(V^{\#})$? If this isomorphism exists, then the sesquilinear forms which describe quantized fields at points may be considered as elements of OpV equipped with the weak topology.

In general, for a given PIP space V, whenever OpV is weakly sequentially complete, we have the following relation between OpV and $\widetilde{L^+(V^{\#})^w}$:

$$\widetilde{L^+(V^{\#})^w} \subseteq \text{OpV} \subseteq \widehat{L^+(V^{\#})^w} \simeq S_{V^{\#}}.$$

We show that this relation holds if in particular $\langle V^{\#}, V \rangle$ is reflexive dual pair. Indeed we have the following:

PROPOSITION 3.1. — Let V be a PIP space. If $\langle V^{\#}, V \rangle$ is a reflexive dual pair, then OpV is weakly sequentially complete.

Proof. — Let $\{ T_n \}$ be a weak Cauchy sequence in OpV, i. e. $\forall f \in V^{\#}$, $\{ T_n f \}$ is a weak Cauchy sequence in V.

Since $\langle V^{\#}, V \rangle$ is reflexive, it follows that $V^{\#}$ and V are quasi complete (i.e. closed bounded sets are complete) with respect to the weak topology and therefore $V^{\#}$ and V are weakly sequentially complete i. e. $w\text{-}\lim_{n\to\infty} T_n f = Tf \in V$. This shows that T is a map from $V^{\#}$ into V.

In order to show that T is continuous from $V^{\#}[\tau(V^{\#}, V)]$ to $V[\tau(V, V^{\#})]$, one uses the dual mapping theory [*18*]. □

REMARK 3.2. — For Reg $V \simeq L^{+}(V^{\#})$ one could also try to perform the same proof as in Proposition 3.1, but in general we do not have $D(T) = D(T^*) = V$. So, in general $L^{+}(V^{\#})$ is not weakly sequentially complete. Actually this fits with results of [*3*] where it is shown that $\widetilde{L^{+}(V^{\#})}^{w}$ may contain elements which are not operators.

The condition of reflexivity of the dual pair $\langle V^{\#}, V \rangle$ is weak enough to cover most spaces of practical interest, in particular, all spaces of distributions.

Typical instances when the dual pair $\langle V^{\#}, V \rangle$ is reflexive are [*7*]:

. $V^{\#}$ is a Hilbert space; then so is V.

. $V^{\#}$ is a reflexive Banach space; then so is V.

. $V^{\#}$ is a reflexive Fréchet space; V is the a reflexive complete DF-space [*18*].

. $V^{\#}$ is a Montel space; then so is V.

Now, given a PIP space V, when is OpV contained in $\widetilde{L^{+}(V)}^{w}$? Let $A \in \text{OpV}$, $V^{\#}$ separable, e_ν an orthonormal basis in $V^{\#}$ and $P_\nu = | e_\nu \rangle \langle e_\nu |$ the orthogonal projection on e_ν. In the terminology of [*5*], P_ν is a very regular operator.

Consider the operator $P_j A P_{j'}$. Obviously this operator is regular, since the operator itself as well as its adjoint leave $V^{\#}$ invariant. Let B_{nm} be the sequence in $L^{+}(V^{\#})$ defined by $B_{nm} = \sum_{j=1}^{n} \sum_{j'=1}^{m} P_j A P_{j'}$.

Since $\{ e_\nu \}$ is an orthonormal basis, for all $f \in V^{\#}$ we have $\sum_{\nu} P_\nu f = f$, and consequently: $\forall f, g \in V^{\#}$ we get:

$$\lim_{m\to\infty} \langle B_{nm} f, g \rangle = \lim_{m\to\infty} \left\langle \sum_{j=1}^{n} \sum_{j'=1}^{m} P_j A P_{j'} f, g \right\rangle = \lim_{m\to\infty} \left\langle \sum_{j'=1}^{m} A P_{j'} f, \sum_{j=1}^{n} P_j g \right\rangle =$$

$$= \lim_{m\to\infty} \left\langle \sum_{j'=1}^{m} A P_{j'} f, g \right\rangle = \lim_{m\to\infty} \left\langle \sum_{j'=1}^{m} P_{j'} f, A^* g \right\rangle = \langle f, A^* g \rangle = \langle Af, g \rangle$$

Thus, the arbitrary element $A \in \mathrm{Op}V$ is the weak limit of a weakly convergent (hence a weak Cauchy) sequence of $L^+(V^\#)$, i. e. $A \in \widetilde{L^+(V^\#)}^w$.

We summarize this analysis in the following:

PROPOSITION 3.3. — If V is a PIP space, then $\mathrm{Op}V \subset \widetilde{L^+(V^\#)}^w$.

Now, putting together Propositions 3.1 and 3.3 we can state our main result (which shows in particular that a quantized field at a point may be considered as an element of OpV).

PROPOSITION 3.4. — Let V be a PIP space. If $\langle V^\#, V \rangle$ is a reflexive dual pair then OpV is isomorphic to $\widetilde{L^+(V^\#)}^w$.

4. OpV-VALUED FIELDS AND WIGHTMAN FIELDS

In this section we discuss the concepts of OpV-valued and Wightman fields and in particular using some results of Ref [*15*] we compare our approach to the work of Fredenhagen and Hertel [*8*].

DEFINITION 4.1. — We call *OpV-valued field* any mapping A from the Minkowski space-time M into OpV, satisfying the following axioms:

1. *Translation invariance:* There exists in the central Hilbert space $\mathscr{H}$ a strongly continuous unitary representation U of the group of translations of M such that $\forall a \in M$, $U(a)V^\# \subset V^\#$ and

$$U(a)A(x)U(a)^{-1} = A(x + a)\,; \qquad x \in M\,.$$

2. *Existence of a translation invariant vacuum:* There exists a vector $\Omega \in V^\#$ such that $\forall a \in M$,

$$U(a)\Omega = \Omega\,.$$

3. *Spectral postulate:* The eigenvalues of the energy-momentum operator P^n do not lie outside the forward light cone.

DEFINITION 4.2. — We call (scalar) *Wightman field* over $V^\#$ a mapping A from $\mathscr{S}(M)$ into $L^+(V^\#)$ such that $\forall \phi, \psi \in V^\#$, the mapping from $\mathscr{S}(M)$ into $\mathbb{C}$ defined by $f \mapsto \langle A(f)\phi, \psi \rangle$ is a tempered distribution i. e. it is continuous.

We assume that the Wightman field satisfies the following axioms:

W1: Translation invariance

W2: Existence of a translation invariant vacuum

W3: *Cyclicity of the vacuum:* Ω is a cyclic vector for the algebra generated by the set of operators $\{ A(f) \mid f \in \mathscr{S}(M) \}$.

In [8] a field at a point is defined as being a sesquilinear form on $V^\# \times V^\#$ satisfying a H-*bound condition* i. e. it is assumed that there exists a natural number k such that $R^k A(x) R^k$, with $R = (1 + H)^{-1}$ ($H = P^0$ is the energy operator in $\mathscr{H}$) is a bounded operator in $V^\#$.

DEFINITION 4.3. — A point-like field $A(x)$ is said to belong to *the class* $\mathscr{F}$ [8] if for some $k \in \mathbb{N}$, the operator $R^k A(0) R^k$, with $A(0) = U(-x)A(x)U(x)$, is a bounded operator.

In order to compare our approach to the one developed in [8] we will restrict ourselves to a special $V^\#$, namely

$$V^\# = \mathscr{D}^\infty(H) = \bigcap_{n>0} D(H^n).$$

We will consider on $V^\#$ the t_H-topology defined by the seminorms:

$$\phi \to \| H^n \phi \|, \qquad n \in N.$$

Then, $V^\# [t_H]$ is a reflexive Fréchet space.

PROPOSITION 4.4. — If $x \to A(x)$ is an OpV-valued field with $V^\# = \mathscr{D}^\infty(H)$, then $A(x)$ satisfies a H-bound condition.

Proof. — See e. g. [15, Proposition 6].

COROLLARY 4.5. — If $V = \mathscr{D}^\infty(H)$, then every OpV-valued field belongs to the class $\mathscr{F}$.

PROPOSITION 4.6. — Let $V^\# = \mathscr{D}^\infty(H)$ and $x \to A(x)$ be an OpV-valued field.

Then $\forall \phi, \psi \in V^\#$ and $f \in \mathscr{S}(M)$, the integral

$$\langle A(f)\phi, \psi \rangle = \int d^4x f(x) \langle A(x)\phi, \psi \rangle$$

converges and defines a Wightman field i. e. $A(f) \in L^+(V^\#)$.

Proof. — See e. g. [15, Proposition 7].

As a consequence of Proposition 4.6, our approach may be considered as equivalent to that of Ref [8].

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