ANNALES DE L'I. H. P., SECTION A

H. BACRY

The position operator revisited

Annales de l'I. H. P., section A, tome 49, n° 2 (1988), p. 245-255

<http://www.numdam.org/item?id=AIHPA_1988__49_2_245_0>

Ann. Inst. Henri Poincaré,
Vol. 49, n° 2, 1988, p. 245-255. *Physique théorique*

The position operator revisited

by

H. BACRY (*)
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT. — A position operator is proposed for any kind of particle. The components of this operator are non-commuting for spinning particles. The properties of such an operator are examined.

RÉSUMÉ. — Nous introduisons un opérateur de position pour tous les types de particules. Pour des particules à spin, les différentes composantes de cet opérateur ne commutent pas et nous étudions les propriétés de cet opérateur.

I. INTRODUCTION

A few years ago I proposed [*1*] a position operator for massless particles with helicity. There has been a challenge for such particles since the work of Newton and Wigner [*2*] who concluded that these particles cannot be localized and, as a consequence, do not possess a position operator. The operator proposed in Ref. 1 had non-commuting components. It was suggested by a formal analogy between the massless representations of the Poincaré group and a group theoretical treatment of the Dirac magnetic monopole. In that paper, I underlined some physical interpretations of such an operator. I was led to a further investigation by the enlightening discussions I had on that subject with A. Connes. Without them, I would never have been led to the present paper. It is perhaps interesting to say in a few words how I arrived at the proposal made in the present work.

(*) Permanent address: Centre de Physique Théorique, Université de Marseille, Case 907, 13288, Marseille Cedex 09, France.

Accepting the operator introduced in Ref. 1 for massless spinning particles would mean the rejection of the Newton-Wigner postulates which led them to conclude that these particles have no position operator. Giving up these postulates for only these particles would be nonsense. Therefore the electron had also to have non-commuting position operator components. Such a situation could be accepted if we recall the fact that the electron in a homogeneous magnetic field has a circular trajectory and that the coordinates of the centre of this trajectory do not commute [3]. Therefore, the average position cannot be localized for this rotating charge. Why would a spinning electron not have the same property?

Consequently, I generalized the position operator of Ref. 1 to all kinds of particles. On applying it to the Dirac particle, I realized that I was arriving at the operator introduced by Schrödinger [4]. It took some time for me to obtain this result because *i*) my approach was through group theory instead of field equations and *ii*) there are many ways of writing the Schrödinger operator (I will say why). This coincidence has an explanation which will be given here. For the time being, let us explain in a slightly different way the work of Schrödinger.

In 1930, Schrödinger [4], in introducing his new position operator, wanted to avoid two difficulties of the Dirac equation, namely:

i) the presence of negative energy states;

ii) the fact that the speed of the of the particle is $\pm c$ in any direction, since

$$\frac{dx}{dt} = -i[x, \mathrm{H}] = c\alpha_x \tag{1}$$

where

$$\mathrm{H} = c\vec{\alpha}\cdot\vec{p} + \beta mc^2 . \tag{2}$$

His operator $\vec{x}_S$ was just what he called the even part of $\vec{x}$

$$\vec{x}_S = \frac{1}{2}(\pi_+ \vec{x}\pi_+ + \pi_- \vec{x}\pi_-) \tag{3}$$

where $\pi_\pm$ is the Hermitian projection onto the space of positive (respectively negative) energy states. From a group theory point of view, Schrödinger had taken the restriction of the operator $\vec{x}$ to the subspaces describing irreducible representation spaces of the Poincaré group. Today, physicists remember the « lost part of $\vec{x}$ », that is $\vec{\xi} = \vec{x} - \vec{x}_S$ as the *zitterbewegung*, a term introduced by Schrödinger. The two difficulties mentioned above were solved since.

i) If we add to the Dirac Hamiltonian a potential written as a function of $\vec{x}_S$ instead of $\vec{x}$, the change is quite small (as shown by Schrödinger himself) and it has the advantage that a positive energy state will stay in the positive energy region during its evolution.

ii) If we compute the speed of $\vec{x}_S$, we obtain a satisfactory answer, namely

$$\frac{d\vec{x}_S}{dt} = -i[\vec{x}_S, H] = \frac{c^2\vec{p}}{H}. \tag{4}$$

In Sec. II, we will generalize the Schrödinger approach for any particle described by a wave equation of *first order*. We will treat separately the case of the photon from Maxwell's equations in Sec. III. This calculation leads to the operator introduced in Ref. 1. In Sec. IV, we will give the general (group theoretical) definition of the position operator. Many comments on this proposal are made in Sec. V and, finally, we will give, as a conclusion, the main advantages of this new operator.

II. THE POSITION OPERATOR A LA SCHRÖDINGER (WAVE EQUATIONS OF FIRST ORDER)

There are essentially two ways of approaching the problem of the position operator, either by defining the concept of the particle from the Poincaré group, that is the Wigner point of view [5], or by using a field equation. Let us first give a few comments on the first approach.

From the Wigner point of view, the states of a given particle are given abstractly since they span a representation space of the Poincaré group. The variables x^μ do not enter this description and, if we want to define a position operator acting on this space, we have to define explicitly the properties required of this operator. This is, in principle, the way followed by Newton and Wigner [2], except that in studying the case of spinning particles they used the Bargmann-Wigner equations rather than representation theory alone. This work was improved by Wightman [6] who set the problem in the Poincaré group representation context.

In their postulates, Newton and Wigner wanted to define « localized states » *from which a position operator would be derived.* It follows from their approach that if a particle has no localized state, no position operator can be defined for it. A localized state is a state for which a measurement of the coordinates x, y, z is possible. It implies that x, y, z are the eigenvalues of three commuting operators, as is the case in non-relativistic quantum mechanics.

The conclusions of the (beautiful) works of Newton-Wigner and Wightman were disappointing in many respects. The worst one concerned the massless particles with non zero helicity which had no localized states [7], and therefore no position operator, *in the Newton-Wigner sense.* That the photon has no localized states can be easily seen with the aid of dimensional analysis. If the electric vector $\vec{E}$, or the combination $\vec{B} - i\vec{E}$ or the vector potential $\vec{A}$ was given a probability amplitude interpretation for

the photon, then $\vec{E}^2$ or $\vec{B}^2 + \vec{E}^2$ or $\vec{A}^2$, multiplied by some universal constant $\hbar^\alpha c^\beta e^\gamma$, would have the dimensions of a (volume)$^{-1}$. That this is impossible is easy to check. This is sufficient to prove that the photon is not localizable.

For the discussion we will have later, I mention here the relationship between the Newton-Wigner position operator $\vec{q}$ for spinning particles of mass m and the generators of the Poincaré group.

rotation generators $\vec{J} = \vec{q} \wedge \vec{p} + \vec{s}$

boost generators

$$\vec{K} = \frac{1}{2}(\sqrt{p^2 + m^2c^2}\,\vec{q} + \vec{q}\sqrt{p^2 + m^2c^2}) + \frac{\vec{p} \wedge \vec{s}}{mc + \sqrt{p^2 + m^2c^2}} \tag{5}$$

space translations $\vec{P} = \vec{p}$

time translations $P_0 = \sqrt{p^2 + m^2c^2}$.

These expressions are known as the Foldy « canonical representation » of the infinitesimal generators of the Poincaré group. These generators act on the Hilbert space spanned by the sections of the vector bundle which has the mass shell (of mass m) as a base and the $2s + 1$ dimensional spin representation space as a fiber. The symbol $\vec{s}$ represents three $(2s + 1) \times (2s + 1)$-matrices generating an irreducible representation of the spin group.

Let us now examine the case where the spinning particle is described by a wave function with n components. This number n is loosely related with the spin s of the particle. It could be $n = 2s + 1$, as it is the case for the neutrino described by the Weyl equation, or $n = 2(2s + 1)$ as in the Dirac case; but for the photon, the field is supposed to obey extra conditions and $n = 6$ if we adopt the Maxwell field description. In all cases, the wave function obeys a first order equation of the type

$$cp_0\psi = H\psi \tag{6}$$

where $cp_0 = i\dfrac{\partial}{\partial t}$ and H is an operator expressed in terms of $\vec{p}$ and « internal » matrices. The generators of the Poincaré group are

$$\begin{aligned} \vec{J} &= \vec{x} \wedge \vec{p} + \vec{\Sigma} \\ \vec{K} &= p_0\vec{x} + ct\vec{p} + \vec{\Sigma}' \\ \vec{p} &= \vec{p} \\ P_0 &= p_0 . \end{aligned} \tag{7}$$

Here, the $n \times n$ matrices $\vec{\Sigma}, \vec{\Sigma}'$ generate the Lorentz « internal » group which makes the n-dimensional field covariant. Obviously, Eq. (6) must imply

$$p_0^2 - \vec{p}^2 = m^2c^2. \tag{8}$$

If m is not zero and if the parity is implemented then the equation implies both a positive and a negative mass as is the case in the Dirac equation.

Let us denote by E the space of the n component fields and by E_i the subspaces of the solutions of (6) associated with the irreducible representation ρ_i of the Poincaré group. Of course

$$\bigoplus_i E_i = \text{Ker}\,(cp_0 - H)\,. \tag{9}$$

Let us denote by π_i the Hermitian projection operator on E_i

$$E_i = \pi_i E\,. \tag{10}$$

The difficulty is that for spinning particles, $\vec{\Sigma}' \neq \vec{0}$ and the operator $\vec{x}$ does not leave E_i invariant. If we follow Schrödinger, we would define as a new position operator

$$\vec{R} = \bigoplus_i \pi_i \vec{x} \pi_i\,. \tag{11}$$

Due to the fact that

$$\vec{p} = \bigoplus_i \pi_i \vec{p} \pi_i$$

$$H = \bigoplus_i \pi_i H \pi_i \tag{12}$$

we get

$$[\vec{R}, H] = \bigoplus_i [\pi_i \vec{x} \pi_i, H] = \bigoplus_i \pi_i [\vec{x}, H] \pi_i = \bigoplus_i \pi_i [\vec{x}, cp_0] \pi_i$$

$$= i\pi \frac{c\vec{p}}{p_0} \pi = \frac{ic^2 \vec{p}}{H} \tag{13}$$

as obtained by Schrödinger in the Dirac case.

As an easy application, we apply the procedure to the 2-component neutrino equation. In this case, we have only one irreducible representation. The equation is

$$(p_0 - \vec{\sigma} \cdot \vec{p})\psi = 0\,. \tag{14}$$

The Hamiltonian $H = \vec{\sigma} \cdot \vec{p}$. The only projection π is given by

$$\pi = \frac{1}{2}\left(1 + \frac{\vec{\sigma} \cdot \vec{p}}{p}\right) \tag{15}$$

where $p = \sqrt{\vec{p}^2}$. We have

$$\vec{R} = \pi \vec{x} \pi = \pi\left(\vec{x} + \frac{i\vec{\sigma}}{2p} - \frac{i\vec{p}}{2p^2}\right). \tag{16}$$

III. THE PHOTON CASE

The photon case is slightly different from the general case studied in the last section. This is due to the transversality conditions. In fact, the photon field $\vec{B} - i\vec{E}$ obeys not only

$$i\frac{\partial \vec{F}}{\partial t} = c \operatorname{rot} \vec{F}, \tag{17}$$

that is

$$(p_0 - \vec{S}\cdot\vec{p})\vec{F} = 0 \tag{18}$$

where [8]

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

but also the transversality equation

$$\vec{p}\cdot\vec{F} = 0 \tag{20}$$

which is said to discard the longitudinal photons.

The 3×3 matrix $\vec{S}\cdot\vec{p}$ has three eigenvalues, namely p, $-p$ and zero, the zero eigenvalue corresponding to the non-existing longitudinal states. The projection on the other kinds of states (helicity ± 1) is given by

$$\pi = 1 - \frac{1}{p^2}\begin{pmatrix} p_x^2 & p_xp_y & p_xp_z \\ p_xp_y & p_y^2 & p_yp_z \\ p_xp_z & p_yp_z & p_z^2 \end{pmatrix} = 1 - \pi_0 \tag{21}$$

for a given value of $\vec{p}$. We can check that π has a trace equal to two corresponding to the two possible states of polarization of the photon.

This projection operator is the one we have to use because it projects on the set of all possible states of the photons (we are permitted to combine linearly the ± 1 helicity states). Therefore, our position operator is

$$\vec{R} = \pi\vec{x}\pi = \vec{x} + \frac{\vec{p}\wedge\vec{S}}{p^2}. \tag{22}$$

(If the reader is going to check this result, he must be aware that he could arrive at a different answer; the reason being that there are many ways of writing an operator for which only a restriction to some subspace is specified. If many expressions can be given for the Schrödinger operator in the Dirac case, here the situation provides us with even more freedom, due to the transversality conditions).

If is interesting to compute the commutation relations between the components of $\vec{R}$. We find, for instance,

$$[R_x, R_y] = -i \frac{\vec{S} \cdot \vec{p}}{p^4} p_z \tag{23}$$

or, since

$$(\vec{S} \cdot \vec{p})\vec{F} = \pm p\vec{F},$$

$$[R_x, R_y] = \mp i \frac{p_z}{p^3} \tag{25}$$

depending on the helicity. This relation (25) is exactly the one obtained by the « monopole approach » [*1*].

To conclude this section, it is interesting to give the expression for the most general field which is an eigenfunction of the operator R_z. It is, in the momentum space,

$$F_x(\vec{p}) = \frac{\pm ipp_y - p_x p_z}{p(p_x^2 + p_y^2)} G(p_x, p_y) e^{-iap_z}$$

$$F_y(\vec{p}) = \frac{\mp ipp_x - p_y p_z}{p(p_x^2 + p_y^2)} G(p_x, p_y) e^{-iap_z} \tag{26}$$

$$F_z(\vec{p}) = \frac{1}{p} G(p_x, p_y) e^{-iap_z}.$$

Here, G is an arbitrary function and a is the eigenvalue of R_z. We readily see that a complete set of observables involving R_z is given by $\{ R_z, p_x^2 + p_y^2, xp_y - yp_x + S_z \}$.

We will make the appropriate comments about these results in the last section.

IV. THE POSITION OPERATOR VIEWED FROM THE POINCARÉ GROUP

Obviously we could imagine, in principle, the last calculation made on the wave equation satisfied by the four vector potential A_μ. The calculation would be more difficult, not only because the order of the partial differential equation is higher but also because the space on which the Poincaré group is acting is in fact a factor space since A_μ and $A_\mu + \partial_\mu \chi$ must be considered as equivalent.

Since we are interested in free fields only, the wave equation to be used is irrelevant and all equations leading to the same representation of the Poincaré group have the *same physical contents.* Therefore, it is more natural to give a group theoretical definition of the position operator $\vec{R}$.

It is clear that $\vec{R}$ must belong to the envelopping algebra (in a wide sense) of the Poincaré Lie algebra. Such a property guarantees that $\vec{R}$

will act on any state of the Hilbert space. The definition induced by the article [*1*] is defined from the boosts and the time translations generator P_0 only by

$$\vec{R} = \frac{1}{2P_0}\vec{K} + \vec{K}\frac{1}{2P_0}. \tag{27}$$

It readily follows that

$$\vec{K} = \frac{1}{2}(P_0\vec{R} + \vec{R}P_0). \tag{28}$$

In particular, for a massive particle, we get, from Eq. (5)

$$\vec{R} = \vec{q} + \frac{\vec{p} \wedge \vec{S}}{(mc + \sqrt{p^2 + m^2c^2})\sqrt{p^2 + m^2c^2}} \tag{29}$$

which shows the relationship between our position operator and the Newton-Wigner one.

V. CONCLUSIONS

We could have studied the non-relativistic Schrödinger equation with the aid of the Galileo group. In that case, even with a spinning particle, the restrictions of the operator $\vec{x}$ to the space of solutions would have given $\vec{x}$ itself, as expected. In fact, if we perform a *contraction*, an operation which permits us to go from the Poincaré group to the Galilei group, both the Newman-Wigner $\vec{q}$ and the operator $\vec{R}$ become $\vec{x}$. This does not mean that we cannot obtain something else near the *non relativistic limit*, that is in the case of small momenta (without making $c \to \infty$). We will examine this point at the end of this article.

We will now present three arguments in favour of our position operator.

1. *The photon localizability.*

It is clear that the fact that we are able to associate a position operator with the photon does not ensure the measurability of its position, because we cannot measure more than one position coordinate at a time. To see the advantage of such a situation, we examine the case where a photon has essentially a momentum in the z direction, then the uncertainty relation for the other components x and y gives

$$\Delta x \Delta y \sim \frac{\hbar^2}{p^2} \tag{30}$$

which means that the uncertainty in measuring the position in an orthogonal direction is of the order of the wave length. This is satisfactory since it means that the localizability increases with the frequency of light.

As a consequence, we get a better explanation of the electron slit experiment [*10*]. It is well-known that if we use an intense light source to know which slit the electron went through, the interference pattern disappears. If we decrease the intensity of light, the fringes reappear progressively. To explain why, we usually need two kinds of arguments depending on the way the intensity of the light source decreases; *i*) if we keep fixed the frequency of the light and decrease the number of photons emitted per second, then the probability of a photon-electron scattering diminishes and there are electrons which will not be seen and will contribute to the interference pattern. *ii*) If we make the intensity of light diminish by increasing its wave length then we need an argument *taken from classical electrodynamics:* the image of an electron will not be a point but a spot, the width of which is proportional to the wave length. Therefore, it will be more and more difficult to know which slit the electron went through.

It is not a satisfactory situation to call on the classical theory to interpret the electron slit experiment. Our formula (30) permits us to provide a purely quantum interpretation.

2. *All particles are equal.*

One of the important ideas of the century was that of de Broglie, who proposed in 1923 to put matter and radiation on the same footing. An irony of history was that this « democracy » was destroyed by the Born statistical interpretation of the wave function, an interpretation that the photon field cannot have. Obviously, such an interpretation was a non-relativistic property but it was believed that it was valid also for relativistic waves.

With our proposal, the symmetry between all kinds of particles is recovered. First each kind of particle has a position operator. All spinning particles have in common the property of not being localizable. Since all stable particles are spinning, this provides the spin with a fundamental character.

3. *The non-relativistic limit.*

We have already mentioned that the electron has, as a possible position operator,

$$\vec{R} = \vec{q} + \frac{\vec{p} \wedge \vec{s}}{\sqrt{p^2 + m^2c^2}(\sqrt{p^2 + m^2c^2} + mc)}. \tag{31}$$

In the non-relativistic limit, we know that the Newton-Wigner operator $\vec{q}$ becomes the standard non relativistic Schrödinger position operator $\vec{x}$. It is clear that if we make c go to infinity the second term in Eq. (31) vanishes and the limit of $\vec{R}$ is also $\vec{x}$. But this limit is the *Galilean* limit to be distinguished from the *non-relativistic* limit, which corresponds to the case

of small linear momenta. Therefore, the non-relativistic approximation of the operator $\vec{R}$ is

$$\vec{R} \approx \vec{x} + \frac{\vec{p} \wedge \vec{s}}{2m^2c^2}. \tag{32}$$

It is natural to replace in the Schrödinger-Pauli equation the potential $V(\vec{x})$ by $V(\vec{R})$. For a spherical potential, we get as a first approximation

$$R^2 \approx r^2 + \frac{\vec{L} \cdot \vec{s}}{m^2c^2}, \qquad r^2 = \vec{x}^2$$

and

$$V(R) \approx V(r) + V'(r) \frac{\vec{L} \cdot \vec{s}}{2m^2c^2r} \tag{33}$$

which is the spin-orbit coupling with the right Thomas factor $\frac{1}{2}$.

This is, in our opinion, the best argument in favour of the operator $\vec{R}$. It is also an encouragement in favour of a new investigation of the two-component Schrödinger-Pauli equation theory with the systematic replacement of the potential described by Eq. (33).

To conclude, it is perhaps useful to mention that the non-localizability of stable particles must have some consequences in our formulation of quantum field theory. We hope to discuss this point in a forthcoming paper.

ACKNOWLEDGEMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He would like to express his gratitude to Professors A. J. Bracken, L. C. Biedenharn, L. Michel and J. Zak for discussions on the present work. He would also like to thank Professor A. Connes for stimulating and enlightening discussions and Professor A. J. Bracken for his help in the preparation of the manuscript.

REFERENCES

[1] H. BACRY, *J. Phys.*, t. **A 14**, 1981, L73.
[2] T. D. NEWTON and E. P. WIGNER, *Rev. Mod. Phys.*, t. **21**, 1949, p. 400.
[3] L. D. LANDAU and E. M. LIFSHITZ, *Quantum Mechanics*, p. 475 (Pergamon Press, 1959).
[4] E. SCHRÖDINGER, *Collected Papers*, Vol. III, p. 394-435 (Friedr. Wieweg, Wien, 1984).
[5] E. P. WIGNER, *Ann. Math.*, t. **40**, 1939, p. 149.
[6] A. S. WIGHTMAN, *Rev. Mod. Phys.*, t. **34**, 1962, p. 845.

[7] Except for a four-component neutrino. Note that massless particles with zero helicity are localizable.

[8] In fact, we have to double the number of components and take a 6-dimensional vector $(\vec{F}, \vec{F}^*) = (\vec{B} - i\vec{E}, \vec{B} + i\vec{E})$. See, for instance, Ref. [9].

[9] H. BACRY, *Nuovo Cimento*, t. **32 A**, 1976, p. 448.

[10] R. P. FEYNMAN, *Lecture Notes in Physics*, Vol. III, Chapter 1 (Addison-Wesley, 1966).

(Article reçu le 28 janvier 1988)

(Version révisée reçue le 23 mars 1988)