ANNALES DE L'I. H. P., SECTION A

E. C. G. SUDARSHAN
TOM D. IMBO
CHANDNI SHAH IMBO

Topological and algebraic aspects of quantization : symmetries and statistics

Annales de l'I. H. P., section A, tome 49, nº 3 (1988), p. 387-396
<http://www.numdam.org/item?id=AIHPA_1988__49_3_387_0>

Ann. Inst. Henri Poincaré,
Vol. 49, n° 3, 1988, p. 387-396. *Physique théorique*

Topological and algebraic aspects of quantization: symmetries and statistics

by

E. C. G. SUDARSHAN (*), **Tom D. IMBO** (*) and **Chandni Shah IMBO** (†)
Center for Particle Theory (*) and Department of Mathematics (†),
University of Texas at Austin, Austin, Texas 78712

ABSTRACT. — We discuss the inequivalent quantizations of a physical system with configuration space Q. A brief review is given of rigorous results concerning the number and type of quantizations available to such a system. The example of n identical particles moving on an arbitrary manifold is considered in some detail.

RÉSUMÉ. — Nous discutons les quantifications inéquivalentes d'un système physique à espace de configuration Q. Nous présentons une brève revue des résultats rigoureux concernant le nombre et le type de quantifications possibles pour un tel système. L'exemple de n particules identiques en mouvement sur une variété arbitraire est considéré en détail.

I. INTRODUCTION

When constructing a quantum theory from a classical dynamical system with configuration space [*1*] Q, the standard procedure is to choose the fixed-time quantum mechanical state vectors as functions from Q into the complex numbers $\mathbb{C}$. However, one can use a much more general notion of a state vector $\Psi(q)$. First, Ψ may be « multiple-valued » on Q; and second, Ψ may take values in any $\mathbb{C}^N$, $N \geq 1$. More specifically, $\Psi(q)$

can have N-components $\Psi_n(q)$, $1 \leq n \leq N$, and when q is taken around a generic loop l in Q, we may have [2]

$$\Psi_n(q) \xrightarrow{l} \sum_{m=1}^{N} V_{nm}([l])\Psi_m(q) \tag{1}$$

where $V([l])$ is an $N \times N$ unitary matrix depending only on the homotopy class of l, denoted by $[l]$. (Note that $\Psi^\dagger\Psi$ must be single-valued.) Moreover for any two loops l_1 and l_2 we require

$$V([l_1])V([l_2]) = V([l_1 l_2]) \tag{2}$$

where $l_1 l_2$ denotes the standard product loop. Eqs. (1) and (2) imply that V provides an N-dimensional unitary representation of $\pi_1(Q)$, the fundamental group of Q [3].

If Ψ and Ψ' are two such N-component objects « transforming under loops » according to V and V′ respectively, then $\Psi + \Psi'$ is an acceptable state vector if and only if $V = V'$ (as representations of $\pi_1(Q)$). Thus the total Hilbert space $\mathscr{H}$ of states breaks up into a direct sum of subspaces $\{\mathscr{H}_\rho\}$ where each $\mathscr{H}_\rho$ only contains states which transform according to the fixed representation ρ of $\pi_1(Q)$. If ρ is reducible, then $\mathscr{H}_\rho$ breaks up further into a direct sum of subspaces $\{\mathscr{H}_{\rho_i}\}$ where the ρ_i's are the irreducible components of ρ. Therefore we can achieve a decomposition of $\mathscr{H}$ into superselection sectors, each labelled by a finite-dimensional irreducible unitary representation (IUR) of $\pi_1(Q)$ [4]. If we let $\mathscr{R}(\pi_1(Q))$ denote the set of all finite-dimensional IUR's of $\pi_1(Q)$, then the quantum theories defined by each of the sectors $\mathscr{H}_\alpha$, $\alpha \in \mathscr{R}(\pi_1(Q))$, represent the « prime » quantizations of the original system. Clearly, $\mathscr{R}(\pi_1(Q))$ contains at least one element, namely the trivial IUR, and the corresponding quantum theory has ordinary complex-valued functions as state vectors. However, in general $\mathscr{R}(\pi_1(Q))$ will contain more than one element revealing the essential « kinematical ambiguity » in quantizing a classical system [6].

The quantizations corresponding to IUR's of degree 1, the so-called scalar quantizations, are simply labelled by the character group Ω of $\pi_1(Q)$ [7], [8]:

$$\Omega = \mathrm{Hom}\,(\pi_1(Q), U(1)) \cong \mathrm{Hom}\,(H_1(Q), U(1)) \tag{3}$$

where $H_1(Q)$ is the first (integral) homology group of Q. Again, there is always at least one scalar quantization. The quantizations associated with IUR's of degree > 1 are of a qualitatively different nature. They possess an « internal symmetry » of topological origin associated with the entire system [9]. In this paper, we review various results concerning when a system has a unique quantization, a unique scalar quantization, or *only* scalar quantizations. We then turn our attention to a class of systems which almost always possesses nonscalar quantizations; namely,

n identical particles moving on a manifold M. Here the different choices of quantum representations of the system are related to the different possible types of statistics available for the n identical particles. Much work has been done on the scalar quantizations of this class of systems, and their relationship to statistics. Here we survey these studies and proceed to derive some new results concerning nonscalar quantizations.

II. SURVEY OF RIGOROUS RESULTS

In what follows, by a Q-system we will mean a physical system with configuration space Q. One natural question concerning the above classification of inequivalent quantizations of a Q-system is:

A) When does a Q-system have a unique prime quantization?

For scalar quantizations, this question is answered in Ref. [*10*].

THEOREM 1. — A Q-system has a unique scalar quantization if and only if $\pi_1(Q)$ is a perfect group [*11*], or equivalently, $H_1(Q)$ is trivial.

More generally, if we call a group G with no nontrivial finite-dimensional IUR's a U-inert group, then by definition a Q-system has a unique prime quantization if and only if $\pi_1(Q)$ is U-inert. (Clearly U-inert groups must be perfect.) A characterization of finitely generated U-inert groups is given in Ref. [*5*]:

THEOREM 2. — A finitely generated group G is U-inert if and only if G has no nontrivial finite quotient groups.

The restriction to finitely generated groups is not severe since almost all spaces Q of interest in physics have $\pi_1(Q)$ finitely generated. There are many nontrivial examples of U-inert groups [*5*] and the possibility of finding physically interesting Q-systems with $\pi_1(Q)$ U-inert (and nontrivial) is addressed in Ref. [*5*] with some success. We therefore see, contrary to the usual intuition, that there exist physical systems with *multiply-connected* configuration spaces Q which nonetheless have a unique quantization.

Another natural question is the following:

B) When are all the prime quantizations of a Q-system scalar?

Let us call a group U-scalar if it has no finite-dimensional IUR's of degree > 1. Then by definition, all the prime quantizations of a Q-system are scalar if and only if $\pi_1(Q)$ is U-scalar. Clearly, all abelian groups are U-scalar. In Ref. [*5*] it is shown that:

THEOREM 3. — A finitely generated group G is U-scalar if and only if G has no finite nonabelian quotient groups.

There are many examples of nonabelian U-scalar groups (for example, all nontrivial U-inert groups), and such groups *can* be realized as $\pi_1(\mathrm{Q})$ of interesting Q-systems [*5*]. For more properties of U-inert and U-scalar groups, see Ref. [*5*].

III. STATISTICS OF IDENTICAL PARTICLES

An interesting example of a class of physical systems which almost always possesses nonscalar quantizations is that of n identical particles moving on an arbitrary path-connected manifold M (without boundary). The relevant configuration space is the orbit space [*8*]

$$\mathrm{Q}_n(\mathrm{M}) = (\mathrm{M}^n - \Delta)/\mathrm{S}_n \tag{4}$$

where M^n represents the n-fold Cartesian product of M with itself, Δ is the subcomplex of M^n for which two or more (particle) coordinates are the same, and S_n is the permutation group on n symbols with the obvious action on $\mathrm{M}^n - \Delta$. Clearly this action is free (i. e. without fixed points), yielding the following fibration [*12*]

$$\begin{array}{ccc} \mathrm{S}_n & \hookrightarrow & \mathrm{M}^n - \Delta \\ & & \downarrow \\ & & \mathrm{Q}_n(\mathrm{M}) . \end{array}$$

The long exact homotopy sequence of this fibration gives the following five term short exact sequence for $\pi_1(\mathrm{Q}_n(\mathrm{M}))$:

$$\{ e \} \to \pi_1(\mathrm{M}^n - \Delta) \to \pi_1(\mathrm{Q}_n(\mathrm{M})) \to \mathrm{S}_n \to \{ e \} . \tag{5}$$

The group $\pi_1(\mathrm{Q}_n(\mathrm{M}))$ is called the n-string braid group of M in the mathematical literature [*12*], [*13*], and is usually denoted by $\mathrm{B}_n(\mathrm{M})$. (Note that $\mathrm{B}_1(\mathrm{M}) \cong \pi_1(\mathrm{M})$.) The prime quantizations of our system are labelled by $\mathscr{R}(\mathrm{B}_n(\mathrm{M}))$. $\mathscr{R}(\mathrm{B}_n(\mathrm{M}))$ contains at least as many elements as $\mathscr{R}(\mathrm{S}_n)$. More specifically, for any IUR ρ of S_n of degree m, there is a corresponding IUR of $\mathrm{B}_n(\mathrm{M})$ of degree m constructed by « lifting » ρ. So since S_n, $n \geq 3$, always has an IUR of degree > 1, we see that there is always a nonscalar quantization of our system if $n \geq 3$ (for any M).

It is clear that the different quantizations of the above system are related to the different possible statistics for the n identical particles ($n \geq 2$). However, one must be careful not to overcount. There is, in general, a quantization ambiguity already present for $n = 1$ (and therefore having nothing to do with statistics) which will manifest itself again in $\mathscr{R}(\mathrm{B}_n(\mathrm{M}))$ for any n. So in order to get the set which labels the different choices of statistics one must take $\mathscr{R}(\mathrm{B}_n(\mathrm{M}))$ and « mod out » by $\mathscr{R}(\mathrm{B}_1(\mathrm{M}))$ in an appropriate way. We will denote the formal quotient set $\mathscr{R}(\mathrm{B}_n(\mathrm{M}))/\mathscr{R}(\mathrm{B}_1(\mathrm{M}))$

by $\mathscr{C}_n(M)$. For simply-connected manifolds M we have $\mathscr{C}_n(M) = \mathscr{R}(B_n(M))$. We now turn to a discussion of scalar quantizations and the corresponding choices of statistics.

IV. SCALAR STATISTICS

The scalar quantizations of n identical particles moving on a manifold M are labelled by

$$\Omega_n(M) = \mathrm{Hom}\,(\pi_1(Q_n(M)), U(1)) \cong \mathrm{Hom}\,(H_1(Q_n(M)), U(1)). \qquad (6)$$

In Ref. [14] it is shown, using standard techniques in algebraic topology and homological algebra, that if dim M ≥ 3 or if M is a closed 2-manifold not equal to S^2, then

$$H_1(Q_n(M)) \cong H_1(M) \oplus \mathbb{Z}_2, \qquad n \geq 2. \qquad (7)$$

For these manifolds we then have

$$\Omega_n(M) \cong \Omega_1(M) \oplus \mathbb{Z}_2, \qquad n \geq 2. \qquad (8)$$

The different statistics associated with these scalar quantizations (« scalar statistics ») are then labelled by $\Omega_n(M)/\Omega_1(M) \cong \mathbb{Z}_2$ and it is easy to see that these two choices correspond to Bose and Fermi statistics. So there are no exotic *scalar* statistics available to n identical particles moving on the above manifolds.

If $M = \mathbb{R}^2$ the situation is different. We have [15] ($n \geq 2$)

$$H_1(Q_n(\mathbb{R}^2)) \cong \mathbb{Z}, \qquad \Omega_n(\mathbb{R}^2) \cong U(1). \qquad (9)$$

So we see that the choices of statistics for n identical particles moving on $\mathbb{R}^2$ are labelled by an angle θ, $0 \leq \theta < 2\pi$. This angle smoothly interpolates between quantizations with Bose ($\theta = 0$) and Fermi ($\theta = \pi$) statistics. The new statistics are often called θ-statistics (or fractional statistics) and seem to be relevant in theoretical interpretations of the Fractional Quantum Hall Effect [16]. Finally, for $M = S^2$ one has [16], [17] ($n \geq 2$)

$$H_1(Q_n(S^2)) \cong \Omega_n(S^2) \cong \mathbb{Z}_{2n-2}. \qquad (10)$$

Note that the number of possible scalar statistics grows as n grows. This peculiar n-dependence of the set of scalar statistics seems to be unique to the case $M = S^2$.

V. GENERAL STATISTICS ON SIMPLY CONNECTED MANIFOLDS

In this section we consider the possible statistics for n identical particles moving on a simply connected manifold M. If dim M ≥ 3 we have [8] $\pi_1(M^n - \Delta) = \{ e \}$ and Eq. (5) yields

$$B_n(M) \cong S_n. \qquad (11)$$

The representations of S_n are well studied. The number of IUR's of S_n is equal to the number of partitions of the integer n, denoted by $p(n)$. So there are $p(n)$ choices of statistics for n identical particles moving on a simply connected manifold M of three or more dimensions. Some values of $p(n)$ are given below [*18*].

n	$p(n)$	n	$p(n)$
2	2	10	42
3	3	20	627
4	5	50	204, 266
5	7	100	190, 569, 292

Note that $p(n)$ grows rapidly as n increases. For large n, one can use the Hardy-Ramanujan asymptotic formula [*18*]

$$p(n) \simeq \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} . \tag{12}$$

For any $n \geq 2$, there are only two IUR's of degree 1. Namely the trivial IUR and the IUR sending all even permutations to $+1$ and all odd permutations to -1. The corresponding statistics are Bose and Fermi respectively (see Section IV). The (nonscalar) statistics associated with the remaining IUR's represent a generalization of these [*5*], [*19*].

The situation in dimension two is much more complex. $B_n(\mathbb{R}^2)$ $(n \geq 2)$ is an infinite group known as the n-string Artin braid group [*20*] and has many applications in knot theory. It can be defined by the following presentation

$$\begin{aligned} B_n(\mathbb{R}^2) = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2 \,; \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2 \rangle . \end{aligned} \tag{13}$$

$B_2(\mathbb{R}^2) \cong \mathbb{Z}$ and therefore there are no nonscalar quantizations of 2 identical particles on $\mathbb{R}^2$. For $n \geq 3$, $B_n(\mathbb{R}^2)$ is infinite and nonabelian. We shall concentrate on the case $n = 3$. A slightly more useful presentation of $B_3(\mathbb{R}^2)$ is [*20*], [*21*]

$$B_3(\mathbb{R}^2) = \langle a, b \,|\, a^3 = b^2 \rangle \tag{14}$$

where $a = \sigma_1\sigma_2$ and $b = \sigma_1\sigma_2\sigma_1$. The degrees of the finite dimensional IUR's of $B_3(\mathbb{R}^2)$ are unbounded. This can be seen as follows. It is known [*22*] that

$$\pi_1((\mathbb{R}^2)^3 - \Delta) \cong F_2 \times \mathbb{Z} = \langle x, y, z \,|\, xz = zx, yz = zy \rangle \tag{15}$$

where F_2 is the free group on two generators (x and y). It is easy to see that $F_2 \times \mathbb{Z}$ has an IUR in every positive dimension m. Just send x to any diagonal $m \times m$ unitary matrix all of whose eigenvalues are distinct, and send y to any $m \times m$ unitary matrix all of whose off diagonal elements are nonzero. Finally, send z to any unitary scalar matrix. These three matrices will generate a unitary representation ρ of $F_2 \times \mathbb{Z}$, and since the only matrices commuting with all of them are the scalar matrices, ρ is irreducible. Now by Eq. (5) we have that $F_2 \times \mathbb{Z}$ is a normal subgroup of *finite index* in $B_3(\mathbb{R}^2)$. It is a well known result in the theory of induced representations [*23*] that if a group G has a normal subgroup of finite index with an IUR of degree m, then G *must* have an IUR of finite degree $\geq m$. We therefore have that the degrees of the finite-dimensional IUR's of $B_3(\mathbb{R}^2)$ are unbounded. So we see that there are an infinite number of « types » of nonscalar statistics for 3 particles on $\mathbb{R}^2$.

All the two-dimensional IUR's of $B_3(\mathbb{R}^2)$ are easily found. They are labelled by two angles ϕ and θ, $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi/2$, and are generated by (see Eq. (15))

$$a = e^{2i\phi}\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \qquad b = e^{3i\phi}\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \tag{16}$$

where $\omega = e^{2\pi i/3}$. (For $\theta = 0$, one must restrict the range of ϕ to $0 \leq \phi < \pi$, in order not to overcount.) The case $\phi = \pi/3$, $\theta = 0$ corresponds to the « lift » of the two-dimensional IUR of S_3. A 3-parameter family of three-dimensional IUR's of $B_3(\mathbb{R}^2)$ is generated by $(0 \leq \phi < 2\pi)$

$$a = e^{2i\phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (b)_{jk} = e^{3i\phi}(-\delta_{jk} + 2n_j n_k), \quad 1 \leq j, k \leq 3, \tag{17}$$

where δ_{jk} is the Kronecker delta symbol and $\vec{n} = (n_1, n_2, n_3)$ is a real positive unit vector all of whose components are nonzero. (If $n_1 = n_2 = n_3 = 1/\sqrt{3}$, then one must restrict the range of ϕ to $0 \leq \phi < 2\pi/3$ so as not to overcount.) These exhaust the three dimensional IUR's of $B_3(\mathbb{R}^2)$. The higher dimensional IUR's of $B_3(\mathbb{R}^2)$ can also be explicitly constructed although the procedure is much more tedious. Representations of $B_n(\mathbb{R}^2)$, $n \geq 4$, are also more difficult to construct.

Finally, we consider $M = S^2$. We have [*24*]

$$\begin{aligned} B_n(S^2) = \langle\, \delta_1, \delta_2, \ldots, \delta_{n-1} \mid \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}, \quad 1 \leq i \leq n-2\,; \\ \delta_i\delta_j = \delta_j\delta_i, \quad |i-j| \geq 2\,; \quad \delta_1\delta_2 \ldots \delta_{n-2}\delta_{n-1}^2\delta_{n-2} \ldots \delta_2\delta_1 = 1 \,\rangle . \end{aligned} \tag{18}$$

$B_2(S^2) \cong \mathbb{Z}_2$ and there are only the Bose and Fermi quantizations. $B_3(S^2)$ can be presented as

$$B_3(S^2) = \langle\, c, d \mid c^3 = d^2, d = cdc \,\rangle \tag{19}$$

where $c = \delta_1\delta_2$ and $d = \delta_1\delta_2\delta_1$. $B_3(S^2)$ has order 12 since [*24*] $\pi_1((S^2)^3 - \Delta) \cong \mathbb{Z}_2$ (see Eq. (5)). There are six IUR's of $B_3(S^2)$; four of them have degree 1 (see Section IV) and the remaining two have degree 2. They are given by

$$c = \lambda^2 \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad d = \lambda^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{20}$$

with $\lambda = 1$ and $\lambda = i$. The groups $B_n(S^2)$, $n \geq 4$, are infinite, nonabelian [*24*] and more difficult to deal with.

VI. CONCLUSION

In this paper we discussed various topological and algebraic aspects of the process of quantizing a classical system. First, the classification of the prime quantizations of a Q-system by finite-dimensional IUR's of $\pi_1(Q)$ was reviewed and then three recent theorems concerning the nature of the quantizations available to a given Q-system were stated. Specifically, we saw that there exist Q-systems with nonabelian $\pi_1(Q)$ which have only scalar quantizations, or even a unique quantization. Finally, we considered in some detail the prime quantizations of n identical particles moving on an arbitrary manifold M and their relationship to statistics. In particular the possible scalar statistics of n identical particles on manifolds of dimension ≥ 3, as well as $\mathbb{R}^2$ and all closed 2-manifolds, were completely given, followed by a discussion of nonscalar statistics on all simply connected manifolds of dimension ≥ 2.

In closing, we would like to say that throughout his career, Jean-Pierre Vigier has always been open to new ideas, no matter how bizarre they may have initially appeared. This immense curiosity has often paid dividends. When the above nonstandard scalar quantizations utilizing multiple-valued state vectors first appeared [*25*], they must also have seemed very strange; a mere formal curiosity. However, with the discovery of « θ-vacua » in nonabelian gauge theories, gravitational theories and nonlinear sigma models [*2*], as well as fractional statistics in condensed matter physics, the new scalar quantizations have entered the mainstream of physics, despite their initial exotic appearance. It is our sincere hope that the (even stranger) *nonscalar* quantizations discussed here will enjoy a similar fate [*26*].

ACKNOWLEDGEMENTS

It is our pleasure to thank T. R. Govindarajan, Duane Dicus, Steve Della Pietra, John Moody, Gary Hamrick, Cecile Dewitt-Morette, Yong-Shi Wu, Lorenzo Sadun and Greg Nagao for interesting discussions and useful comments. One of us (T. I.) would like to acknowledge the Graduate

School of the University of Texas at Austin for support through a fellowship. This work was supported in part by the U. S. Department of Energy under grant number DE-FG05-85ER40200.

REFERENCES

[1] We assume that Q is a path-connected, smooth manifold. We also assume that the system is not interacting with any external field.

[2] C. J. ISHAM, in *Relativity, Groups and Topology II*, edited by B. S. DeWitt and R. Stora (Elsevier, New York, 1984), and references therein.

[3] The quantum theory does not distinguish between V's which form equivalent representations of π_1 (Q). Therefore in what follows, by a « unitary representation » we will always mean a « unitary equivalence class of unitary representations ».

[4] It is unclear whether one can construct consistent dynamical quantum theories based on infinite-dimensional IUR's of π_1 (Q), i. e. using infinite-component state vectors. For more on this point, see Ref. 5. In what follows, the statements we make concerning the classification of quantizations should be understood as modulo these possible « infinite-dimensional » prime quantizations.

[5] T. D. IMBO, C. SHAH IMBO and E. C. G. SUDARSHAN, *Center for Particle Theory*. Report, No. DOE-ER40200-142, 1988.

[6] In this paper we will ignore possible quantization ambiguities of dynamical origin. Also, an alternative, geometrical (and more precise) way of viewing the state vectors in $\mathscr{H}_\alpha$, $\alpha \in \mathscr{R}(\pi_1(Q))$, is as sections of an irreducible $\mathbb{C}^N$-bundle over Q (N = dim α) which possesses a natural, flat U(N) connection whose holonomy gives rise to the representation α of $\pi_1(Q)$. See Refs. 2 and 5.

[7] B. KOSTANT, in *Lecture Notes in Mathematics*, vol. 170 (Springer Verlag, Berlin, 1970).

[8] M. G. G. LAIDLAW and C. M. DEWITT, *Phys. Rev.*, t. **D 3**, 1971, p. 1375; M. G. G. Laidlaw, Ph. D. Thesis, University of North Carolina, 1971.

[9] E. C. G. SUDARSHAN, T. D. IMBO and T. R. GOVINDARAJAN, *Phys. Lett.*, t. **B 213**, 1988, p. 471.

[10] T. D. IMBO and E. C. G. SUDARSHAN, *Phys. Rev. Lett.*, t. **60**, 1988, p. 481.

[11] A group G is called perfect if [G, G] = G where [G, G] is the commutator (or derived) subgroup of G. See, for example, D. J. S. ROBINSON, *A Course in the Theory of Groups* (Springer Verlag, New York, 1982).

[12] E. FADELL and L. NEUWIRTH, *Math. Scand.*, t. **10**, 1962, p. 111.

[13] R. FOX and L. NEUWIRTH, *Math. Scand.*, t. **10**, 1962, p. 119; J. S. BIRMAN, *Braids, Links and Mapping Class Groups* (Princeton University Press, Princeton, 1974), and references therein.

[14] T. D. IMBO and C. SHAH IMBO, *Center for Particle Theory*. Report, in preparation.

[15] Y.-S. WU, *Phys. Rev. Lett.*, t. **52**, 1984, p. 2103.

[16] D. J. THOULESS and Y.-S. WU, *Phys. Rev.*, t. **B 31**, 1985, p. 1191.

[17] J. S. DOWKER, *J. Phys.*, t. **A 18**, 1985, p. 3521.

[18] J. S. LOMONT, *Applications of Finite Groups* (Academic Press, New York, 1959).

[19] F. J. BLOORE, I. BRATLEY and J. M. SELIG, *J. Phys.*, t. **A 16**, 1983, p. 729; R. D. SORKIN, in *Topological Properties and Global Structure of Space-Time*, edited by P. G. Bergmann and V. de Sabbata (Plenum, New York, 1986); H. S. Green, *Phys. Rev.*, t. **90**, 1953, p. 270.

[20] E. ARTIN, *Abh. Math. Sem. Hamburg*, t. **4**, 1926, p. 47.

[21] $B_3(\mathbb{R}_2)$ is torsion-free and is also isomorphic to the group of the trefoil knot. For more on braids and knots see D. L. JOHNSON, *Topics in the Theory of Group Presentations* (Cambridge University Press, Cambridge, 1980), and J. S. BIRMAN in Ref. 13.

[22] J. A. H. SHEPPERD, *Proc. Roy. Soc. London*, t. **A 265**, 1962, p. 229.
[23] A. H. CLIFFORD, *Ann. Math.*, t. **38**, 1937, p. 533.
[24] E. FADELL and J. VAN BUSKIRK, *Duke Math. Jour*, t. **29**, 1962, p. 243 ; J. VAN BUSKIRK, *Trans. Amer. Math. Soc.*, t. **122**, 1966, p. 81.
[25] See Refs. 7 and 8, as well as D. FINKELSTEIN, *J. Math. Phys.*, t. **7**, 1966, p. 1218 ; D. FINKELSTEIN and J. RUBINSTEIN, *J. Math. Phys.*, t. **9**, 1968, p. 1762 ; L. S. SCHULMAN, *Phys. Rev.*, t. **176**, 1968, p. 1558 ; *J. Math. Phys.*, t. **12**, 1971, p. 304 ; J. S. DOWKER, *J. Phys.*, t. **A 5**, 1972, p. 936.
[26] Further references on nonscalar quantizations are, A. P. BALACHANDRAN, *Nucl. Phys.*, t. **B 271**, 1986, p. 227 ; Syracuse University Report No. SU-4428-361, 1987 ; Syracuse University Report No. SU-4428-373, 1988 ; as well as Ref. 2.

(Manuscrit reçu le 10 juillet 1988)