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Semiclassical resolvent estimates

by

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ABSTRACT. — We prove estimates in the semiclassical regime of small $h$ on the boundary values of the resolvent of the Schrödinger operator: $H(h) = -h^2 \Delta + V$ in a neighborhood of a non-trapping energy $E$. The potential $V$ is bounded, but not necessarily decaying with derivatives decaying at infinity. The method also applies to potentials with local singularities and to a family of Stark Hamiltonians. The proof is based on Mourre theory and decay estimates for wave packets in the classically forbidden region.

RÉSUMÉ. — Dans le régime semi-classique (petit $h$), nous estimons les valeurs au bord de la résolvante de l'opérateur de Schrödinger $H(h) = -h^2 \Delta + V$ dans un voisinage d'une énergie non liante $E$. Le potentiel $V$ est borné mais n'est pas nécessairement décroissant mais ou avec des dérivées décroissantes à l'infini. La méthode s'applique aussi à des potentiels avec des singularités locales et à une famille d'Hamiltoniens de Stark. La preuve repose sur la théorie de Mourre et des estimations de décroissance des paquets d’ondes dans la zone classiquement interdite.

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1. INTRODUCTION

Semiclassical estimates on the resolvent of Schrödinger operators are an important technical tool for studying the behavior of observables like the scattering matrix and the total cross-section ([RT-1], [RT-2], [Y], see also [N-1] for an application to the shape resonances). In this note, we give a simple proof of these estimates for a large class of potentials. We give the details for reasonably smooth potentials and discuss the generalization in Section 4. We consider the following conditions:

CONDITION (A). V is a real valued function such that $V = V_1 + V_2$ with $V_i \in C^i(\mathbb{R}^n)$, $i = 1, 2$ and

$$
\left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{-1 - |\alpha|} \quad \text{for} \quad |\alpha| = 0, 1, 2
$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum \alpha_i$.

We will consider fixed energy $E \in \mathbb{R}$ and let $G(E) = \{ x \in \mathbb{R}^n \mid V(x) - E > 0 \}$.

CONDITION (B). There are constants $\delta, \varepsilon_0 > 0$ and a $C^3$-vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

(i) $\left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{-1 - |\alpha|} \quad \text{for} \quad |\alpha| \leq 2$

and $|\Delta (\nabla f)(x)| \leq C$;

(ii) $2(\inf_{\xi \in \mathbb{R}^n} |\xi|^2 - \langle \xi, J_f(x) \xi \rangle)(E - V(x)) - f(x) \cdot \nabla V(x) \geq \varepsilon_0$ \quad (1.1)

for any $x \in G_\varepsilon(E + \delta) = \mathbb{R}^n \setminus G(E + \delta)$, where $J_f$ is the Jacobian of $f$ and $\langle \ldots \rangle$ denotes the Euclidean inner product.

Condition (A) implies $H(h)$ is self-adjoint on $H^2(\mathbb{R}^n)$ (the Sobolev space of order 2). Our main result is:

THEOREM. Let $H(h) = -h^2 \Delta + V$ and suppose that $V$ satisfies Conditions (A) and (B) at energy $E$. Then there is an open interval $I \ni E$ such that for any $\alpha > 1/2$ and $\lambda \in I$,

$$
\lim_{\varepsilon \to 0} \langle x \rangle^{-\alpha}(H - \lambda \pm i\varepsilon)^{-1} \langle x \rangle^{-\alpha}\nexists, \quad \text{and}
$$

$$
\| \langle x \rangle^{-\alpha}(H - \lambda \pm i0)^{-1} \langle x \rangle^{-\alpha} \| \leq C h^{-1} \quad (\lambda \in I) \quad (1.2)
$$

if $h$ is sufficiently small.
This result is a key ingredient in the estimation of the semiclassical behavior of the scattering cross-section $\sigma_h(E, \omega)$, $E > 0$, $\omega \in S^{n-1}$. For potentials $V(x) = O(\langle x \rangle^{-\alpha})$, $\alpha > \frac{n+1}{2}$, and energies $E$ such that (1.1) holds on $\mathbb{R}^n$ with $f(x) = x$, the leading behavior of $\sigma_h(E, \omega)$ is $O(h^{-\gamma})$, where $\gamma \equiv (n-1)(\alpha-1)^{-1}$ (cf. [RT-2], [Y]). Using the above theorem, it should be possible to extend [RT-2] to the more general situation where $V$ satisfies Condition (A) (with possible local singularities, see Section 4) and is non-trapping in the sense of Condition (B). A similar result may hold for Stark Hamiltonians discussed in Section 4. We also remark that our methods apply to generalized N-body Schrödinger operators, although the potential $V$ does not satisfy Condition (A). The potential $V = \sum_{j=1}^{N} V_j \pi_j$, where $\{\pi_j\}_{j=1}^{N}$ is a set of mutually orthogonal projections in $\mathbb{R}^n$. We assume that each $V_j$ satisfies Condition (A) on $\pi_j(\mathbb{R}^n)$. Then, if we take $f(x) = x$ in Condition (B) and consider energies $E$ for which the resulting nontrapping condition (1.1) holds on $\mathbb{R}^n$, the analog of the above theorem holds for $H = -\hbar^2 \Delta + V$. To see this, we simply note that all the remainder terms in (3.3)-(3.5) vanish except for $(x \cdot \nabla) V$ and $(x \cdot V)^2 V$ because $\partial^2 f_j/(\partial x_j \partial x_k) = 0$. (Jensen [J] has recently obtained similar results in this case).

Our proof of this theorem is given in Sections 2-3. In Section 4, we discuss generalizations to potentials with singularities and to Stark Hamiltonians. Our method of proof utilizes the local positive commutator approach of E. Mourre ([M], [CFKS]) to obtain estimates in the nontrapping region $\mathbb{R}^n \setminus G(E + \delta)$ and semiclassical decay estimates on wave packets localized to $G(E + \delta)$ (cf. the Appendix).

Some results on semiclassical resolvent estimates are known. These first appeared in a paper by Robert and Tamura [RT-1] who consider nontrapping potential $V \in C_0^\infty(\mathbb{R}^n)$. Later, in [RT-2] they obtained semiclassical resolvent estimates at (classically) nontrapping energy $E$ for smooth potentials decaying at infinity as $\langle x \rangle^{-\rho}$, $\rho > 0$, using both Mourre theory and Fourier integral methods. We note that Condition (B) implies the classical condition of [RT-1], [RT-2]. A shorter proof of their result was given by Gérard and Martinez [GM] who constructed an escape function $a(x, \rho)$ such that the Poisson bracket $\{h, a\}$ is globally positive. Yafaev [Y] also used Mourre theory to obtain semiclassical resolvent estimates in the high energy regime for potentials $C^2$ in the $\|x\|$-variable and satisfying $|x| \left| \left( \frac{\partial}{\partial |x|} \right)^k V(x) \right| \leq C(k = 0, 1, 2)$. A method of Lavine [L] was also applied to prove estimate (1.2) for decaying potentials under nontrapping condition (1.1) with $f(x) = x[N/1]$. 

We note that the semiclassical resolvent estimate is closely related to the absence of resonances near the real axis in the semiclassical limit. Our nontrapping condition (1.2) first appeared in a proof of the absence of resonance in [N-2] (see also [BCD-1], [DeBH], [HeSj], [K], [S-1]).

2. SEMICLASSICAL MOURRE ESTIMATES

We restate the standard assumptions of the Mourre theory for a self-adjoint operator $H$ and a skew-operator $A$ in an $h$-dependent manner. For $s \geq 0$, let $\mathcal{H}_s := D((|H| + 1)^{s/2})$ with the norm $\|\psi\|_s := \|(|H| + 1)^{s/2} \psi\|$, and $\mathcal{H}_{-s} := \mathcal{H}_s^*$. $\cdot_\cdot_s$ denotes the norm of the maps from $\mathcal{H}_s$ to $\mathcal{H}_{-s}$. We let $C$ denote a $h$-independent constant whose value may change from line to line.

(M1) $D(A) \cap \mathcal{H}_2$ is dense in $\mathcal{H}_2$. 
(M2) The form $[H, A]$ defined on $D(A) \cap \mathcal{H}_2$ extends to a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_{-1}$ and $\|[H, A]\|_2, -1 \leq Ch$.
(M3) There exists a self-adjoint operator $H_0$ with $D(H_0) = D(H)$ such that $[H_0, A]$ extends to a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_{0}$, $\|[H_0, A](H_0 + i)^{-1}\| \leq C$, $\|H(H_0 + i)^{-1}\| \leq C$ and $D(A) \cap D(H_0 A)$ is a core for $H_0$.
(M4) The form $[[H, A], A]$ where $[H, A]$ is as in (M2) extends from $D(A) \cap D(HA)$ to a bounded operator from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$ and $\|[H, A], A\|_2, -2 \leq Ch$.

DEFINITION (The semiclassical Mourre estimate). — Let $g$ be a function such that $g \in C_0^\infty(\mathbb{R})$, $0 \leq g(x) \leq 1$ and $g=1$ on a neighborhood of an interval $I$. We say that the semiclassical Mourre estimate holds on $I$ if there exist such $g \in C_0^\infty(\mathbb{R})$, an operator $K(h)$ from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$ with $\|K(h)\|_{2, -2} \to 0$ as $h \to 0$ and $\alpha_0 > 0$ such that

$$M^2 := g(H)[H, A]g(H) \geq \alpha_0 h g(H)^2 + hg(H) K(h) g(H) \tag{2.1}$$

PROPOSITION 2.1. — Let $H(h)$ be a self-adjoint operator and $A(h)$ a skew-adjoint operator satisfying (M1)-(M4), and suppose the Mourre estimate (2.1) holds on $I \subset \mathbb{R}$. Then there exist $h_0 > 0$ such that for any $\alpha > 1/2$, $h \in (0, h_0)$ and $E \in I$, $\lim_{\epsilon \to 0} \langle A \rangle^{-\alpha}(H - E \pm i \epsilon)^{-1} \langle A \rangle^{-\alpha}$ exists and

$$\|\langle A \rangle^{-\alpha}(H - E \pm i 0)^{-1} \langle A \rangle^{-\alpha}\| \leq C h^{-1}. \tag{2.2}$$

Proof. — (1) We retrace the proof of Mourre as presented in [CFKS] and [PSS] keeping track of the $h$-dependence, and we refer Section 4.3 of [CFKS] for details. At first we remark that if $h$ is sufficiently small, the second term of the RHS of (2.1) is dominated by the first term, and hence it can be omitted.
For \( \varepsilon > 0 \) let \( G_\varepsilon(z) := (H - i\varepsilon M^2 - z)^{-1} \) which is analytic in \( z \) for \( \text{Re} \, z \in I \) and \( \text{Im} \, z > 0 \). Then we obtain the following estimates (cf. Lemma 4.14 of [CFKS]):

\[
\| g(H) G_\varepsilon(z) \phi \| \leq (2\varepsilon \alpha_0 h)^{-1/2} \| \phi \|, \\
\| (1 - g(H)) G_\varepsilon(z) \| \leq C(1 + \varepsilon h \| G_\varepsilon(z) \|), \\
\| G_\varepsilon(z) \| \leq C(\varepsilon \alpha_0 h)^{-1},
\]

(2.3) (2.4) (2.5)

If \( \varepsilon \) is sufficiently small. It follows in the same way as in [CFKS] that the bounds (2.3), (2.4) and (2.5) hold with \( \| \cdot \|_{0,2} \) replacing \( \| \cdot \| \).

(2) Let \( D_\varepsilon := (1 + |A|)^{-\alpha}(\varepsilon |A| + 1)^{\alpha-1} \) for \( \alpha \in (1/2, 1], \varepsilon > 0 \) and let \( F_\varepsilon(z) := D_\varepsilon G_\varepsilon(z) D_\varepsilon \) for \( z : \text{Re} \, z \in I, \text{Im} \, z > 0 \). By (2.5) and the definition of \( F_\varepsilon(z) \),

\[
\| F_\varepsilon(z) \| \leq \| D_\varepsilon \|^2 \cdot \| G_\varepsilon(z) \| \leq C(\varepsilon \alpha_0 h)^{-1}.
\]

(2.6)

From (2.3) and (2.4) with \( \phi = D_\varepsilon \psi \), we have

\[
\| G_\varepsilon D_\varepsilon \| \leq C((\alpha_0 \varepsilon h)^{-1/2} \| F_\varepsilon \|^{1/2} + 1).
\]

The derivative of \( F_\varepsilon(z) \) in \( \varepsilon \) is estimated using (2.3)-(2.6) (cf. [CFKS], Lemma 4.15), and we obtain

\[
\left\| \frac{dF_\varepsilon}{d\varepsilon} \right\| \leq C \varepsilon^{\alpha-1} (1 + (\alpha_0 \varepsilon h)^{-1/2} \| F_\varepsilon \|^{1/2} + \| F_\varepsilon \|).
\]

(2.8)

It follows from (2.6) and (2.8) that there exists \( C > 0 \) such that

\[
\limsup_{\varepsilon \downarrow 0} \sup_{\lambda \in I} \| \langle A \rangle^{-\alpha}(H - \lambda \pm i\varepsilon)\langle A \rangle^{-\alpha} \| \leq C h^{-1}
\]

(2.9)

after integrating a finite number of times ([CFKS], Proposition 4.11).

(3) By differentiating \( F_\varepsilon(z) \) in \( z \), we have

\[
\| F_\varepsilon(z) - F_\varepsilon(z') \| \leq |z - z'| \sup_z \| D_\varepsilon G_\varepsilon(z)^2 D_\varepsilon \| \leq C \varepsilon^{-1} |z - z'|
\]

(2.10)

for sufficiently small fixed \( h \). Here we used estimates (2.7) and \( \| F_\varepsilon \| \leq C \). (2.8) and (2.9) imply

\[
\| F_0(z) - F_0(z') \| \leq \| F_0(z) - F_\varepsilon(z) \| + \| F_\varepsilon(z) - F_\varepsilon(z') \| + \| F_\varepsilon(z') - F_0(z') \| \\
\leq C \varepsilon^{\alpha-1/2} + \varepsilon^{-1} |z - z'|.
\]

(2.11)

If we set \( \varepsilon = |z - z'|^\beta \) with \( \beta = (\alpha - 1/2)^{-1} \), then we obtain the Hölder continuity of order \( (\alpha - 1/2)/(\alpha + 1/2) \) for \( F_0(z) \). The existence of the limit of \( F_0(z) \) as \( \text{Im} \, z \to 0 \), \( \text{Re} \, z \in I \) follows from this. Consequently, (2.2) follows from (2.9). ■

Remark 2.2. — It follows from (M2), (M4) and Lemma 4.12 of [CFKS], i.e. that \( \| [A, g(H)]\|_{-1,1} \leq C \) in our situation, that for any \( g \in C^\infty_0(\mathbb{R}) \), \( [g(H)[H, A]g(H), A] \) extends to a bounded operator and is
O(h). As an alternative to (M4) we can take

\[ \text{(M4')} \quad \text{for any } g \in C_0^\infty(\mathbb{R}), \quad \| [g(H), A] g(H), A \| \leq C \hbar. \]

3. PROOF OF THEOREM

In this section, we prove that Conditions (A) and (B) imply that H(h) satisfies (M1)-(M4) and the semiclassical Mourre estimate for supp g sufficiently small and containing the nontrapping energy E. The conjugate operator is

\[ A := \frac{\hbar}{2} [\nabla \cdot f(x) + f(x) \cdot \nabla] \quad (3.1) \]

where \( f \) is the vector field of Condition B.

**Lemma 3.1.** Let \( H(h) := -\hbar^2 \Delta + V \) where \( V \) satisfies Conditions (A) and (B). Let \( g \in C_0^\infty(I), I \subset \mathbb{R} \) compact and \( E \in I \). Then

(i) \( A \) and \( H \) satisfy (M1)-(M4) with \( H_0 := H \) in (M3).

(ii) There exist \( \alpha_0 > 0 \) and a bounded operator \( K(h) \) with \( \| K(h) \| \to 0 \) as \( h \to 0 \) such that for sufficiently small,

\[ g(H) [H, A] g(H) \geq \alpha_0 h g(H)^2 + h g(H) K(h) g(H). \quad (3.2) \]

The operator \( K(h) \) is given explicitly in (3.8) below.

In the proof of this lemma, we use a decay result for wave packets in the classically forbidden region \( G(E) \). This result, in its optimal form due to [BCD-2], is discussed in the Appendix.

Let \( \delta \) be as in Condition (B). The function:

\[ K(x) := \inf \{ |\xi|^{-2} \langle \xi, J(x)\xi \rangle \} \]

is easily seen to be uniformly Lipschitz continuous, and let \( c_0 \) be the Lipschitz constant.

**Lemma 3.2.** Let \( K(x) \) be as above and \( \varepsilon_0 \) be as in (1.1). Then there exists \( \tilde{K}(x) \in C^\infty(\mathbb{R}^n) \) such that

(i) \( \tilde{K}(x) \leq K(x), x \in \mathbb{R}^n; \)

(ii) \( 2 \tilde{K}(x)(V(x) - E) - f(x) \cdot \nabla V(x) \geq \varepsilon_0/2, x \in G_{\varepsilon}(E + \delta). \)

**Proof.** Let \( c_\kappa \) be a mollifier: \( c_\kappa \in C_0^\infty(\{ |x| \leq \kappa \}), \int c_\kappa(x) dx = 1. \) Let \( K_\kappa := c_\kappa * K \), so \( K_\kappa \in C^\infty \). Since \( K \) is uniformly Lipschitz, it follows that

\[ K(x) - c_0 \kappa \leq K_\kappa(x) < K(x) + c_0 \kappa \]

for \( x \in G_{\varepsilon}(E + \delta). \) Set \( \tilde{K}(x) := K_\kappa(x) - c_0 \kappa \), then this proves (i). For (ii),

\[ 2 \tilde{K}(x)(V(x) - E) - f(x) \cdot \nabla V(x) \geq \varepsilon_0 - 2 c_0 \kappa (V(x) - E) \]
for $x \in G_c(E + \delta)$. If $\kappa < \varepsilon_0 (4 c_0 \sup |V(x) - E|)^{-1}$, (ii) holds. ■

**Proof of lemma 3.1.**  — (1) Since $C_0^\infty (\mathbb{R}^n)$ is a common core for $D(H)$, $D(A)$, etc., it is sufficient to prove the estimates. By a simple calculation, as a quadratic form on $C_0^\infty (\mathbb{R}^n)$:

$$[H, A] = h \left\{ 2 h^2 p J_f p - f \cdot \nabla V - \frac{h^2}{2} \Delta (\nabla \cdot f) \right\}$$  \hspace{1cm} (3.3)

where $J_f = (\frac{\partial f_j}{\partial x_i})$ is the Jacobian matrix of $f$ and $p = -i \nabla$. By Conditions (A) and (B), $\| [H, A] \|_{2,0} \leq c h$, hence (M1)-(M3) are satisfied. As for (M4), $[[H, A], A]$ as a quadratic from on $C_0^\infty (\mathbb{R}^n)$ has the form:

$$[[H, A], A] = h^2 \left\{ -2 p_i J_{ij,k} f_{k,i} + 2 p_k J_{kl,i} f_{l,ij} + 2 p_i J_{ij,k} p_k + i (f_{k,ik} J_{ij,k} - p_i J_{ij,k}) \right\}$$

$$= -[(f \cdot \nabla V), f \cdot \nabla] + \frac{h^2}{2} [(f \cdot \nabla), \Delta (\nabla \cdot f)]$$  \hspace{1cm} (3.4)

where $J_{ij,k} = \frac{\partial}{\partial x_k} (J_f)_{ij}$, $f_{i,k} = (\frac{\partial f_i}{\partial x_k})$, etc. The term $h^2 \{ \ldots \}$ is clearly uniformly bounded by $H$, and the last is also uniformly bounded by $H$. The second term is

$$- h^2 [(f \cdot \nabla V), f \cdot \nabla] = h^2 \left\{ f \cdot \nabla (f \cdot \nabla V_2) \right.$$  \hspace{1cm} (3.5)

$$+ [\nabla_p f_j (f \cdot \nabla V_1)] - (\nabla \cdot f) (f \cdot \nabla V_1) \right\}$$

$$= h^2 \left\{ f \cdot \nabla (f \cdot \nabla V_2) - (\nabla \cdot f) (f \cdot \nabla V_1) \right\}$$

$$= h\left\{ [h \nabla V], f_j (f \cdot \nabla V_1) \right\} = I_1 + I_2.$$  

Clearly, $I_1$ is $O(h^2)$, and $(H + i)^{-1} I_2 (H + i)^{-1}$ is $O(h)$ since $h \cdot \nabla J$ is uniformly $H$-bounded. Thus $\| [[H, A], A] \|_{2,2} = O(h)$.

(2) In the sense of quadratic forms, it follows from Lemma 3.2 that $p J_f p \geq p K p \geq p \bar{K} p$ and $2 \cdot p \bar{K} p = \bar{K} p^2 + p^2 \bar{K} + \Delta \bar{K}$. We obtain from (3.3):

$$[H, A] \geq h \left[ \bar{K} (H - E) (H - E) \bar{K} + 2 \bar{K} (E - V) - f \cdot \nabla V \right.$$  \hspace{1cm} (3.6)

$$+ h^2 \left\{ \Delta \bar{K} - \frac{1}{2} \Delta (\nabla \cdot f) \right\} \right].$$

Let $g \in C_0^\infty (I)$, $E \in I$ and let $\chi$ be the characteristic function of $G_c(E + \delta)$. By Lemma 3.2,

$$(2 \bar{K} (E - V) - f \cdot \nabla V) \chi \geq (\varepsilon_0 / 2) \chi.$$  

Let $\beta : = \sup_{x \in G(E+\delta)} 2 \tilde{K}(E-V) - f \cdot \nabla V$ and $\gamma = \sup_{x \in \mathbb{R}^n} |\tilde{K}(x)|$. Then for $|I|$ sufficiently small,

$$g(H)[H, A]g(H) \geq h \left( \frac{\varepsilon_0}{2} - 2\gamma |1| \right) g(H)^2$$

$$- hg(H) \left[ (1-\chi) \left( \beta + \frac{\varepsilon_0}{2} \right) + h^2 \Delta (\nabla f) - h^2 \Delta \tilde{K} \right] g(H)$$

$$\geq \frac{h \varepsilon_0}{4} g(H)^2 + hg(H) K(h) g(H) \quad (3.7)$$

where

$$K(h) := \left( \beta + \frac{\varepsilon_0}{2} \right) E_1(H)(\chi - 1) E_1(H) - \frac{h^2}{2} \Delta (\nabla f) + h^2 \Delta \tilde{K} \quad (3.8)$$

and $E_1(H)$ is the spectral projection for $H$ and $I$. By Lemma A.2, $\| E_1(H)(\chi - 1) \| = O(h^N)$ for any $N$, so we have $\| K(h) \| = O(h^2)$. This completes the proof.

**Proof of Theorem.** — By Lemma 3.1, the hypothesis of Proposition 2.1 are satisfied, so the resolvent of $H(h)$ satisifies (2.2). To pass to (1.3), we use the fact there exists a constant $C$ independent of $h$ such that

$$\| \langle x \rangle^{-\alpha} (H+i)^{-1} \langle A \rangle^{\alpha} \| \leq C \quad (3.9)$$

for $\alpha \in [0,1]$ (cf. Lemma 8.2 of [PSS]). Estimate (3.9) is proved directly for $\alpha = 1$ using the fact $|\langle x \rangle^{-1} f(x)| \leq C$ which follows from Condition (B), and extended by complex interpolation.

**Remark 3.3.** — In certain cases, a more precise propagation estimate results from (2.2) if we replace $\langle A \rangle^{-\alpha}$ by $\langle f \rangle^{-\alpha}$. This is the case when $f$ vanishes on some unbounded set.

**Remark 3.4.** — Instead of Lemma A.2, we can also apply the cut and paste technique (or so-called geometric method) to isolate the classically forbidden region. In fact, if the semiclassical resolvent estimate is proved for $H$ on $L^2(G_c(E+\delta))$, the estimate on $L^2(\mathbb{R}^n)$ follows (cf. (A.5) or [BCD-2]). Since nontrapping inequality (1.1) holds globally on $G_c(E+\delta)$, the semiclassical resolvent estimate on $L^2(G_c(E+\delta))$ can be proved by the above argument.
4. GENERALIZATIONS

A. Stark Hamiltonians

The methods developed here can be extended to a class of Stark Hamiltonians as we now indicate. In place of Condition (A) we assume

\[ V \in C^2(\mathbb{R}^n), \quad |V(x)| \leq C \langle x \rangle \quad \text{and} \quad \left| \left( \frac{\partial}{\partial x} \right)^\alpha V(x) \right| \leq C, \quad |\alpha| = 1, 2. \]

The vector field in Condition (B) must satisfy \( f \in C^4(\mathbb{R}^n), \quad |f(x)| \leq C \) and

\[ \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| \leq C \langle x \rangle^{-1} \quad \text{for} \quad |\alpha| = 1, 2, 3, 4. \]

The nontrapping condition is as in (1.1). Note that the proof of Lemma 3.2 must be improved to show that \( |K(x) - \tilde{K}(x)| \leq \kappa \langle x \rangle^{-1} \) with small \( \kappa > 0 \) using the fact that \( K(x) = 0 \langle \langle x \rangle^{-1} \rangle \). We also need the following lemma:

**Lemma 4.1.** Let \( V \in C(\mathbb{R}^n) \) and suppose that \( |V(x)| < C \langle x \rangle^\gamma \) for some \( \gamma : 0 \leq \gamma \leq 2 \). Then \( -\hbar^2 \langle x \rangle^{-\gamma} \Delta \) is relatively \( H(\hbar) \)-bounded uniformly in \( h \).

It follows from the assumptions and this lemma that \( \| [H, A](H + i)^{-1} \| = O(\hbar) \) and \( \| [[H, A], A](H + i)^{-1} \| = O(\hbar^2) \). With these modifications, one proves (M1)-(M4) and the semiclassical Mourre estimate (2.1). As a consequence, we obtain the semiclassical resolvent estimate

\[ \|(H - \varepsilon)^{-1} \|_{B(H^1, H^{-1})} \leq C \hbar^{-1} \]

where \( H^1(\mathbb{R}^n) \) is the usual Sobolev space with norm \( \| \varphi \|_{H^1}^2 = \| \varphi \|^2 + \hbar^2 \| \nabla \varphi \|^2 \). Here we used the fact that \( D(A) \rightarrow H^1(\mathbb{R}^n) \), and the inclusion map is bounded uniformly in \( h \).

B. Local Singularities

The results of Section 3 apply if \( V \) is singular in the classically forbidden region for an interval of nontrapping energies around \( E \). In this case, we require \( V \in L^p(G(E + \delta)) \) for \( \delta \) as in Condition (B), with \( p = 2 \) for \( n \leq 3 \) and \( p > n/2 \) for \( n \geq 4 \). As is easily seen from the proof, we only need \( V \) to be bounded away from \( G(E + \delta) \) so the decay estimate \( \| (1 - \chi) E_i(H) \| = O(\hbar^n) \) holds for this class of potentials.
C. Exploding potentials

We can also treat potentials of the type

\[ V \in C^2(\mathbb{R}^n), \quad \left( \frac{\partial}{\partial x} \right)^\alpha V(x) \leq C \langle x \rangle^{2-\alpha} \left| \alpha \right| \leq 2, \quad \text{and} \quad V(x) \to -\infty \]

as \( |x| \to \infty \). Again, we must take vector fields \( f \) such that \( f \in C^4(\mathbb{R}^n) \) and

\[ \left| \frac{\partial}{\partial x} \right|^\alpha f(x) \leq C \langle x \rangle^{-1-\alpha} \left| \alpha \right| \leq 4. \]

Following modifications similar to those described in Part A above, we obtain a semiclassical Mourre estimate and the result that

\[ \| (H - E \pm i\theta)^{-1} \|_{B(H^1, H^{-1})} \leq C h^{-1}. \]

APPENDIX

Decay of wave packets

The purpose of this section is to prove Lemma A.2 the result of which is used in equation (3.8). We use a perturbation idea of [BCD-2] and a simple iteration argument on the localized resolvent. Although Lemma A.2 is sufficient for our purposes, we mention a result of [BCD-2] which states that there exists \( \sigma > 0 \), where \( \sigma \) is described in terms of a distance in the Agmon metric, such that \( \| (1 - \chi) E_1(H) \| = O(e^{-\sigma/h}) \).

**Lemma A.1.** — Suppose \( F > E \) and \( \sup \| \nabla V \| = C < \infty \). Then

\[ \text{dist}(G(F), G_\epsilon(E)) \geq C^{-1} (F - E) \quad (A.1) \]

where \( G_\epsilon(E) = \mathbb{R}^n \setminus G(E) \) and \( \text{dist}(\ldots, \ldots) \) is the Euclidean distance.

**Proof.** — Let \( x \in G(F) \), \( y \in G_\epsilon(E) \), then

\[ F - E \leq V(x) - V(y) = \int_0^1 \frac{d}{dt} V(\gamma(t)) \, dt \quad (A.2) \]

for the path \( \gamma : \gamma(t) = tx + (1-t)y \). By the assumption,

\[ \text{[the RHS of (A.2)]} = \int_0^1 \frac{d}{dt} \left( \nabla V(\gamma(t)) \right) dt \leq C \int_0^1 \left| \frac{d}{dt} \right| dt = C \text{ dist}(x, y). \quad (A.3) \]

This proves the lemma. ■

We note that the assumption \( \sup \| \nabla V \| < \infty \) is necessary only on the convex hull of \( G(E) \) in order to apply the method to exploding potentials [Section 4 (C)].

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**Lemma A.2.** Suppose \( \sup |V V| < \infty \). Let \( \chi \) be the characteristic function of \( G_{c}(F) \) and \( I = [D, E] \) with \( D < E < F \). Then
\[
\| (1 - \chi) E_{i}(H) \| \leq C_{N} \cdot h^{N} \tag{A.4}
\]
for any \( N \), where \( E_{i}(H) \) is the spectral projection of \( H \).

**Proof.** Let \( \varepsilon := (F - E)/(2N + 4) \). By virtue of Lemma A.1, there exist \( C^{\infty} \)-functions \( \{ J_{j} \}_{j=1,...,N} \) such that (i) \( 0 \leq J_{j}(x) \leq 1 \); (ii) \( \sup \| V J_{j}(x) \| < \infty \); (iii) \( J_{j}(x) = 1 \) if \( x \in G(F - 2j \varepsilon) \) and \( = 0 \) if \( x \in G_{e}(F - (2j + 1) \varepsilon) \). Let \( V_{0}(x) := \max \{ V(x), E + 2 \varepsilon \} \), and let \( H_{0} = -h^{2} \Delta + V_{0}(x) \). Then \( \sigma(H_{0}) \subset [E + 2 \varepsilon, \infty) \). We have the geometric resolvent equation:
\[
J_{N} R(z) = R_{0}(z) J_{N} + R_{0}(z) M_{N} R(z) \tag{A.5}
\]
where
\[
R(z) = (H - z)^{-1}, \quad R_{0}(z) = (H_{0} - z)^{-1}, \quad \text{and} \quad M_{j} = -h^{2} \{ \nabla (\nabla J_{j}) + (\nabla J_{j}) \nabla \}.
\]
It is easy to see \( \supp M_{j} \subset G(F - (2j + 1) \varepsilon) \cap G_{e}(F - 2j \varepsilon) \), and hence \( M_{j+1} J_{j} = 0 \). Using this identity, we obtain
\[
(1 - \chi) R_{0}(z) M_{N} R(z) = (1 - \chi) J_{N-1} R_{0}(z) M_{N} R(z) = (1 - \chi) [J_{N-1}, R_{0}(z)] M_{N} R(z) = (1 - \chi) R_{0}(z) M_{N-1} R_{0}(z) M_{N} R(z) = (1 - \chi) R_{0}(z) M_{1} R_{0}(z) M_{2} \ldots R_{0}(z) M_{N} R(z). \tag{A.6}
\]
Let \( \Gamma \) be a positively oriented, simple closed around \( I \), and away from \( [E + 2 \varepsilon, \infty) \). Then, as the first term of the RHS of (A.5) is analytic in \( \Gamma \), we conclude
\[
(1 - \chi) E_{i}(H) = -\frac{1}{2 \pi i} \int_{\Gamma} (1 - \chi) J_{N} R(z) E_{i}(H) \, dz
\]
\[
= -\frac{1}{2 \pi i} \int_{\Gamma} (1 - \chi) R_{0}(z) M_{1} \ldots R_{0}(z) M_{N} R(z) E_{i}(H) \, dz. \tag{A.7}
\]
Now, since \( \| M_{j} R_{0}(z) \| = O(h) \) and \( \| R(z) E_{i}(H) \| \leq C \) on \( \Gamma \), it follows immediately from (A.7) that \( \| (1 - \chi) E_{i}(H) \| = O(h^{N}) \). 

**References**


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