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P. CARBONARO

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High frequency waves in relativistic ideal fluiddynamics

by

P. CARBONARO

Dipartimento di Matematica,
Viale A.-Doria 6, Catania, Italy

ABSTRACT. — An exhaustive analysis of the asymptotic waves in one-dimensional relativistic fluiddynamics is presented. This has been done by applying a theory, recently developed by A. Majda and R. Rosales, concerning the propagation and interaction of high frequency weakly non-linear waves The theory permits one to reduce the original system to three coupled model equations. In this paper all the coupling coefficients have been calculated and some of their interesting features have been pointed out.

RÉSUMÉ. — Nous présentons une analyse exhaustive des ondes asymptotiques dans des fluides relativistes en dimension un. Cette analyse repose sur une théorie récemment développée par A. Majda et R. Rosales qui concerne la propagation et l'intéraction d'ondes de haute fréquence faiblement non linéaires. Cette théorie permet de réduire le système original à trois équations modèles couplées. Dans cet article tous les coefficients de couplage sont calculés et nous montrons certaines propriétés intéressantes.

1. INTRODUCTION

The idea of looking for solutions of a linear differential equation which consist of a high frequency carrier modulated by a slow varying amplitude has had many applications both in physics and in mathematics. This method is generally known as geometrical optics approximation or W.K.B. method since it has been used to obtain semiclassical solutions of the Schrödinger equation.

Its generalization to systems of non-linear partial derivative equations (the so-called "Method of asymptotic waves") is due to I. Choquet-Bruhat who also gave many significant applications ([1], [2]). Then, every time that the problem consents to identify two scales of variation, one for the amplitude, the other for the oscillation, a perturbation technique reduces a non-linear system, no matter how complicated it is, to a single equation which, though more tractable, contains many of the characteristics of the original system.

In particular, the asymptotic expansion, when applied to hyperbolic quasi-linear equations, leads, at the zero order of approximation, to the equation of propagation of the wave front (the eikonal equation); at the main order, to an evolution equation of the type

$$\pi_t + a\,\pi\pi_x + b\,\pi = 0, \qquad \pi = \pi(t, x) \tag{1}$$

while all the higher order approximations give linear evolution equations ([1]). Thus the effects of non-linearity are described by the model equation (1) and the structural peculiarities of the system are contained in someway in the coefficient a, which has already been calculated for a large variety of hyperbolic systems both in classical and in relativistic contexts (*cf.* refs. [1], [2], [3] and [10]).

The limit of these approaches is that they enable one to treat only the propagation of a single wave of the system. In other words, equation (1) does not account for the possible non-linear interaction with other waves which an initial perturbation excites in general in a medium. This interaction may produce effects of the same order as the self-interaction does, on condition that some resonance relations are verified. In any case to solve a complete Cauchy problem one must give a criterion to develop the initial datum in all the "modes" of propagation, which, evidently, differs from that used for the linear equations.

I. Choquet-Bruhat has shown that this is possible for a 2×2 system of quasi-linear hyperbolic equations, in practice for off-resonance waves [4]. Recently a systematic asymptotic theory for resonantly interacting weakly non-linear waves has been developed by A. Majda and R. Rosales [5]; it includes, as a particular case, the Choquet-Bruhat's method, and also permits to take into consideration situations when more than two waves coexist and resonances may occur.

In the present paper the method expounded in reference [5] is applied to the equations of the one-dimensional relativistic fluiddynamics. This system occurs in the following conservation law form

$$\partial_\alpha \mathscr{H}^\alpha(\mathrm{U})=0; \qquad \left(\alpha=0,1;\ \partial_\alpha=\frac{\partial}{\partial x^\alpha};\ x^0=t, x^1=x\right) \tag{2}$$

where $\mathrm{U}=\mathrm{U}(\underline{x})$ is a n-component column vector, $\underline{x}$ denotes the time and spatial coordinates, $\underline{x}=(t, x)$, $\mathscr{H}^\alpha$ are analytic functions of their argument, such that admit the Taylor development around an unperturbed state, say, $\mathrm{U}=0$:

$$\mathscr{H}^\alpha(\mathrm{U})=\mathscr{H}^\alpha(0)+\mathscr{A}^\alpha.\mathrm{U}+\frac{1}{2}\mathscr{B}^\alpha(\mathrm{U},\mathrm{U})+0(|\mathrm{U}|^3) \tag{3}$$

with $\mathscr{A}^\alpha=\nabla\mathscr{H}^\alpha$ Jacobian matrices, and $\mathscr{B}^\alpha(\mathrm{U}, \mathrm{U})$ symmetric bilinear forms of the collumn vector U.

The system (2) is supposed to be strictly hyperbolic, then the matrix $(\mathscr{A}^1-\lambda\mathscr{A}^0)$ possesses n distinct eigenvalues λ_k and n corresponding independent left and right eigenvectors which will be indicated respectively with L_k and R_k, $(k=1, \ldots, n)$.

The asymptotic solution we are looking for has the form

$$\mathrm{U}(\underline{x}, \underline{\theta})=\varepsilon\sum_k \pi^k(\underline{\mathrm{X}}, \theta_k)\mathrm{R}_k+\varepsilon^2\sum_k \sigma^k(\underline{x}, \underline{\mathrm{X}})\mathrm{R}_k+\ldots \tag{4}$$

where $\varepsilon \ll 1$ is a perturbation parameter, $\underline{\mathrm{X}}=(\mathrm{T}, \mathrm{X})=(\varepsilon t, \varepsilon x)$ denotes slowly varying variables and $\underline{\theta}$ is a vector whose components are the n phases $\theta_k=x-\lambda_k t$, π^k and σ^k are respectively the first and the second order perturbation amplitudes. In the framework of the theory of A. Majda and R. Rosales the π^k turn out to satisfy the following system of coupled equations

$$\frac{\partial\pi^k}{\partial \mathrm{T}}+\lambda_k\frac{\partial\pi^k}{\partial \mathrm{X}}+\frac{\partial}{\partial\theta_k}\left[\frac{1}{2}\mathrm{D}^k_{kk}(\pi^k)^2+\sum_{i<j}\mathrm{D}^k_{ij}\lim_{\mathrm{T}\to\infty}\frac{1}{2\mathrm{T}}\right.$$
$$\left.\times\int_{-\mathrm{T}}^{\mathrm{T}}\pi^i(\underline{\mathrm{X}}, \varphi)\,\pi^j\left(\underline{\mathrm{X}}, \frac{\lambda_j-\lambda_i}{\lambda_k-\lambda_i}\theta_k+\frac{\lambda_k-\lambda_j}{\lambda_k-\lambda_i}\varphi\right)d\varphi\right]=0 \tag{5}$$

where no sum is taken over the index k, and

$$\mathrm{D}^k_{ij}=\mathrm{L}_k[\mathscr{B}^1(\mathrm{R}_i, \mathrm{R}_j)-\lambda_k\mathscr{B}^0(\mathrm{R}_i, \mathrm{R}_j)], \qquad (i, j, k=1, \ldots, n) \tag{6}$$

are symmetric (in i and j) interaction coefficients. They measure the strength of the coupling, between the i-th and the j-th wave, which can give origin to a third wave.

The plan of the paper is as follows: in section 2 the equations of the one-dimensional relativistic fluiddynamics are written in a convenient form for our purpose, in section 3 the eigenvalues and the eigenvectors of the problem are reported, in section 4 all the coefficients of non-linear interaction are presented and finally, in section 5, some comments on the coupling coefficients are made.

2. THE FIELD EQUATIONS IN MINKOWSKI SPACE

The fluid is assumed to be ideal, namely with zero viscosity and thermal conductivity, then it is described by the set of equations [1] [6]

$$\nabla_\alpha (r u^\alpha) = 0 \tag{7}$$

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{8}$$

where ∇_α denotes the covariant derivative, r is the proper matter density, $\underline{u}$ the 4-velocity ($u^\alpha u_\alpha = -c^2$), $\underline{T}$ the energy-momentum tensor defined by

$$\underline{T} = r f \underline{u} \otimes \underline{u} + p \underline{g}$$

with

$$f = 1 + h/c^2, \qquad h = e + p/r,$$

here f is the so-called "index" of the fluid [8], h is the classical enthalpy, p the pressure and e is the specific internal energy which is related to the total energy density by

$$\rho = r(c^2 + e), \tag{9}$$

$\underline{g}$ is the metric tensor.

From now on we release the covariant formalism and we carry out the calculations in a 3-vector notation. This facilitates the physical interpretation of the results and permits us to make an immediate comparison step by step with the non-relativistic fluid treated in reference [5]. The passage from the 4-vector notation to the 3-vector one is accomplished by representing a 4-vector with a column or a row depending on whether it is a contravariant vector or a covariant one, *i. e.*

$$x^\alpha = \begin{pmatrix} ct \\ \underline{x} \end{pmatrix}, \qquad x_\alpha = (-ct, \underline{x})$$

$$\nabla^\alpha = \begin{pmatrix} -c^{-1}\,\partial/\partial t \\ \nabla \end{pmatrix}, \qquad \nabla_\alpha = \left(c^{-1} \frac{\partial}{\partial t}, \nabla \right)$$

and, since we are going to consider the one-dimensional motion in Minkowski space-time, we also put

$$u^\alpha = \gamma \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix}, \qquad u_\alpha = \gamma(-c, v, 0, 0)$$

where v is the relative velocity in a given Lorentz frame along the x-direction, $\underset{\sim}{x}$ is the vector of cartesian components (x, y, z) and $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$ is the Lorentz factor.

As dependent variables we will choose r, v and s (the specific entropy), then we assume that p, e and h are smooth functions of r and s, satisfying the thermodynamic relation (6)

$$de = \mathrm{T}\, ds - p\, d(1/r) \tag{10}$$

implying

$$e_r = p/r^2, \qquad e_s = \mathrm{T}, \qquad p_s = r^2\, \mathrm{T}_r \tag{11}$$

With the above hypotheses and after some manipulations equations (7), (8), (9) and (10) read

$$\begin{aligned} & r_t + \frac{\gamma^2}{c^2} r v v_t + v r_x + \gamma^2 r v_x = 0 \\ & \frac{1}{c^2} v p_r r_t + \gamma^2 f r v_t + \frac{v}{c^2} p_s s_t + p_r r_x + \gamma^2 f r v v_x + p_s s_x = 0 \\ & h r_t + \frac{\gamma^2}{c^2} r v h v_t + r \mathrm{T} s_t + v h r_x + \gamma^2 r h v_x + r v \mathrm{T} s_x = 0 \end{aligned} \tag{12}$$

If one introduces the field vector

$$\mathrm{U} = \begin{pmatrix} r \\ v \\ s \end{pmatrix}$$

then the system (12) can be written in the following matricial form

$$\mathscr{A}^0 \mathrm{U}_t + \mathscr{A}^1 \mathrm{U}_x = 0 \tag{13}$$

with

$$\mathscr{A}^0 = \begin{pmatrix} 1 & \frac{\gamma^2}{c^2} r v & 0 \\ \frac{1}{c^2} v p_r & f \gamma^2 r & \frac{1}{c^2} v p_s \\ h & \frac{\gamma^2}{c^2} r v h & r \mathrm{T} \end{pmatrix}, \qquad \mathscr{A}^1 = \begin{pmatrix} v & \gamma^2 r & 0 \\ p_r & \gamma^2 f r v & p_s \\ v h & \gamma^2 r h & r v \mathrm{T} \end{pmatrix}$$

3. EIGENVALUES AND EIGENVECTORS OF THE PROBLEM

The roots of $D=\det(\mathscr{A}^1-\lambda\mathscr{A}^0)=0$ yield the velocities of propagation of the three waves of the system. We have

$$D=\gamma^2\left(1-\frac{\lambda v}{c^2}\right)^3 r \begin{vmatrix} v\dot{-}\lambda & 1 & 0 \\ p_r & f(v\dot{+}\lambda) & p_s \\ h(v\dot{-}\lambda) & h & rT(v\dot{-}\lambda) \end{vmatrix} \tag{14}$$

where we indicate with a dot the relativistic sum, *i. e.* $v\dot{-}\lambda=\dfrac{v-\lambda}{1-v\lambda/c^2}$.

We also define (7)

$$c_s^2=p_r/f, \tag{15}$$

then it is easily seen that the zeros of the determinant in (14) are

$$\lambda_1=v\dot{-}c_s, \qquad \lambda_2=v, \qquad \lambda_3=v\dot{+}c_s. \tag{16}$$

λ_1 and λ_3 are the sound speeds in a fluid moving with velocity v, modified according to the relativistic Doppler effect; λ_2 is the velocity of the so-called "material" wave.

For the sake of simplicity we are going to consider the propagation through the constant state

$$\underset{(0)}{U}=\begin{pmatrix} r_0 \\ 0 \\ s_0 \end{pmatrix} \tag{17}$$

then $\mathscr{A}^0$ and $\mathscr{A}^1$ take the simple form

$$\underset{(0)}{\mathscr{A}^0}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & fr & 0 \\ h & 0 & rT \end{pmatrix}_0, \qquad \underset{(0)}{\mathscr{A}^1}=\begin{pmatrix} 0 & r & 0 \\ p_r & 0 & p_s \\ 0 & rh & 0 \end{pmatrix}_0. \tag{18}$$

and

$$\underset{(0)}{(\mathscr{A}^0)^{-1}}=\frac{1}{r_0 T_0}\begin{pmatrix} rT & 0 & 0 \\ 0 & T/f & 0 \\ -h & 0 & 1 \end{pmatrix}_0, \tag{19}$$

where the subscript $_0$ indicates that all the elements of the matrices (18) and (19) are evaluated for $r=r_0$ and $s=s_0$. When the unperturbed background is the constant state (17) the eigenvalues of the problem are simply

$$\lambda_1=-c_s, \qquad \lambda_2=0, \qquad \lambda_3=c_s \tag{20}$$

with c_s evaluated in $r=r_0$ and $s=s_0$, while the correspondent right and left eigenvectors are respectively

$$\mathrm{R}_1=\begin{pmatrix} r \\ -c_s \\ 0 \end{pmatrix}_0, \qquad \mathrm{R}_2=\begin{pmatrix} p_s \\ 0 \\ -p_r \end{pmatrix}_0, \qquad \mathrm{R}_3=\begin{pmatrix} r \\ c_s \\ 0 \end{pmatrix}_0 \tag{21}$$

and

$$\begin{gathered} \mathrm{L}_1=\mathcal{N}_1(p_r\mathrm{T}-hp_s/r,\ -c_s\mathrm{T},\ p_s/r)_0 \\ \mathrm{L}_2=\mathcal{N}_2(h,\ 0,\ -1)_0 \\ \mathrm{L}_3=\mathcal{N}_3(p_r\mathrm{T}-hp_s/r,\ c_s\mathrm{T},\ p_s/r)_0 \end{gathered} \tag{22}$$

where $\mathcal{N}_i$, the normalization constants obtained by putting $\mathrm{L}_i \underset{(0)}{\mathcal{A}^0} \mathrm{R}_i=1$, are

$$(\mathcal{N}_1)^{-1}=2(\mathcal{N}_2)^{-1}=(\mathcal{N}_3)^{-1}=2(r\mathrm{T}p_r)_0. \tag{23}$$

4. THE COUPLING COEFFICIENTS

The conservation law form of the field equations (7) and (8) in the 3-vector notation is:

$$\partial_\alpha \mathcal{H}^\alpha(\mathrm{U})=0$$

with

$$\mathcal{H}^0=\begin{pmatrix} \gamma r \\ f\gamma^2 rv \\ \gamma^2\left(\rho+\dfrac{pv^2}{c^2}\right)-\gamma c^2 r \end{pmatrix}, \qquad \mathcal{H}^1=\begin{pmatrix} \gamma rv \\ f\gamma^2 rv^2+p \\ \gamma^2(\rho+p)v-\gamma c^2 rv \end{pmatrix}.$$

The divergence of $\mathcal{H}^\alpha$ is given in (18), while the vector valued bilinear forms $\mathcal{B}^0(\mathrm{V},\mathrm{W})$ and $\mathcal{B}^1(\mathrm{V},\mathrm{W})$ defined in (3) turn out to be

$$\mathcal{B}^0(\mathrm{V},\mathrm{W})=\begin{pmatrix} \dfrac{1}{c^2}r\mathrm{V}_2\mathrm{W}_2 \\ \left(1+\dfrac{h+p_r}{c^2}\right)(\mathrm{V}_2\mathrm{W}_1+\mathrm{V}_1\mathrm{W}_2) \\ \qquad +\dfrac{1}{c^2}(r\mathrm{T}+p_s)(\mathrm{V}_3\mathrm{W}_2+\mathrm{V}_2\mathrm{W}_3) \\ r^{-1}p_r\mathrm{V}_1\mathrm{W}_1+r\left(1+\dfrac{2h}{c^2}\right)\mathrm{V}_2\mathrm{W}_2+r\mathrm{T}_s\mathrm{V}_3\mathrm{W}_3 \\ \qquad +(\mathrm{T}+r\mathrm{T}_r)(\mathrm{V}_3\mathrm{W}_1+\mathrm{V}_1\mathrm{W}_3) \end{pmatrix}$$

$$\mathscr{B}^1(\mathrm{V},\mathrm{W})=\begin{pmatrix} \mathrm{V}_2\mathrm{W}_1+\mathrm{V}_1\mathrm{W}_2 \\ p_{rr}\mathrm{V}_1\mathrm{W}_1+2r\left(1+\dfrac{h}{c^2}\right)\mathrm{V}_2\mathrm{W}_2 \\ \qquad +p_{ss}\mathrm{V}_3\mathrm{W}_3+p_{rs}(\mathrm{V}_1\mathrm{W}_3+\mathrm{W}_1\mathrm{V}_3) \\ (h+p_r)(\mathrm{V}_2\mathrm{W}_1+\mathrm{V}_1\mathrm{W}_2)+(r\mathrm{T}+p_s)(\mathrm{V}_3\mathrm{W}_2+\mathrm{V}_2\mathrm{W}_3) \end{pmatrix}$$

Where V_1, V_2, V_3 and W_1, W_2, W_3 are the components respectively of the vectors V and W.

Whereupon, making use of (21) and (22), it is a simple matter to get the interaction coefficients defined in (6):

$$\mathrm{D}^1_{11}=-\frac{f^2}{2rc_s}\left\{\left(\frac{r}{f}\right)^3 h_r\right\}_r$$

$$\mathrm{D}^1_{12}=\mathrm{D}^1_{21}=\frac{c_s^3}{2}\left\{r\left[f\frac{\mathrm{T}_r}{h_r}-\frac{\mathrm{T}}{c^2}\right]\right\}_r$$

$$\mathrm{D}^1_{13}=\mathrm{D}^1_{31}=-\frac{1}{2rc_s}\left\{r\left[\frac{f}{h_r}-\frac{r}{c^2}\right]\right\}_r$$

$$\mathrm{D}^1_{22}=\frac{r\mathrm{T}c_s p_r}{2}\left\{\frac{1}{\mathrm{T}}\left[\frac{r\mathrm{T}_r^2}{h_r}-\mathrm{T}_s\right]\right\}_r$$

$$\mathrm{D}^1_{23}=\mathrm{D}^1_{32}=\frac{r^2}{2}c_s^3\left\{\frac{1}{r}\left[f\frac{\mathrm{T}_r}{h_r}-\frac{\mathrm{T}}{c^2}\right]\right\}_r$$

$$\mathrm{D}^1_{33}=-\frac{c_s\mathrm{T}^4}{2rfp_r}\left\{fh_r\frac{r^3}{\mathrm{T}^4}\right\}_r$$

As far as the coefficients of the type D^3_{ij} are concerned, one gas

$$\mathrm{D}^3_{33}=-\mathrm{D}^1_{11},\qquad \mathrm{D}^3_{23}=-\mathrm{D}^1_{21},\qquad \mathrm{D}^3_{13}=-\mathrm{D}^1_{31},$$
$$\mathrm{D}^3_{21}=-\mathrm{D}^1_{23},\qquad \mathrm{D}^3_{22}=-\mathrm{D}^1_{22},$$

while the coefficients giving the amount of material wave produced by the interactions are simply:

$$\mathrm{D}^2_{11}=-\mathrm{D}^2_{33}=\frac{2c_s}{\mathrm{T}},\qquad \mathrm{D}^2_{22}=\mathrm{D}^2_{31}=\mathrm{D}^2_{13}=0,$$
$$\mathrm{D}^2_{23}=-\mathrm{D}^2_{21}=c_s.$$

It may be useful to have alternative expressions for D^1_{11}, D^1_{13} and D^1_{33} in terms of the total energy density ρ. Because

$$p_r=\frac{p+\rho}{r}p_\rho \qquad \text{and} \qquad p_\rho=\frac{c_s^2}{c^2}\ [7],$$

we immediately obtain

$$\mathrm{D}_{11}^{1} = -\frac{c_s}{2}\frac{1}{p_\rho}\{(p+\rho)p_{\rho\rho}+2p_\rho(1-p_\rho)\},$$

$$\mathrm{D}_{13}^{1} = -\frac{c_s}{2}\frac{1}{p_\rho}\{(p+\rho)p_{\rho\rho}-2p_\rho(1-p_\rho)\},$$

and, when the temperature T is constant,

$$\mathrm{D}_{33}^{1} = -\frac{c_s}{2}\frac{1}{p_\rho}\{(p+\rho)p_{\rho\rho}+2p_\rho(1+p_\rho)\}.$$

5. THE COEFFICIENTS D_{11}^{1} AND D_{13}^{1}

The self-interaction coefficient D_{11}^{1} has already been found [1]; it vanishes when

$$p_\rho = 1 \qquad \text{and} \qquad p_{\rho\rho} = 0$$

namely when the fluid is relativistically incompressible and the pressure law is

$$p = \rho + \text{Const.}$$

This is the only solution to equation $\mathrm{D}_{11}^{1} = 0$ compatible with the conditions that an equation of state characterizing a real fluid must satisfy in order not to violate the causality principle (1 to 6):

$$p_\rho \leqq 1 \qquad \text{and} \qquad p_{\rho\rho} \geqq 0 \tag{24}$$

If the fluid is isentropic, the evolution equations (5) are decoupled when $\mathrm{D}_{13}^{1} = 0$, *i. e.*

$$\{(p+\rho)p_{\rho\rho}-2p_\rho(1-p_\rho)\} = 0.$$

The above equation admits two solutions satisfying the inequalities (24):

(i) the incompressible fluid equation of state

$$p = \rho + \text{Const.} \tag{25}$$

(ii) the parametric solution

$$\left.\begin{aligned} p(t) &= k(\sinh t - t) \\ \rho(t) &= k(\sinh t + t) \qquad (k \text{ is a constant}), \end{aligned}\right\} \tag{26}$$

from which one gets the sound velocity

$$c_s^2/c^2 = p_\rho = \mathrm{tgh}^2(t/2) < 1.$$

We notice that solution (25) is nothing more than the limit of the parametric function (26) when $t \to \infty$.

As a function of the proper matter density r, the pressure p takes the form (8)

$$p(r)=\frac{1}{2}\left\{r\left(1+\frac{r^2}{R^2}\right)^{1/2}-R\ln\left[\frac{r}{R}+\left(1+\frac{r^2}{R^2}\right)^{1/2}\right]\right\},\qquad (R=\text{Const.}) \quad (27)$$

which in the non-relativistic limit ($R \to \infty$) tends to the polytrope $p \propto r^3$ and in the ultrarelativistic limit (for $R \to 0$) tends to $p \propto r^2$.

The fact deserves mention that with the pressure law (26) or (27) it is possible to find two new field variables, in place of r and v, which allow the 2×2 original system (of the isentropic fluiddynamics) to be separated into two independent equations. As new dependent variables, the velocities of propagation of the acoustic waves may be chosen: $\lambda_1 = v \dot{-} c_s$ and $\lambda_3 = v \dot{+} c_s$ $[c_s = c_s(r) = (p_r/f)^{1/2}]$ given in (16). Then, some manipulations lead to the split system

$$\begin{aligned}\lambda_{1t}+\lambda_1\lambda_{1x}&=0\\ \lambda_{3t}+\lambda_3\lambda_{3x}&=0.\end{aligned}$$

Finally we notice that, when $T = \text{Const.}$, a comparison between the coefficients D^1_{11} and D^1_{33} shows that, at variance with the non-relativistic case, $D^1_{11} \neq D^1_{33}$. Furthermore, while $D^1_{11}=0$ admits the solution (25), the equation

$$D^1_{33} = -\frac{c_s}{2p_\rho}\{(p+\rho)p_{\rho\rho}+2p_\rho(1+p_\rho)\}=0$$

does not admit any physical solution, *i. e.* compatible with the condition (24). In the A. Majda and R. Rosales theory the coefficient D^1_{33} as well as D^1_{22} (and of course D^3_{11} and D^3_{22}) are of less importance than the others because they do not enter into the determination of the evolution equation governing the wave amplitude Π^k; they contribute only to the higher order correction of the approximated solution in (4).

6. CONCLUSIONS

Despite several applications of the asymptotic method to the relativistic fluid being available in literature ([1], [2], [3], [10]), a complete study, taking into account the possibility that many waves coexist and interact, has never been accomplished. The important paper by A. Majada and R. Rosales [5], in which a systematic theory of resonantly intaracting waves is developed, provides us with a framework to carry out an exhaustive asymptotic analysis of such a fluid in one spatial dimension. In particular, we have calculated all the possible coupling coefficients, *i. e.*

those which account for the self-interaction (already known) and those characterizing the process of mutual interaction among the different waves of the system, in our case the two countertravelling acoustic waves and the so-called "material" one.

We have pointed out that the coefficient of mutual interaction between two acoustic waves vanishes in correspondence with a non-trivial equation of state which is compatible with the causality principle and thus, probably, of physiscal interest.

Of course, the most interesting situation occurs when both the self-interaction and the mutual interaction coefficients are different from zero. In this case the integro-differential equations in (5) represent a generalization to the genuinely non-linear case of the classical three-wave resonant-interaction equations.

Future studies will be devoted to investigate the behaviour of the coupling coefficients in correspondence with the different equations of state which one encounters in relativistic context, and to take into consideration the possibility of integrating numerically the model equation we have derived.

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REFERENCES

[1] Y. Choquet-Bruhat, Ondes asymptotiques et approchées pour les systèmes d'équations aux dérivées partielles non-linéaires, *J. Maths. Pures Appl.*, T. **48**, 1969, p. 117.

[2] Y. Choquet-Bruhat, Couplage d'ondes gravitationnelles et électromagnétiques à haute fréquence in *Ondes et radiations gravitationnelles*, Institut Henry-Poincaré, Paris, 18-22 juin 1973, C.N.R.S., 1974, p. 85.

[3] A. M. Anile, Non linear waves in relativistic cosmology, *Rend. Acc. Naz. Lincei, Scienze Fisiche*, T. **63**, 1977, p. 375.

[4] Y. Choquet-Bruhat, Problème de Cauchy oscillatoire pour un système de deux équations quasi linéaires à deux inconnues, *C. R. Acad. Sci. Paris*, **268**, Série A, 1969, p. 1560.

[5] A. Majda and R. Rosales, Resonantly interacting weakly non-linear hyperbolic waves. I. A single space variable. *Stud. Appl. Math.*, T. **71**, 1984, p. 149.

[6] A. H. Taub, Relativistic fluid mechanics, *Ann. Rev. Fluid Mech.*, T. **10**, 1978, p. 301.

[7] A. Lichnerowicz, Shock waves in relativistic magnetohydrodynamics under general assumptions, *J. Math. Phys.*, T. **17**, 1976, p. 2135.

[8] P. CARBONARO, Exceptional relativistic gasdynamics, *Physics Letters A*, T. **129**, 1988, p. 372.
[9] G. BOILLAT, *La propagation des ondes*, Gauthier-Villars, Paris, 1965.
[10] A. M. ANILE, *Relativistic fluids and magnetogluids*, Cambridge University Press, Cambridge, 1989.

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