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<http://www.numdam.org/item?id=AIHPA_1989__51_4_335_0>

Curves of maximum modulus in coherent state representations

by

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ABSTRACT. — We consider functions of the form $e^{-\alpha |z|^2} f(z)$, f entire analytic (Bargmann case), and $y^\alpha f(z)$, f analytic in $y > 0$ (Bergman case). We describe curves along which the moduli of such functions attain local maxima. Such a curve determines uniquely (up to a constant factor) a function f. If the curve is closed and if f has exactly one zero in the domain bounded by the curve, then the curve is a quantized circle. In this way we obtain the quantized classical orbits of the harmonic oscillator and the hydrogen atom, their wave functions and the correct spectra. In the Bargmann case the family of curves of maximum modulus is shown to consist of all straight lines and the quantized circles, while in the Bergman case this family is found to be much richer. It is believed that this represents an interesting approach to the study of some quantum mechanical operators.

RÉSUMÉ. — Nous considérons des fonctions $e^{-\alpha |z|^2} f(z)$, f entière analytique (cas de Bargmann), et $y^\alpha f(z)$, f analytique pour $y > 0$ (cas de Bergman). Nous décrivons des courbes le long desquelles les modules de ces fonctions atteignent des maximums locaux. Une telle courbe détermine uniquement une fonction f (à un facteur constant près). Si la courbe est fermée et que f ait exactement un zéro dans le domaine borné par la courbe, alors celle-ci est un cercle quantifié. Ainsi, nous obtenons les orbites classiques quantifiées de l'oscillateur harmonique et de l'atome d'hydrogène, leurs fonctions ondulatoires et les spectres corrects. Dans le cas de Bargmann, nous montrons que la famille de courbes de module maximal consiste de toutes les droites et les cercles quatifiés, tandis que

dans le cas de Bergman, cette famille est beaucoup plus riche. On croit que cela est une façon intéressante d'aborder l'étude de certaines opérateurs de la mécanique quantique.

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1. INTRODUCTION

This paper presents a geometrical approach to the study of some fundamental operators in quantum mechanics. Actually, it is first of all a story of geometry, the quantum mechanics comes into the game as an apparently miraculous by-product. As a story of quantum mechanics this paper may be taken as an argument for the usefulness of coherent state representations.

We shall work with two representations of coherent states, the canonical case of Bargmann (*see* e. g. [3]) and the Bergman case as studied by Th. Paul [4], the latter case being related to the radial harmonic oscillator and the hydrogen atom.

The geometrical setting is as follows. Given some complex domain Ω, the class of functions f analytic in Ω and some positive weight function w which here is associated to a characteristic kernel of the domain Ω. We look for possible curves Γ (called M-curves below) in Ω such that to a curve Γ there exists an analytic function f with the property that $w(z)|f(z)|$ attains maxima along Γ.

It turns out that the family of such curves is rather restricted and we find that certain natural subfamilies give all information about eigenfunctions and spectra of fundamental quantum mechanical operators like the Hamiltonians of the harmonic oscillator and the coulomb potential problem of the hydrogen atom (*see* Theorem 4.1 and Theorem 5.2). Such subfamilies will be the quantized classical orbits of the operators in question. One should notice that in this way we obtain the orbits without leaving the framework of quantum mechanics, that is, without going to some asymptotic limit.

Before starting the discussion, let us remark that this is the first paper in a planned series of works related to the above idea. The aspects that will be explored are generalizations to higher dimensions, perturbations and further quantum mechanical interpretations. As pointed out below there are also some problems presented in the present text that will be investigated further.

2. DEFINITION OF M-CURVES

In the Bargmann case we are concerned with functions

$$S(z) = e^{-\alpha|z|^2} f(z) \tag{1}$$

where $\alpha > 0$ is some fixed number and $f(z)$ is an entire function. In the Bergman case our objects of interest are functions

$$S(z) = y^\alpha f(z), \tag{2}$$

again with some fixed $\alpha > 0$ and now $f(z)$ analytic in the upper half-plane $U = \{z = x + iy: y > 0\}$. It is often convenient to switch to the unit disk $\Delta = \{z : |z| < 1\}$ in which case we consider functions

$$S(z) = (1 - |z|^2)^\alpha f(z) \tag{3}$$

with $f(z)$ analytic in Δ. If $S(z)$ is of the form (3) then

$$\tilde{S}(z) = y^\alpha \left(\frac{2}{z+i}\right)^{2\alpha} f\left(\frac{z-i}{z+i}\right) \tag{4}$$

is of the form (2) and

$$|\tilde{S}(z)| = \left| S\left(\frac{z-i}{z+i}\right) \right|. \tag{5}$$

For our purposes this relation tells us that the two cases are equivalent and that we are allowed to move freely between them.

We shall be interested in curves of the following kind.

DEFINITION. — *A curve Γ is an M-curve of $S \neq 0$ if* $\dfrac{\partial |S|}{\partial x} = \dfrac{\partial |S|}{\partial y} = 0$
along Γ, where S is of the form (1), (2) *or* (3).

We shall often find it convenient to say that Γ is an M-curve of f when Γ by this definition is an M-curve of S and f are related by (1), (2) or (3).

It is costumary to impose some growth restriction on the functions, e. g. that functions of the form (1), (2) or (3) belong to L^2 or L^∞ over the domain in question. We find however such a restriction somewhat unnecessary since in all our examples the functions S will be bounded. These examples include all possible M-curves in the Bargmann case, in which case there is definitely no need for such a limitation. We conjecture that a function of the form (2) or (3) is bounded if it has an M-curve.

3. ELEMENTARY PROPERTIES OF M-CURVES

We start by listing some elementary facts about M-curves. This gives useful information in the Bergman case but may to some extent seem

superfluous in the Bargmann case in view of the simple characterization we shall give in the next section. We have included both cases here since it can be done at hardly any extra cost and since we by this emphasize some important common characteristics.

PROPOSITION 3.1. — *Being in either the Bargmann or the Bergman case, suppose S has an M-curve* Γ. *Then*
(i) $|S|$ *is constant along* Γ;
(ii) $|S|$ *attains local maxima along* Γ;
(iii) S *is determined up to a constant factor by any subset of* Γ *containing an accumulation point.*

Proof. — (i) is obvious since $d|S| = \dfrac{\partial |S|}{\partial x} dx + \dfrac{\partial |S|}{\partial y} dy = 0$ along Γ. We prove (ii) and (iii) for the Bargmann case. By (i) and analyticity, $S \neq 0$ along Γ. If we write $f(z) = |f(z)| e^{i \Phi(z)}$ we have by the Cauchy-Riemann equations

$$\frac{\partial \Phi}{\partial x} = -2 \alpha y - \frac{\partial \ln |S|}{\partial y} \tag{6}$$

$$\frac{\partial \Phi}{\partial y} = 2 \alpha x + \frac{\partial \ln |S|}{\partial x} \tag{7}$$

wherever $S \neq 0$. Along Γ we thus have

$$\frac{d}{dz} \ln f = 2 \alpha \bar{z}. \tag{8}$$

It will be shown later that an M-curve is differentiable (actually analytic). We can hence compute

$$\frac{d^2}{dz^2} \ln f = 2 \alpha \frac{1 - i (dy/dx)}{1 + i (dy/dx)} \tag{9}$$

$\left(\text{with an obvious modification if } \dfrac{dy}{dx} \text{ does not exist}\right)$. Since

$$\frac{d^2}{dz^2} \ln f = \frac{\partial^2 \Phi}{\partial x \, \partial y} + i \frac{\partial^2 \Phi}{\partial^2 x} \tag{10}$$

we have by (9)

$$\frac{\partial^2 \Phi}{\partial x \, \partial y} = 2 \alpha \frac{1 - (dy/dx)^2}{1 + (dy/dx)^2} \tag{11}$$

along Γ. Putting this into (6) and (7) leads to

$$\frac{\partial^2}{\partial^2 x} \ln |S| = \frac{-4\alpha}{1+(dy/dx)^2} \tag{12}$$

$$\frac{\partial^2}{\partial^2 y} \ln |S| = \frac{-4\alpha\,(dy/dx)^2}{1+(dy/dx)^2} \tag{13}$$

which in conjunction with (i) show that $|S|$ attains local maxima along Γ.

To prove (iii), let z_0 be an accumulation point of some subset $\tilde{\Gamma}$ of Γ. Then there exists a simply connected domain Ω containing z_0 such that $\dfrac{d}{dz} \ln f = \dfrac{f'(z)}{f(z)}$ is analytic in Ω, which means that $\dfrac{d}{dz} \ln f$ is the derivative of some function in Ω. By (8), $\dfrac{d}{dz} \ln f$ is given at $\tilde{\Gamma}$. Hence by the assumption on $\tilde{\Gamma}$, $\tilde{\Gamma}$ determines $\dfrac{d}{dz} \ln f$ in Ω, thus $\ln f$ up to an additive constant in Ω and f itself up to a constant factor. This completes the proof in the Bargmann case.

The proof in the Bergman case is completely analogous and is omitted. $\square$

Remark. $-$ Notice the content of (iii). Simply the set of points Γ determines S. Given Γ and the value of f at some point z_0 of Γ we find by the above proof that f can be found by the formula

$$f(z) = f(z_0)\, e^{2\,\alpha \int_{z_0}^{z} z'\,dz'} \tag{14}$$

along Γ in the Bargmann case and (*see* below)

$$f(z) = f(z_0)\, e^{i\,\alpha \int_{z_0}^{z} (1/y')\,dz'} \tag{15}$$

along Γ in the half-plane Bergman case.

Notice that (6) and (7) implies

$$\Delta \ln |S| = -4\alpha \tag{16}$$

$\ln |S|$ is hence superharmonic wherever it is defined and the only local minima of $|S|$ are the zeros. In particular, if an M-curve Γ is a closed path then S has at least one zero in the region bounded by Γ.

We shall need the analogues of (6) and (7) for the Bergman case. If we use the half-plane model then

$$\frac{\partial \Phi}{\partial x} = \frac{\alpha}{y} - \frac{\partial \ln |S|}{\partial y} \tag{17}$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \ln |S|}{\partial x}. \tag{18}$$

This gives

$$\frac{d}{dz} \ln f = \frac{i\alpha}{y} \tag{19}$$

along Γ and

$$\Delta \ln |S| = -\frac{\alpha}{y^2} \tag{20}$$

so what was just said about the Bargmann case is also valid for the Bergman case.

If we use the disk model we find

$$\frac{\partial \Phi}{\partial x} = \frac{-2\alpha y}{1 - |z|^2} - \frac{\partial \ln |S|}{\partial y} \tag{21}$$

$$\frac{\partial \Phi}{\partial y} = \frac{2\alpha x}{1 - |z|^2} + \frac{\partial \ln |S|}{\partial x} \tag{22}$$

and thus

$$\frac{d}{dz} \ln f = \frac{2\alpha \bar{z}}{1 - |z|^2} \tag{23}$$

along Γ. Let us stress the importance of (8), (19) and (23).

PROPOSITION 3.2. — *A curve Γ is an M-curve of S if and only if*
 (i) *(8) holds along Γ in the Bargmann case;*
 (ii) *(19) holds along Γ in the half-plane Bergman case;*
 (iii) *(23) holds along Γ in the disk Bergman case.*

4. M-CURVES IN THE BARGMANN CASE

It turns out that Proposition 3.2 enables us characterize all possible M-curves in the Bargmann case.

THEOREM 4.1. — *In the Bargmann case, a curve is an M-curve if and only if it is a straight line or a circle with radius $\sqrt{\dfrac{n}{2\alpha}}$, n a positive integer.*

Proof. — We let f be any entire function having an M-curve Γ. Writing

$$F(z) = \frac{1}{2\alpha} \frac{f'(z)}{f(z)} \tag{24}$$

we have

$$F(z) = \bar{z} \tag{25}$$

along Γ by Proposition 3.2. Associated to F we introduce the mero-morphic function $\bar{F}$ defined in the usual way by

$$\bar{F}(z) = \overline{F(\bar{z})}. \tag{26}$$

The function $(\bar{F} F)(z) = \bar{F}(F(z))$ is analytic in the whole complex plane except possibly at an infinite set of singular points. Since

$$(\bar{F}F)(z) = z \tag{27}$$

along Γ, $(\bar{F}F)(z) = z$ in its whole domain of analyticity. It follows that the singularities of $\bar{F}F$ are removable and that (28) is valid for all z in the extended complex plane. From (28) we see that F is a one-to-one confor-mal map of the extended complex plane onto itself, and F must thus be a Möbius transformation. By the requirement $F^{-1} = \bar{F}$ we hence have

$$F(z) = \frac{\bar{a}z + b}{cz - a} \tag{29}$$

with b, c real.

We finally check these candidates. First, if $c = 0$, (29) becomes

$$F(z) = -\frac{\bar{a}}{a} z - \frac{b}{a}. \tag{30}$$

Then $F(z) = \bar{z}$ along the line $z = iat - b/\bar{a}$, t real, and any straight line can of course be parametrized in this way. Second, if $c \neq 0$, we can assume $c = 1$ and write (29) as

$$F(z) = \bar{a} + \frac{|a|^2 + b}{z - a}. \tag{31}$$

Clearly, $|a|^2 + b$ must equal a positive integer times 2α, and we see that $F(z) = \bar{z}$ along $|z - a|^2 = |a|^2 + b$. $\square$

From the above proof we may now extract the entire functions corres-ponding to the curves in the theorem.

COROLLARY 4.2. — *The functions corresponding to the curves in Theorem 4.1 are the following. For the circles centered at a they are, modulo constant factors,*

$$f(z) = e^{2\alpha \bar{a}z}(z - a)^n. \tag{32}$$

The function associated to a straight line parametrized by $z = at + ib/\bar{a}$ (b real) is, modulo a constant factor,

$$f(z) = e^{\alpha(\bar{a}/a) z^2 + i 2\alpha(b/a) z}. \tag{33}$$

5. M-CURVES IN THE BERGMAN CASE

In the Bergman case it is useful to notice the following invariance property of M-curves.

PROPOSITION 5.1. — *In the Bergman case the set of M-curves is invariant under Möbius self-maps. If Γ is an M-curve of $f(z)$ then $T^{-1}(\Gamma)$ is an M-curve of $(T'(z))^{\alpha} f(T z)$ for any Möbius self-map T.*

Remark. — We remind the reader that a Möbius self-map T of the upper half-plane is a linear transformation of the form

$$T(z) = \frac{az+b}{cz+d} \qquad (34)$$

with a, b, c, $d \in \mathbf{R}$ and $ad - bc > 0$. A Möbius self-map T of the unit disk is a linear transformation of the form

$$T z = \frac{az+b}{bz+\bar{a}} \qquad (35)$$

with $|a|^2 - |b|^2 > 0$.

Closely related to the groups of Möbius self-maps are the Bergman kernels which we use in the below proof. The Bergman kernel of U is

$$B(z, \zeta) = -\frac{1}{\pi} \frac{1}{(z-\bar{\zeta})^2} \qquad (36)$$

and the Bergman kernel of Δ is

$$B(z, \zeta) = \frac{1}{\pi} \frac{1}{(1-z\bar{\zeta})^2}. \qquad (37)$$

For an introduction to such kernels and the associated geometry, *see* [1].

Proof. — Both in the half-plane and in the disk case our functions are of the form

$$S(z) = C \cdot B(z, z)^{-(\alpha/2)} f(z) \qquad (38)$$

for some constant C where $B(z, \zeta)$ denotes the Bergman kernel of the domain in question. The proposition follows by the elementary property of the Bergman kernel

$$B(T z, T \zeta) \, T' z \overline{T' \zeta} = B(z, \zeta), \qquad (39)$$

valid for all Möbius self-maps T of the half-plane or the disk. $\square$

Our first example of M-curves in the Bergman case is provided by the following theorem.

THEOREM 5.2. — *In the Bergman case, assume Γ is a closed path and an M-curve of f and that f has exactly one zero z_0 in the region bounded by*

Γ, *the zero being of order n. Then Γ is the circle of hyperbolic radius*
$2 \ln \left(\sqrt{\dfrac{n}{2\alpha} + 1} + \sqrt{\dfrac{n}{2\alpha}} \right)$ *with hyperbolic center z_0.*

Conversely, there are no other circles (properly contained in the domain in question) that are M-curves.

Proof. — We use the disk model, and so by Proposition 5.1 we can assume without any loss of generality that $z_0 = 0$. Then $zf'(z)/f(z)$ is analytic in the region Ω bounded by Γ and by (23)

$$z \frac{f'(z)}{f(z)} = \frac{2\alpha |z|^2}{1 - |z|^2} \qquad (40)$$

along Γ. Thus the imaginary part of the analytic function $zf'(z)/f(z)$ vanishes on the boundary of Ω and consequently, by the maximum principle, it vanishes throughout Ω. This means that $zf'(z)/f(z)$ is a constant which by assumption must equal n. By (40) $|z|^2/(1 - |z|^2) = n/2\alpha$ along Γ which proves the first part of the theorem.

The second part is trivial. For if Γ was a circle of radius R centered at 0 we would have by (23)

$$\frac{f'(z)}{f(z)} = \frac{2\alpha R^2}{1 - R^2} \frac{1}{z}. \qquad (41)$$

But for f to be analytic we must require $2\alpha R^2/(1 - R^2)$ to be a positive integer. $\square$

We see that during the above proof we have found the functions f corresponding to the circles in Theorem 5.2.

COROLLARY 5.3. — *If $z_0 = 0$ the functions associated to the circles in Theorem 5.2 are, modulo constant factors, the powers z^n.*

It is now natural to ask if there are other closed M-curves, in other words if we can find M-curves enclosing several zeros of their corresponding functions. The next theorem tells us that this is indeed possible.

THEOREM 5.4. — *In the Bergman case we can find an M-curve Γ such that its corresponding function f has two zeros in the domain bounded by Γ.*

We give the proof in the appendix since it may have an independent interest. It shows how to solve certain "free boundary value problems". Let us also remark that for this proof we will make use of the last theorem of this paper.

We find the above existence result for the Bergman case interesting but as it stands it is of course unsatisfactory. We intend to explore this case carefully in a forthcoming paper: describe the quantization, discuss the geometry of the curves and study their quantum mechanical significance. We will then also consider the case of more than two zeros.

Let us now leave the closed M-curves and look at some examples.

Straight lines as M-curves in the half plane Bergman case

Let us first treat the case of horizontal lines $y = \alpha/\omega$. Then by (19)

$$\frac{f'(z)}{f(z)} = i\,\omega \tag{42}$$

along the line and consequently

$$f(z) = C \cdot e^{i\,\omega\,z}. \tag{43}$$

Let next $\Gamma(a)$ be the straight line $x = ay$, $a \in \mathbf{R}$. Then by (19)

$$\frac{f'(z)}{f(z)} = \frac{i\alpha}{z}\,(a+i) \tag{44}$$

along $\Gamma(a)$ and thus

$$f(z) = C \cdot z^{-\alpha + i\,\alpha\,a}. \tag{45}$$

Notice that by Proposition 5.1 and Theorem 5.2 we have now treated all possible circles in the Bergman case.

To be able to generate more examples we give a recipe for checking if a given curve is an M-curve or not.

We consider now the half-plane model. Define

$$F(z) = \frac{1}{i\alpha}\,\frac{f'(z)}{f(z)}. \tag{46}$$

Let z_0 be any point of Γ such that $F'(z_0) \neq 0$. By the inverse function theorem there exists thus a neighbourhood Ω of z_0 such that F is invertible in Ω, say with inverse function F^{-1} analytic in $F(\Omega)$. Along $\Gamma \cap \Omega$, corresponding to some interval $1/t_2 < y < 1/t_1$, we have by (19)

$$z = F^{-1}\left(\frac{1}{y}\right). \tag{47}$$

The function $H(\zeta) = F^{-1}(\zeta) - i\,\dfrac{1}{\zeta}$ is analytic in some neighbourhood containing (t_1, t_2) and for $t \in (t_1, t_2)$

$$x(t) = H(t). \tag{48}$$

We have thus found that $\Gamma \cap \Omega$ admits the parametrization

$$y(t) = \frac{1}{t}, \qquad x(t) = H(t), \qquad t \in (t_1, t_2) \tag{49}$$

where $H(z)$ is analytic in $F(\Omega)$.

Γ is thus an analytic Jordan arc and in particular, the tangent is well defined at each point of Γ. A necessary and sufficient condition for $F'(z_0) \neq 0$ is seen to be that the tangent at z_0 be non-horizontal. We see that we have proved the following.

THEOREM 5.5. — *A curve Γ is an* M-*curve in the half-plane Bergman case if and only if the following holds. Γ is an analytic Jordan arc and around any point z_0 of Γ where the tangent is non-horizontal Γ admits a parametrization of the form (49). The inverse function of $i \dfrac{1}{\zeta} + \mathrm{H}(\zeta)$ can be continued meromorphically throughout* U, *the result* F (z) *being independent of z_0 and satisfying (46) for some function $f(z)$ analytic in* U.

Notice that the last requirement tells us that $F(z)$ can only have simple poles, each of residue a positive integer times $1/i\alpha$.

The assumption about a non-horizontal tangent is of course not essential. We could have assumed a non-vertical tangent and derived a similar result.

Let us now check a few families of curves with the help of the above theorem.

The half-plane Bergman case, $x = a/y$, $a \neq 0$

We apply Theorem 5.5. We find $\mathrm{H}(\zeta) = a\zeta$, and thus to find the right inverse function we must find ζ from the equation

$$z = i \frac{1}{\zeta} + a\zeta, \tag{50}$$

that is

$$\zeta(z) = \frac{1}{2a}\left(-z + \sqrt{z^2 - 4ai}\,\right). \tag{51}$$

But this function cannot be continued meromorphically throughout U since $z^2 - 4ai$ will have a zero in U for all $a \neq 0$. We conclude that the curves $x = a/y$ are not M-curves.

The half-plane Bergman case, $x = ay^2$, $a \neq 0$

Our task is now to solve

$$z = i \frac{1}{\zeta} + a \frac{1}{\zeta^2}$$

with respect to ζ, giving

$$\zeta(z) = \frac{1}{z}(i + \sqrt{4az - 1}). \tag{53}$$

Now $4az - 1$ never vanishes for $z \in U$, thus (53) can be continued analytically throughout U, and the curve is an M-curve by Theorem 5.5. We see that the square root must be in U and after some computation we find from (53) that

$$f(z) = C . z^{-(\alpha/2)} \left(\frac{\sqrt{4az - 1} + i}{\sqrt{4az - 1} - i} \right)^{\alpha/2} e^{i\alpha\sqrt{4az - 1}}. \tag{54}$$

By the same reasoning as above we find that the parabolas $x = a\sqrt{y}$, $a \neq 0$, are not M-curves and that circles are the only ellipses that can be M-curves.

6. M-CURVES AND DIFFERENTIAL OPERATORS

We shall close by very briefly pointing out the announced link between M-curves and quantum mechanical operators (more will be said about this in a forthcoming paper).

Proposition 3.2 tells us that to an M-curve Γ there exists a unique homogeneous linear differential equation of order one whose solution is the function f corresponding to Γ. The families of circles and straight lines above define likewise differential operators and the individual M-curves give their eigenfunctions and their eigenvalues.

It is of course well-known that the Hamiltonian of the harmonic oscillator takes the form $z \dfrac{d}{dz}$ in the Bargmann case, which as we have seen is the differential operator corresponding to circles centered at 0. These M-curves give hence directly the eigenfunctions and the spectrum of this Hamiltonian. Using the results in [4], one may check the following. The differential operator corresponding to circles with the same Euclidean center in the half-plane Bergman case is, for appropriate choices of α, the Hamiltonian of the Coulomb potential problem of the hydrogen atom. Circles with the same hyperbolic center, again by using [4], are found to correspond to the Hamiltonian of the radial harmonic oscillator.

7. APPENDIX

Proof of Theorem 5.4

For this proof we will use the disk model.

We have confined ourselves to discuss only the case $n=2$ but it is natural to start with an arbitrary $n>1$. That is, we assume there exists a closed M-curve Γ whose corresponding function f has exactly n zeros in the domain Ω bounded by Γ. We shall show how his leads us to an actual construction of such a curve for the case $n=2$.

By Theorem 5.2 and induction on n we may assume Ω to be simply connected, indeed a Jordan domain by Theorem 5.5. Also, by Proposition 5.1, we may assume that f has zeros at distinct points $0=w_0, w_1, \ldots, w_{n-1}$ with $0, w_1, \ldots, w_{n-1} \in \Omega$. By (23), our first task is therefore to find a meromorphic function $z\,F(z)$ in Ω with exactly $n-1$ poles, all simple and located at the points $w_1, \ldots, w_{n-1}$, such that

$$z\,F(z) = \frac{|z|^2}{1-|z|^2} \qquad (55)$$

along Γ.

We use the Riemann mapping theorem to formulate this in the unit disk Δ. To this end, let Φ be the unique univalent function with $\Phi(0)=0$, $\Phi'(0)>0$ that maps Ω onto Δ. Then since Ω is a Jordan domain, this transformation is one-to-one and continuous in the closure of Ω (see [2], p. 86). We thus seek a function $H(z)$ meromorphic in Δ with exactly $n-1$ poles, which are all simple and located at $\zeta_1=\Phi(w_1), \ldots, \zeta_{n-1}=\Phi(w_{n-1})$, and satisfying

$$H(z) = \frac{|\Phi^{-1}(z)|^2}{1-|\Phi^{-1}(z)|^2} \qquad (56)$$

along the unit circle T.

Let us now "move" all the poles to 0 by considering the function

$$G(z) = H(z) \prod_{k=1}^{n-1} (z-\zeta_k)\left(\frac{1}{z}-\bar{\zeta}_k\right) \qquad (57)$$

for which we have

$$G(z) = \frac{\left|\Phi^{-1}(z) \prod_{k=1}^{n-1} (z-\zeta_k)\right|^2}{1-|\Phi^{-1}(z)|^2} \qquad (58)$$

along T. $G(z)$ has only one pole in Δ, the pole being of order $n-1$ and located at 0. It has thus a Laurent expansion

$$G(z) = a_{-n+1} z^{-n+1} + a_{-n+2} z^{-n+2} \ldots, \tag{59}$$

but by (58) $a_{-k} = \bar{a}_k$ for all k, thus

$$G(z) = a_0 + \sum_{k=1}^{n-1} \left(a_k \frac{1}{z^k} + a_k z^k \right). \tag{60}$$

We will now use (58) and (60) to find Φ^{-1}. We start by rewriting (58) as

$$|\Phi^{-1}(z)|^2 = \frac{|P(z)|}{\displaystyle\prod_{k=1}^{n-1} |1 - \bar{\zeta}_k z|^2 + |P(z)|}. \tag{61}$$

where $P(z)$ is the polynomial

$$P(z) = z^{n-1} G(z). \tag{62}$$

By (58) $P(z)$ has no roots along T. It is easy to see that if ζ is a root of $P(z)$ then $1/\bar{\zeta}$ is also a root of $P(z)$. Similarily, the denominator of (61) may be written as $|Q(z)|$ with $Q(z)$ a polynomial with the same property as $P(z)$: If ζ is a root of $Q(z)$ then $1/\bar{\zeta}$ is also a root of $Q(z)$. Letting $z_1, \ldots, z_{n-1}$ and $\tilde{z}_1, \ldots, \tilde{z}_{n-1}$ denote the roots of $P(z)$ and $Q(z)$, respectively, lying outside of Δ, we can thus write (disregarding the possibility of degree $Q < n-1$)

$$|\Phi^{-1}(z)|^2 = |C|^2 \cdot \left| z \cdot \prod_{k=1}^{n-1} \frac{z - z_k}{z - \tilde{z}_k} \right|^2 \tag{63}$$

along T, with $|C|^2 = |a_{n-1} \tilde{z}_1 \ldots \tilde{z}_{n-1} / \bar{a}_{n-1} z_1 \ldots z_{n-1}|$. The function $\Phi^{-1}(z)/z$ is analytic and by assumption zero-free in Δ. Since then $\ln(\Phi^{-1}(z)/z)$ is analytic in Δ and we know its real part along T, $\Phi^{-1}(z)$ is determined apart from a constant factor of modulus one by (63). Thus

$$\Phi^{-1}(z) = C \cdot z \cdot \prod_{k=1}^{n-1} \frac{z - z_k}{z - \tilde{z}_k}. \tag{64}$$

Let us consider the case $n = 2$ and first carefully describe what restrictions we have.

We must have a corresponding function f analytic in Δ. This means that the residues of $H(\Phi(z))/2\alpha z$ must be positive and integer-valued. We find that

$$\text{Res}(H(\Phi(z))/z)_{z=0} = H(0) = -\bar{a}_1/\zeta \tag{65}$$

and that (putting $\zeta = \zeta_1$)

$$\mathrm{Res}\,(H\,(\Phi\,(z))/z)_{z=w_1} = \frac{(\Phi^{-1})'\,(\zeta)}{\Phi^{-1}\,(\zeta)}\,\mathrm{Res}\,(H\,(z))_{z=\zeta} \tag{66}$$

where

$$\mathrm{Res}\,(H\,(z))_{z=\zeta} = (a_1\,\zeta^2 + a_0\,\zeta + \bar{a}_1)/(1 - |\zeta|^2). \tag{67}$$

We notice that

$$\frac{(\Phi^{-1})'\,(\zeta)}{\Phi^{-1}\,(\zeta)} = \frac{1}{\zeta} + \frac{z_1 - \tilde{z}_1}{(\zeta - z_1)\,(\zeta - \tilde{z}_1)}. \tag{68}$$

Next, the necessary and sufficient condition for univalence is found to be

$$\inf_{|z|=1}\left| \tilde{z}_1\,\frac{z - z_1}{z - \tilde{z}_1} \right| \geq 1. \tag{69}$$

Furthermore, to insure that $H\,(z)$ be positive along T we must require

$$a_0 > 2\,|a_1|. \tag{70}$$

Finally we find

$$\Phi\,(z) = \frac{1}{2\,C}\,(C\,z_1 + z + \sqrt{(C\,z_1 + z)^2 - 4\,C\,\tilde{z}_1\,z}). \tag{71}$$

It remains to be decided if the polynomial

$$R\,(z) = z^2 + 2\,C\,(z_1 - 2\,\tilde{z}_1)\,z + C^2\,z_1^2 \tag{72}$$

can have both its roots outside of Δ while at the same time the residues (65), (66) and (67) are of the right kind and (69) and (70) are both satisfied.

We show that this is possible for convenient choices of the parameters. We go "backwards", showing how one may choose the parameters a_0, a_1 and ζ.

We let $\varepsilon < 1$, M, N be three positive numbers. We put

$$\zeta = \varepsilon, \qquad a_0 = N, \qquad a_1 = -M\,\varepsilon. \tag{73}$$

We see that by this the residue in (65) equals M and so M must be a positive integer times $2\,\alpha$. We notice also that

$$\mathrm{Res}\,(H\,(z))_{z=\zeta} = \frac{N\,\varepsilon - M\,\varepsilon\,(1 + \varepsilon^2)}{1 - \varepsilon^2}. \tag{74}$$

Writing $Q\,(z) = \tilde{a}_1\,z^2 + \tilde{a}_0\,z + \tilde{a}_1$ we find

$$\tilde{a}_0 = a_0 + 1 + \varepsilon^2, \tag{75}$$
$$\tilde{a}_1 = -\varepsilon\,(M + 1). \tag{76}$$