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The Tomita Operator for the free scalar field

by

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ABSTRACT. — An explicit formula for the Tomita operator Δ associated with local algebras of the free scalar field is given, using the second quantization structure of the free fields and the time zero formulation. We use this formula to prove the strong continuity of Δ_m^{it} when m varies in $[0, +\infty)$.

RÉSUMÉ. — En utilisant la structure de seconde quantification et la formulation à temps zéro, nous donnons une formule explicite pour l'opérateur de Tomita Δ associé aux algèbres locales du champ libre. Nous utilisons cette formule pour démontrer la continuité forte de Δ_m^{it} quand m varie dans $[0, +\infty]$.

INTRODUCTION

We present here an explicit formula for the modular operator associated with the local algebras of the free scalar field.

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As it is well known, in the early seventies the Tomita Takesaki theory and a work of Haag, Hugenoltz and Winnik [7] focussed the attention of mathematical physicists on the modular group associated with the algebra of observables of a thermodynamic system and with an equilibrium state. In fact, it was observed that KMS condition implies that the modular group is the time evolution group, and hence $\log \Delta$ is the energy of the system.

Thus the possibility of interpreting the operator $\log \Delta$ associated with an algebra of local observables (and with the vacuum) as a “local energy” emerged, and the problem of the existence of local dynamics, that is a one parameter group of diffeomorphisms of an open set implementing the modular group, was posed.

Bisognano and Wichmann [3] gave an explicit answer for the so called wedge regions, in which pure Lorentz transformations correspond to the modular group. This has a very interesting interpretation due to Sewell [15]. In fact a uniformly accelerated observer looks at the wedge boundary as his horizon, and regards the vacuum as a thermal state (Hawking effect).

The results of Bisognano and Wichmann hold whenever there is a suitable underlying Wightman theory but only for wedge regions, while in a special case, the free massless scalar field, Hislop and Longo [8] succeeded in obtaining a group of diffeomorphisms implementing the modular group for an important class of bounded regions, the double cones with spherical basis, exploiting the conformal symmetry of the theory.

Unfortunately this technique does not work for the massive case. In fact in this case the only global covariant transformations are Poincaré transformations, and a theorem of Zeeman [16] shows that every local transformation which preserves causality can be extended to a global one.

The objective of our work was to try to obtain an explicit formula for Δ instead of the modular group.

We have used the known fact [5] that Δ for a local algebra of the free field is a second quantization operator. Hence the results of Araki [1] and Roberts, Leyland and Testard [12] are applicable. Our notation in this paper is similar to the one in the quoted papers.

As an application of our formula for Δ , we give a proof of the strong continuity of Δ_m^{it} when $m \in [0, +\infty)$.

In fact, although we could easily prove the continuity for $m > 0$ at the second quantization level because the algebra of a region \mathcal{O} with mass m is equivalent to the algebra of a shrunk region with a suitably scaled mass and because the dilation group acts continuously, the continuity for $m \rightarrow 0$ seems to be a less trivial result.

In our proof we use the first quantization language and the explicit form for δ to demonstrate continuity at zero for bounded regions.

1. THE SECOND QUANTIZATION STRUCTURE

In this section we will review some basic notions of the second quantization structure.

A second quantization structure is given by a Hilbert space \mathcal{H} , the so called 1-particle space, and the symmetric Fock space over it:

$$e^{\mathcal{H}} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$$

In $e^{\mathcal{H}}$ a total set is the set of coherent vectors:

$$\left\{ e^h = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes n}, h \in \mathcal{H} \right\}$$

There is also a map $\mathcal{H} \ni h \rightarrow W(h)$, where the $W(h)$'s are the well known Weyl unitaries acting on $e^{\mathcal{H}}$, completely determined by:

(i) CANONICAL COMMUTATION RELATIONS (CCR):

$$W(h)W(k) = \exp\left(-\frac{1}{2}i \operatorname{Im}(h, k)\right)W(h+k); \quad h, k \in \mathcal{H}$$

$$(ii) \quad W(h)e^0 = \exp\left(-\frac{1}{4}\|h\|^2\right)e^{ih/\sqrt{2}}, \quad h \in \mathcal{H}.$$

The vector $\Omega = e^0 \in e^{\mathcal{H}}$ is called the vacuum.

Corresponding to the exponentiation of a Hilbert space it is possible to define an exponentiation of operators. Let A be a closed densely defined linear operator on \mathcal{H} with domain $D(A)$. Then

$$e^A: e^{\mathcal{H}} \rightarrow e^{\mathcal{H}}$$

is the closure of the linear operator acting on linear combinations of coherent vectors with exponent in $D(A)$ such that:

$$e^A e^h = e^{A h}.$$

This exponentiation preserves selfadjointness, positivity, unitarity but not, for example, boundedness⁽¹⁾.

⁽¹⁾ The map from the category of Hilbert spaces with arrows the linear densely defined closed operators to itself, defined by $\mathcal{H} \rightarrow e^{\mathcal{H}}$, $A \rightarrow e^A$ is the so called "second quantization functor" (see [13]).

With each real closed subspace K of \mathcal{H} (we will write $K \leq_{\mathbb{R}} \mathcal{H}$) we associate a von Neumann algebra in $B(e^{\mathcal{H}})$:

$$\mathfrak{R}(K) = \{ W(h), h \in K \}''$$

This algebra is called the *second quantization algebra* of K .

(1.1) DEFINITION. — We will say that $K \leq_{\mathbb{R}} \mathcal{H}$ is standard if the following conditions are fulfilled:

- (i) $K + iK$ is dense in \mathcal{H}
- (ii) $K \cap iK = \{0\}$.

The name standard for such a property is due to the following:

(1.2) THEOREM [1]. — e^0 is cyclic and separating for $\mathfrak{R}(K)$ [i. e. $\mathfrak{R}(K)$ is in standard form with respect to the vacuum] if and only if K is standard.

It is well known that with each von Neumann algebra \mathfrak{R} in standard form w. r. t. the vector Ω , there are associated the Tomita operators S, J, Δ , where S is the closure of the operator S_0 such that:

$$(1.3) \quad \begin{cases} S_0: \mathfrak{R}\Omega \rightarrow \mathfrak{R}\Omega \\ S_0 A \Omega = A^* \Omega \end{cases}$$

and $S = J\Delta^{1/2}$ is its polar decomposition.

One of the most important facts in the Tomita-Takesaki theory applied to free Bose field theory is:

(1.4) THEOREM [5]. — If \mathfrak{R} is the second quantization algebra of $K \leq_{\mathbb{R}} \mathcal{H}$, and K is standard, then the operator S is the second quantization of the closed, densely defined, conjugate linear operator s over \mathcal{H} defined as:

$$(1.5) \quad \begin{cases} s: K + iK \rightarrow K + iK \\ h + ik \rightarrow h - ik \end{cases}$$

Moreover if $s = j\delta^{1/2}$ is the polar decomposition of s , then

$$J = e^j, \quad \Delta = e^\delta.$$

It is well known that the time zero formulation of the free scalar field with mass m in the vacuum representation is given by the net of local algebras

$$\mathbb{R}^3 \ni \mathcal{O} \rightarrow \mathfrak{R}(\mathcal{O}) \subset B(e^{\mathcal{H}})$$

where $\mathcal{H} = L^2(\mathbb{R}^3)$, $\mathfrak{R}(\mathcal{O})$ is the second quantization algebra corresponding to

$$(1.6) \quad K_m(\mathcal{O}) = \{ \omega_m^{-1/2} f - i \omega_m^{1/2} g, f, g \in \mathcal{D}_{\mathbb{R}}(\mathcal{O}) \}^-$$

and

$$(1.7) \quad (\omega_m f)^\wedge(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2} \hat{f}(\mathbf{p})$$

2. THE LOCAL SUBSPACES

With each $m \geq 0$ we can associate the Sobolev spaces:

(2.1) $H_m^\alpha = H_m^\alpha(\mathbb{R}^3)$, which are the completion of $D(\omega_m^\alpha) \subset L^2(\mathbb{R}^3)$ with respect to the norm: $\|x\|_{\alpha, m} = \|\omega_m^\alpha x\|$.

Then the following duality relation holds

$$(H_m^\alpha)^* = H_m^{-\alpha}, \quad \forall \alpha \in \mathbb{R},$$

where the (sesquilinear) pairing is an extension of the scalar product in $L^2(\mathbb{R}^3)$.

We are interested in "local Sobolev spaces". For each region \mathcal{O} contained in \mathbb{R}^3 , we define (2):

$$(2.2) \quad H_m^\alpha(\mathcal{O}) = \overline{L^2(\mathcal{O}) \cap D(\omega_m^\alpha)}^{\|\cdot\|_{\alpha, m}}$$

In fact, the local subspace $K_m(\mathcal{O})$ can be written in terms of these spaces:

$$K_m(\mathcal{O}) = \omega_m^{-1/2} H_{m, \mathbb{R}}^{-1/2}(\mathcal{O}) + i \omega_m^{1/2} H_{m, \mathbb{R}}^{1/2}(\mathcal{O}).$$

We want to investigate the nature of the spaces $H_m^\alpha(\mathcal{O})$ and $H_m^\alpha(\mathbb{R}^3)$. If $m > 0$, we note that, while the definition of $H_m^\alpha(\mathbb{R}^3)$ is the usual definition of a Sobolev space with fractional index, the definition of local Sobolev spaces is no longer the usual one. There, the norm is an infimum taken all over the extensions of a function defined in \mathcal{O} , while we take the norm of the extension to zero. Unfortunately these two definitions do not always coincide, and in particular not in the case in which we are interested, *i. e.*

$\alpha = \pm \frac{1}{2}$, since the extension to zero is not continuous in the usual norm (see e. g. [10]).

In the case $m = 0$, the situation is a bit more complicated, because $H_0^\alpha(\mathbb{R}^3)$ is different from $H_m^\alpha(\mathbb{R}^3)$, when $m > 0$, $\alpha \neq 0$.

However it is possible to prove the following:

(2.3) PROPOSITION. — *If the region \mathcal{O} in \mathbb{R}^3 is bounded, and $\alpha > -3/2$, then:*

$$H_m^\alpha(\mathcal{O}) \cong H_0^\alpha(\mathcal{O}),$$

that is they are the same vector space with equivalent norms.

For the proof see the appendix (A. 2).

Now we state some properties of the local spaces that we need later. They hold for each $m \geq 0$ and thus we drop the subscript m in the following.

(2) From our definition it follows that there is no difference between $H_m^\alpha(\mathcal{O})$ and $H_m^\alpha(\bar{\mathcal{O}})$.

(2.4) PROPOSITION. — For each region \mathcal{O} with C^1 boundary, $|\alpha| < 3/2$,

- (a) $(H^\alpha(\mathcal{O}))^0 = H^{-\alpha}(\mathcal{O}^c)$.
- (b) $\mathcal{D}_\sigma := H^{\sigma/2}(\mathcal{O}) + H^{\sigma/2}(\mathcal{O}^c)$ is dense in $H^{\sigma/2}(\mathbb{R}^3)$ if $\sigma = \pm 1$.
- (c) ω is antilocal, i. e. $\text{supp } f \subset \mathcal{O}, \text{supp } (\omega f) \subset \mathcal{O} \Rightarrow f \equiv 0$.

The property (c) is standard, and can be found in [14] for the case $m > 0$, in [2] for the case $m = 0$. The properties (a) and (b) are more or less well known, but we prove them in the appendix for the sake of completeness.

It is easy to see that if we define

$$(2.5) \quad K_{\sigma/2}(\mathcal{O}) = \omega^{\sigma/2} H^{\sigma/2}(\mathcal{O}),$$

then the following properties hold:

(2.6) Remark:

- (1) $K(\mathcal{O}) = \text{Re } K_{-1/2}(\mathcal{O}) \oplus_{\mathbb{R}} i \text{Re } K_{+1/2}(\mathcal{O})$.
- (2) $(H^{1/2}(\mathcal{O}))^0 = H^{-1/2}(\mathcal{O}^c) \Rightarrow (K_{\sigma/2}(\mathcal{O}))^\perp = K_{-\sigma/2}(\mathcal{O}^c)$
- (3) $K_{\sigma/2}(\mathcal{O}) \cap K_{\sigma/2}(\mathcal{O}^c) = \omega^{\sigma/2} (H^{\sigma/2}(\mathcal{O}) \cap H^{\sigma/2}(\mathcal{O}^c)) = \{0\}$.

We give here a simple proof on the known property [1] that local subspaces are standard. This property, by Theorem (1.2), is equivalent to the Reeh-Schlieder Theorem, which states that the vacuum is cyclic and separating for local algebras.

(2.7) PROPOSITION. — If \mathcal{O} has C^1 boundary and $\mathcal{O} \neq \emptyset, \mathcal{O}^c \neq \emptyset, K(\mathcal{O})$ is standard in $L^2(\mathbb{R}^3)$.

Proof. — First we prove that $K_{+1/2}(\mathcal{O}) \cap K_{-1/2}(\mathcal{O}) = \{0\}$.

In fact, by antilocality, $\omega H^{1/2}(\mathcal{O}) \cap H^{-1/2}(\mathcal{O}) = \{0\}$, and this implies that $K_{+1/2}(\mathcal{O}) \cap K_{-1/2}(\mathcal{O}) = \{0\}$. Then, by taking the orthogonal complement and by the remark (2.6) (2), it is straightforward to prove that $K_{+1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O})$ is dense in \mathcal{H} .

Now we use these two properties of $K_{+1/2}(\mathcal{O})$ and $K_{-1/2}(\mathcal{O})$ to prove the standard property for $K(\mathcal{O})$.

$K(\mathcal{O}) \cap iK(\mathcal{O}) \neq \{0\}$ iff there are $h_\sigma, k_\sigma \in \text{Re } K_{\sigma/2}(\mathcal{O})$ such that $h_+ + ih_- = ik_+ - k_-$. But this is equivalent to $h_+ - ik_+ = -(k_- + ih_-)$, and these latter objects are in $K_{+1/2}(\mathcal{O}), K_{-1/2}(\mathcal{O})$ respectively. Hence $K(\mathcal{O}) \cap iK(\mathcal{O}) = \{0\}$ iff $K_{+1/2}(\mathcal{O}) \cap K_{-1/2}(\mathcal{O}) = \{0\}$.

Moreover

$$K(\mathcal{O}) + iK(\mathcal{O}) = \text{Re } K_{-1/2}(\mathcal{O}) + i \text{Re } K_{+1/2}(\mathcal{O}) + i \text{Re } K_{-1/2}(\mathcal{O}) + \text{Re } K_{+1/2}(\mathcal{O}) = K_{+1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O}). \quad \square$$

Now let $\chi_\mathcal{O}$ be the characteristic function of the region \mathcal{O} . The multiplication operator by this function, which we also denote by $\chi_\mathcal{O}$, is clearly a projection in $\mathcal{H} = L^2(\mathbb{R}^3)$. We want to define an “extension” of $\chi_\mathcal{O}$ to

$H^{\sigma/2}(\mathbb{R}^3)$. For this we pose:

$$(2.8) \quad \begin{cases} P_{\sigma}: \mathcal{D}_{\sigma} \subset H^{\sigma/2}(\mathbb{R}^3) \rightarrow H^{\sigma/2}(\mathbb{R}^3) \\ P_{\sigma} \uparrow_{H^{\sigma/2}(\mathcal{O})} = I, \quad P_{\sigma} \uparrow_{H^{\sigma/2}(\mathcal{O}^c)} = 0. \end{cases}$$

Because $H^{\sigma/2}(\mathcal{O}) \cap H^{\sigma/2}(\mathcal{O}^c) = \{0\}$ and $H^{\sigma/2}(\mathcal{O}) + H^{\sigma/2}(\mathcal{O}^c)$ is dense, P_{σ} is a well defined, densely defined operator, and it is obviously closed and idempotent.

We conclude this section defining the operator A_{σ} , which we will use in our description of δ .

(2.9) DEFINITION:

$$A_{\sigma}: \mathcal{F}_{\sigma} \subset H^{\sigma/2}(\mathcal{O}) \rightarrow H^{-\sigma/2}(\mathcal{O}), \quad \mathcal{F}_{\sigma} = \omega^{-\sigma} \mathcal{D}_{-\sigma} \cap H^{\sigma/2}(\mathcal{O}), \quad A_{\sigma} = P_{-\sigma} \omega^{\sigma} \uparrow_{H^{\sigma/2}(\mathcal{O})}.$$

(2.10) THEOREM. — A_{σ} is a densely defined closed operator and

$$(A_{\sigma})^* = A_{-\sigma}.$$

To prove the theorem we need some Lemmas:

(2.11) LEMMA. — P_{+1} is the transpose of P_{-1} .

Proof. — Let $x \in D({}^tP_{+1})$, $y = y_+ + y_- \in \mathcal{D}_{-1}$ ($y_{\sigma} \in H^{\sigma/2}(\mathcal{O}^c)$), then:

$$\begin{aligned} \langle {}^tP_{+1} x, y_- \rangle &= \langle x, P_{+1} y_- \rangle = 0 \Rightarrow {}^tP_{+1} x \in (H^{1/2}(\mathcal{O}^c))^0 = H^{-1/2}(\mathcal{O}) \\ \langle (I - {}^tP_{+1}) x, y_+ \rangle &= \langle x, (I - P_{+1}) y_+ \rangle = 0 \\ &\Rightarrow (I - {}^tP_{+1}) x \in (H^{1/2}(\mathcal{O}))^0 = H^{-1/2}(\mathcal{O}^c) \end{aligned}$$

then $x \in H^{-1/2}(\mathcal{O}) + H^{-1/2}(\mathcal{O}^c) \Rightarrow D({}^tP_{+1}) \subset \mathcal{D}_{-1}$.

Furthermore if $x \in \mathcal{D}_{-1}$ and $y \in \mathcal{D}_{+1}$ we have:

$$\langle x, P_{+1} y \rangle = \langle x, y_+ \rangle = \langle x_+, y \rangle = \langle P_{-1} x, y \rangle,$$

i. e. $P_{-1} = {}^tP_{+1} \uparrow_{\mathcal{D}_{-1}}$, and from this the result follows. \square

The proof of the two following Lemmas is based on a straightforward calculation which we omit.

(2.12) LEMMA. — If $T: H^{1/2}(\mathbb{R}^3) \rightarrow H^{1/2}(\mathbb{R}^3)$ is a densely defined closed operator, then $D(T^*) = \omega^{-1} D({}^tT)$ and $T^* = \omega^{-1} ({}^tT) \omega$.

(2.13) REMARK:

$$\begin{aligned} (1) \quad (P_{\sigma})^* &= \omega^{-\sigma} P_{-\sigma} \omega^{\sigma} \quad \text{and} \quad D((P_{\sigma})^*) = \omega^{-\sigma} \mathcal{D}_{-\sigma} \\ (2) \quad (P_{\sigma} \omega^{-\sigma})^* &= (\omega^{-\sigma})^* (P_{\sigma})^* = P_{-\sigma} \omega^{\sigma} \quad \text{and} \quad D(P_{\sigma} \omega^{-\sigma}) = \omega^{\sigma} \mathcal{D}_{\sigma}. \end{aligned}$$

(2.14) LEMMA. — Let \mathcal{H}, \mathcal{K} be Hilbert spaces, $T: \mathcal{H} \rightarrow \mathcal{K}$ a densely defined closed operator, M and N subspaces of \mathcal{H} and \mathcal{K} respectively. Suppose also that $T\mathcal{H} \subset N$ and $D(T) \cap M$ is dense in M . Then $T \uparrow_M: M \rightarrow N$ is closed and $(T \uparrow_M)^* = [M] T^* \uparrow_N$, $D(T \uparrow_M)^* = D(T^*) \cap N$, where we denote by $[M]$ the projection on M in \mathcal{H} .

Now we can prove theorem 2. 10:

Proof (2. 10). — The fact that A_σ is closed follows observing that $\omega^\sigma \uparrow_{H^{\sigma/2}(\mathcal{O})}$ is an unitary operator from $H^{\sigma/2}(\mathcal{O})$ into its range, and that $P_{-\sigma}$ is closed. At this point we have only to prove that \mathcal{F}_σ is dense in $H^{\sigma/2}(\mathcal{O})$, and then to use the preceding lemmas. But we have the following “if and only if chain”:

\mathcal{F}_σ is dense in $H^{\sigma/2}(\mathcal{O})$ iff $\omega^{-\sigma} \mathcal{D}_{-\sigma}$ is dense in $H^{\sigma/2}(\mathcal{O})$ iff $(K_{-\sigma/2}(\mathcal{O}) + K_{-\sigma/2}(\mathcal{O}^c)) \cap K_{\sigma/2}(\mathcal{O})$ is dense in $K_{\sigma/2}(\mathcal{O})$ iff $QK_{-\sigma/2}(\mathcal{O})$ is dense in $K_{\sigma/2}(\mathcal{O})$, where $Q = [K_{\sigma/2}(\mathcal{O})]$.

The last property is an immediate consequence of these considerations: $y \in K_{\sigma/2}(\mathcal{O})$ is such that $(y, Qx)_0 = 0, \forall x \in K_{-\sigma/2}(\mathcal{O})$ iff $(Qy, x)_0 = 0, \forall x \in K_{-\sigma/2}(\mathcal{O})$ iff $Qy \in (K_{-\sigma/2}(\mathcal{O}))^\perp = K_{\sigma/2}(\mathcal{O}^c)$ but $y \in K_{\sigma/2}(\mathcal{O})$, and so $y = Qy \in K_{\sigma/2}(\mathcal{O}) \cap K_{\sigma/2}(\mathcal{O}^c) = \{0\}$. \square

3. THE TOMITA OPERATOR

Let s be the Tomita operator defined by (1. 5) and associated with the standard space $K(\mathcal{O})$. It is worth noticing that:

(3. 1) PROPOSITION. — *The domain of s , $D(s)$, equipped with the graph norm $\|\cdot\|_{g(s)}$ of s is isometrically isomorphic to $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ via the unitary operator:*

$$(3. 2) \quad T = \frac{1}{\sqrt{2}} (\omega^{-1/2} \oplus -i\omega^{1/2})$$

Proof. — First we observe that $D(s) = K_{+1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O})$ and that this decomposition is orthogonal in the graph norm:

$$\begin{aligned} \forall k_+ \in K_{+1/2}(\mathcal{O}), \quad k_- \in K_{-1/2}(\mathcal{O}), \\ (k_+, k_-)_{g(s)} = (k_+, k_-)_0 + (sk_-, sk_+)_0 \end{aligned}$$

but $k_\sigma = h_\sigma + il_\sigma$ with $h_\sigma, l_\sigma \in \text{Re } K_{\sigma/2}(\mathcal{O})$ and $sl_- = l_-, sh_- = h_-, s(ih_+) = ih_+, s(il_+) = il_+$, hence:

$$\begin{aligned} (k_+, k_-)_{g(s)} &= (k_+, k_-)_0 + (s(h_- + il_-), s(h_+ + il_+))_0 = \\ &= (k_+, k_-)_0 + (h_- - il_-, -h_+ + il_+)_0 = 0. \end{aligned}$$

We have still to prove that:

$$2\omega^{\sigma/2}: H^{\sigma/2}(\mathcal{O}) \rightarrow (D(s), \|\cdot\|_{g(s)})$$

is an isometry for $\sigma = \pm 1$. For this let: $h + ik \in H^{\sigma/2}(\mathcal{O})$, with h, k “real” ($\Rightarrow (h, k)_\sigma \in \mathbb{R}$), then:

$$\begin{aligned} \frac{1}{2} (\|\omega^{\sigma/2}(h + ik)\|_{g(s)})^2 &= \frac{1}{2} (\|h\|_\sigma^2 + \|k\|_\sigma^2 - 2\text{Im}(h, k)_\sigma + \\ &+ \|h\|_\sigma^2 + \|k\|_\sigma^2 + 2\text{Im}(h, k)_\sigma) = \|h\|_\sigma^2 + \|k\|_\sigma^2 = \|h + ik\|_\sigma^2. \end{aligned}$$

From the fact that $K_\sigma(\mathcal{O}) = \omega^{\sigma/2} H^{\sigma/2}(\mathcal{O})$ we obtain the result. \square

Via the isomorphism in (3.2), the vectors in $D(s)$ which are invariant for s become the invariant vectors for the operator

$$\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}: H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}) \rightarrow H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$$

therefore in this representation we can identify s with that matrix.

Furthermore we note that if we introduce in $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ the quadratic form defined by the operator:

$$(3.3) \quad N = \frac{1}{2} \begin{pmatrix} 1 & -i\omega \\ i\omega^{-1} & 1 \end{pmatrix},$$

the map:

$$T: (H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}), (\cdot, N \cdot)_0) \rightarrow \mathcal{H}$$

results an isometry with dense range, hence we can identify \mathcal{H} with the completion of $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ with respect to the scalar product defined by N , and the operators $s, \delta = s^*s$ with their conjugation by this unitary.

Because s is easily defined on vectors in $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$, and the same holds for s^* w. r. t. vectors in $H^{-1/2}(\mathcal{O}^c) \oplus H^{1/2}(\mathcal{O}^c)$, the problem of an explicit description of $\delta = s^*s$ is to pass from one representation to the other. More precisely we need an expression for T_σ^{-1} and $T_{\sigma^c}^{-1}$. Then,

$$\delta = s^*s = T_\sigma \circ T_{\sigma^c}^{-1} \begin{pmatrix} C & O \\ O & C \end{pmatrix} T_{\sigma^c}^{-1} \circ T_\sigma \begin{pmatrix} C & O \\ O & C \end{pmatrix}.$$

Formally, $T_\sigma^{-1}h = \sqrt{2} \begin{pmatrix} (\chi_{\sigma^c} \omega \chi_\sigma)^{-1} \chi_{\sigma^c} \omega^{-1/2} h \\ i(\chi_{\sigma^c} \omega \chi_\sigma)^{-1} \chi_{\sigma^c} \omega^{+1/2} h \end{pmatrix}$ where $\chi_{\sigma^c} \omega \chi_\sigma$ is invertible thanks to the antilocality property of ω . A similar expression holds for $T_{\sigma^c}^{-1}$.

We can now observe that, if $B = \begin{pmatrix} 0 & iA_{+1} \\ -iA_{-1} & 0 \end{pmatrix}$ we get, using formal manipulations,

$$\delta = \frac{B+1}{B-1}$$

Now we give a rigorous proof of this formula.

(3.4) DEFINITION. — Let B be the selfadjoint operator defined by:

$$B: \mathcal{F}_{-1/2} \oplus \mathcal{F}_{+1/2} \subset H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}) \rightarrow H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$$

$$B = \begin{pmatrix} 0 & iA_{+1} \\ -iA_{-1} & 0 \end{pmatrix}$$

(3.5) PROPOSITION:

$$1 \notin \sigma_p(\mathbf{B})$$

Proof. — Suppose $\mathbf{B}(f \oplus g) = (f \oplus g)$. This condition is equivalent to:

$$\begin{cases} iA_+ f = g \\ -iA_- g = f \end{cases}$$

which implies $A_+ A_- f = f$, iff $\chi_{\mathcal{O}} \omega \chi_{\mathcal{O}} \omega^{-1} \chi_{\mathcal{O}} f = f$ iff $\chi_{\mathcal{O}} \omega (I - \chi_{\mathcal{O}}) \omega^{-1} f = 0$ iff $(I - \chi_{\mathcal{O}}) \omega^{-1} f \in \text{Ker}([I - \chi_{\mathcal{O}}, \omega])$, but $\text{Ker}([I - \chi_{\mathcal{O}}, \omega]) = \{0\}$ by antilocality and $(I - \chi_{\mathcal{O}}) \omega^{-1} f = 0$ means $\text{supp } \omega^{-1} f \subset \mathcal{O}$; but $\text{supp } f \subset \mathcal{O}$ too and then, by applying once more the antilocality property, $f = 0$. \square

(3.6) THEOREM:

$$\delta = \frac{\mathbf{B} + 1}{\mathbf{B} - 1}$$

Proof. — It follows from proposition (3.5) that $\frac{\mathbf{B} + 1}{\mathbf{B} - 1}$ is a well defined densely defined operator.

We note that in (3.3) \mathbf{N} defines on $\mathbf{H}^{-1/2}(\mathcal{O}) \oplus \mathbf{H}^{1/2}(\mathcal{O})$ a quadratic form. However, the operator which is associated with this quadratic form is not \mathbf{N} , but the operator $\frac{1}{2}\mathbf{R}$, where:

$$(3.7) \quad \mathbf{R} = \begin{pmatrix} \mathbf{Q}_- & 0 \\ 0 & \mathbf{Q}_+ \end{pmatrix} \begin{pmatrix} 1 & -i\omega \\ i\omega^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{Q}_- & 0 \\ 0 & \mathbf{Q}_+ \end{pmatrix} = \\ = \begin{pmatrix} 1 & -i\mathbf{Q}_- \omega \mathbf{Q}_+ \\ i\mathbf{Q}_+ \omega^{-1} \mathbf{Q}_- & 1 \end{pmatrix}$$

and we have posed $\mathbf{Q}_\sigma = [\mathbf{H}^{\sigma/2}(\mathcal{O})]$, $\mathbf{H}^{\sigma/2}(\mathcal{O}) \subset \mathbf{H}^{\sigma/2}(\mathbf{R}^3)$.

Let now $f \in \mathbf{H}^{-1/2}(\mathcal{O})$, $g \in \mathbf{H}^{1/2}(\mathcal{O})$, $x = \omega^{-1/2} f - i\omega^{1/2} g \in \mathbf{K}(\mathcal{O})$.

Then:

$$\|f\|_{-1}^2 + \|g\|_{+1}^2 = (\|x\|_{g(s)})^2 = (x, (\mathbf{I} + \delta)x)_0 = \left(f \oplus g, \frac{1}{2}\mathbf{R}(\mathbf{I} + \delta)(f \oplus g) \right).$$

From this we have $\frac{1}{2}\mathbf{R}(\mathbf{I} + \delta) = 1$ Then to prove (i) it is sufficient to show that:

$$(\mathbf{R} - \mathbf{I})\mathbf{B} = \mathbf{B}(\mathbf{R} - \mathbf{I}) = -\mathbf{I},$$

but:

$$-\mathbf{B}(\mathbf{R} - \mathbf{I}) = \begin{pmatrix} \mathbf{A}_+ \mathbf{Q}_+ \omega^{-1} \mathbf{Q}_- & 0 \\ 0 & \mathbf{A}_- \mathbf{Q}_- \omega \mathbf{Q}_+ \end{pmatrix}$$

and for each $f, g \in H^{-1/2}(\mathcal{O})$

$$\begin{aligned} (f, A_+ Q_+ \omega^{-1} Q_- g)_{-1} &= (A_- f, \omega^{-1} g)_{+1} = (A_- f, g)_0 \\ &= (\chi_{\mathcal{O}} \omega^{-1} \chi_{\mathcal{O}} f, g)_0 = (f, g)_{-1} \Rightarrow A_+ Q_+ \omega^{-1} Q_- = I \uparrow_{H^{-1/2}(\mathcal{O})} \end{aligned}$$

Similarly we obtain the result for $A_- Q_- \omega Q_+$ and $(R-I)B$. \square

It is clear from (3.6) that $1 \notin \sigma_p(\delta)$, but it is possible to prove (see [4]) that A_{σ} , hence B , is unbounded, that is $1 \in \sigma(\delta)$. These two properties imply the known fact that $\mathfrak{R}(\mathcal{O})$ is the unique hyperfinite III₁ factor (see [6]).

4. THE CONTINUITY OF THE MODULAR OPERATOR WITH RESPECT TO THE MASS

Up to now we have in a certain sense identified the theories with different masses, because of the property (2.3). But on the other side $K_m(\mathcal{O})$ is a subspace of $L^2(\mathbb{R}^3)$ for any mass, so that the mass m free scalar fields could be seen as different fields acting on the same Fock space. With this point of view we can naturally ask for the continuity of Δ_m in the mass parameter.

The following theorem holds:

(4.1) THEOREM. — $m \rightarrow \delta_m$ is a continuous function in the strong generalized sense.

To prove this theorem we need the following lemma:

(4.2) LEMMA. — If we put $H_m = (\delta_m + 1)^{-1} (\delta_m - I)$, then H_m is a continuous operator on $L^2(\mathbb{R}^3)$ and $m \rightarrow H_m$ is a continuous function in the strong topology.

Proof of (4.1). — We have only to observe that $H_m = I - 2(\delta_m + I)^{-1}$ so that, by (4.2), the resolvent $R_{\lambda}(\delta_m)$, for $\lambda = -1$, is a strong continuous function of the mass. By a well known theorem (see for example [9], cap. 8°, cor. 1.4, p.429), we have that δ_m is strongly continuous in the generalized sense. \square

The proof of lemma (4.2) is a straightforward sequence of estimates, based mainly on the following:

(4.3) LEMMA. — $\lim_{\mu \rightarrow m} \frac{\omega_{\mu}}{\omega_m} = 1$ as an operator in $\mathcal{B}_{\sigma} = B(H_m^{\sigma/2}(\mathcal{O}), H_m^{\sigma/2}(\mathbb{R}^3))$.

This property, non trivial only if $m=0$, is proved in Appendix (A.2) (b).

We are also interested in proving strong resolvent continuity for $\log \delta_m$, that is, by Trotter Kato theorem ([13], Vol. 1, 8.21), strong continuity

for δ_m^{it} . From this, via the properties of the second quantization structure, one obtains the strong continuity of Δ_m^{it} .

(4.4) THEOREM. — *The map $m \rightarrow \delta_m^{it}$ is strongly continuous, uniformly for each t in any finite interval.*

(4.5) COROLLARY. — *The map $m \rightarrow \Delta_m^{it}$ is strongly continuous, uniformly for each t in any finite interval.*

Proof. — Take $F(x) = x^{it}$, $x \in \mathbb{R}_+$, $\rho \notin \sigma_p(\delta_m)$, that is $E_{[0, \rho]}(\delta_m) = E_{[0, \rho]}(\delta_m)$, where E is the spectral measure associated with δ_m . Then set:

$$F_\rho(x) = \begin{cases} F(x), & x \geq \rho \\ F(\rho), & x \leq \rho \end{cases}$$

Remark that $F_\rho(x) \in C_{\text{lim}}(\mathbb{R})$.

Now, if $x \in H_{(\rho, +\infty)}(\delta_\mu) = (I - E_{[0, \rho]}(\delta_\mu)) L^2(\mathbb{R}^3)$, then $F_\rho(\delta_\mu)x = F(\delta_\mu)x$, therefore

$$\begin{aligned} \|(\delta_m^{it} - \delta_\mu^{it})x\|_0 &= \|(F(\delta_m) - F(\delta_\mu))x\|_0 \leq \\ &\leq \|(F_\rho(\delta_m) - F_\rho(\delta_\mu))x\|_0 + \|(F_\rho(\delta_m) - F(\delta_m))(I - E_{[0, \rho]}(\delta_\mu))x\|_0 \leq \\ &\leq \|(F_\rho(\delta_m) - F_\rho(\delta_\mu))x\|_0 + 2\|(E_{[0, \rho]}(\delta_m) - E_{[0, \rho]}(\delta_\mu))x\|_0. \end{aligned}$$

The fact that the two summands of the last inequality go to zero, follows from standard arguments (see e. g. [9], thm. 1.5, cap. 8; [13]). On the other side

$$\bigcup_{\substack{\rho > 0 \\ \rho \notin \sigma_p(\delta_\mu)}} H_{(\rho, +\infty)}(\delta_\mu) = \bigcup_{\rho > 0} H_{(\rho, +\infty)}(\delta_\mu)$$

is dense in $L^2(\mathbb{R}^3)$, and from this the result follows. \square

APPENDIX

Some results about Local Spaces

In this appendix we will state some technical results about the local spaces defined in (2.1), (2.2). The most intriguing case is obviously $m=0$. We prove the following properties:

(A.1) PROPOSITION. — *Let $f \in H_0^\alpha(\mathbb{R}^3)$, then \hat{f} is a measurable function and, if $\alpha < 3/2$, \hat{f} is a tempered distribution. Moreover if $\alpha > -3/2$, $\mathcal{S} \subset H_0^\alpha(\mathbb{R}^3)$.*

Proof. — Such an f is the limit of a Cauchy sequence $\{f_n\}$ of functions in $D(\omega_m^\alpha) \subset L^2(\mathbb{R}^3)$ with respect to the norm $\|f_n\|_{\alpha, m} = \|\mathbf{p}^\alpha \hat{f}_n\|$. Thus we

have that $|\mathbf{p}|^\alpha \hat{f}_n$ is a Cauchy sequence in $L^2(\mathbb{R}^3)$, hence there exists g in $L^2(\mathbb{R}^3)$ which is the limit of $|\mathbf{p}|^\alpha \hat{f}_n$.

It is obvious that $\hat{f} = |\mathbf{p}|^{-\alpha} g$, hence \hat{f} is a measurable function. It is a tempered distribution if $|\mathbf{p}|^{-\alpha} g$ is locally integrable, but:

$$\int_{|\mathbf{p}| < 1} |\mathbf{p}|^{-\alpha} g(\mathbf{p}) d^3 p \leq \left(\int_{|\mathbf{p}| < 1} |\mathbf{p}|^{-2\alpha} d^3 p \right)^{1/2} \|g\| < +\infty$$

if and only if $\alpha < 3/2$. \square

(A. 2) PROPOSITION. — *If $\mathcal{O} \subset \mathbb{R}^3$ is bounded, and $\alpha = \pm 1/2$, then:*

(a)
$$H_m^\alpha(\mathcal{O}) \cong H_0^\alpha(\mathcal{O}),$$

that is they are the same vector space with equivalent norms; more precisely:

(i)
$$\|f\|_{1/2, 0} \leq \|f\|_{1/2, m} \leq c(m, \mathcal{O}) \|f\|_{1/2, 0}$$

(ii)
$$\|f\|_{-1/2, m} \leq \|f\|_{-1/2, 0} \leq c(m, \mathcal{O}) \|f\|_{-1/2, m}$$

(b)
$$\lim_{\mu \rightarrow m} \frac{\omega_\mu}{\omega_m} = 1$$
 as an operator in $\mathcal{B}_\sigma = \mathbf{B}(H_m^{\sigma/2}(\mathcal{O}), H_m^{\sigma/2}(\mathbb{R}^3))$.

Proof. — (a) The first inequalities in (i) and (ii) are obvious. As far as the second ones are concerned, we have:

(i) Let $f \in H_0^{1/2}(\mathcal{O})$. It is easy to see that

$$\sqrt{|\mathbf{p}|^2 + m^2} \leq \sqrt{1 + m^2} |\mathbf{p}| + m \chi_1(\mathbf{p})$$

where $\chi_1(\mathbf{p})$ is the characteristic function of the unit ball in \mathbb{R}^3 , then:

$$\|f\|_m^2 = \int |\hat{f}|^2 \sqrt{|\mathbf{p}|^2 + m^2} d^3 p \leq \sqrt{1 + m^2} \|f\|_0^2 + \int_{|\mathbf{p}| \leq 1} |\hat{f}|^2 d^3 p$$

Now let $\varphi \in \mathcal{D}$ be such that $\varphi \equiv 1$ on $\text{supp } f$.

$$\begin{aligned} \int_{|\mathbf{p}| \leq 1} |\hat{f}|^2 d^3 p &= \int \chi_1(\mathbf{p}) \left| \int \hat{f}(\mathbf{k}) \hat{\varphi}(\mathbf{p} - \mathbf{k}) d^3 k \right|^2 d^3 p = \\ &= \int \chi_1(\mathbf{p}) \left| \int (|\mathbf{k}|^{1/2} \hat{f}(\mathbf{k})) (|\mathbf{k}|^{-1/2} \hat{\varphi}(\mathbf{p} - \mathbf{k})) d^3 k \right|^2 d^3 p \leq \\ &\leq \int \chi_1(\mathbf{p}) \left(\int |\mathbf{k}| \cdot |\hat{f}(\mathbf{k})|^2 d^3 k \right) \left(\int |\mathbf{k}|^{-1} |\hat{\varphi}(\mathbf{p} - \mathbf{k})|^2 d^3 k \right) d^3 p = \\ &= \|f\|_0^2 \int \chi_1(\mathbf{p}) |\mathbf{k}|^{-1} |\hat{\varphi}(\mathbf{p} - \mathbf{k})|^2 d^3 k d^3 p \end{aligned}$$

Thus we get $\|f\|_m \leq c(m, \mathcal{O}) \|f\|_0$, where

$$c(m, \mathcal{O}) = \sqrt{1 + m^2} + \left(m \int \chi_1(\mathbf{p}) |\mathbf{k}|^{-1} |\hat{\varphi}(\mathbf{p} - \mathbf{k})|^2 d^3 k d^3 p \right)^{1/2}$$

With similar techniques we obtain (ii), and this proves (a).

(b) follows easily from (a). In fact let $f \in H_m^\alpha(\mathcal{O})$, then

$$\begin{aligned} \left\| \left[1 - \left(\frac{\omega_\mu}{\omega_m} \right)^{1/2} \right] f \right\|_{1/2, m}^2 &= \| (\omega_\mu^{1/2} - \omega_m^{1/2}) f \|^2 \leq \\ &\leq \left\| \frac{\omega_\mu^{1/2} - \omega_m^{1/2}}{\omega_1^{1/2}} \omega_1^{1/2} f \right\|^2 \leq (\sqrt{m} - \sqrt{\mu})^2 \| f \|_{1/2, 1}^2. \end{aligned}$$

The proof for $-1/2$ is analogous. \square

As far as the Proposition (2.4) (a) is concerned, let us observe that, by definition, $H^\alpha(\mathcal{O})^0 \cong H^{-\alpha}(\mathcal{O}^c)$. Moreover:

(A.3) PROPOSITION:

$$H^{-\alpha}(\mathcal{O}^c)^0 \subset \{ f \in H^\alpha(\mathbb{R}^3) / \text{supp } f \subset \bar{\mathcal{O}} \}, \quad \forall \alpha / |\alpha| < 3/2, \quad \forall m \geq 0.$$

Proof. — $f \in H^{-\alpha}(\mathcal{O}^c)^0$ if and only if $f \in H^\alpha(\mathbb{R}^3)$ and $\langle f, g \rangle = 0, \forall g \in H^{-\alpha}(\mathcal{O}^c)$ if and only if $f \in H^\alpha(\mathbb{R}^3)$ and $\text{supp } f \subset \mathcal{O}$. \square

(A.4) PROPOSITION. — Let $\mathcal{O} \subset \mathbb{R}^3$ be a region with C^1 boundary. Then $\mathcal{D}(\mathcal{O})$ is dense in $\{ f \in H^\alpha(\mathbb{R}^3) / \text{supp } f \subset \mathcal{O} \}$ if $|\alpha| < 3/2, m \geq 0$.

To prove this property, we need two lemmas:

(A.5) LEMMA. — $\forall \varphi \in \mathcal{D}(\mathbb{R}^3)$, the operator $\mathcal{M}_\varphi : H_m^\alpha(\mathcal{O}) \rightarrow H_m^\alpha(\mathcal{O})$, defined by $f \rightarrow \varphi \cdot f$, is continuous in $\| \cdot \|_{\alpha, m}$.

Proof. — Let $m > 0$. We use Peetre inequality (see e. g. [5]):

$$\begin{aligned} (|\mathbf{p} + \mathbf{k}|^2 + m^2)^\alpha &\leq \left(1 + \frac{2}{m^2} \right)^{-\alpha} (|\mathbf{p}|^2 + m^2)^\alpha (|\mathbf{k}|^2 + m^2)^{|\alpha|}, \\ &\forall \alpha \in \mathbb{R}, \quad m \geq 0. \end{aligned}$$

We will pose $\left(1 + \frac{2}{m^2} \right)^{-\alpha} = C(m, \alpha)$. Then:

$$\begin{aligned} \| \varphi f \|_{m, \alpha}^2 &\leq \int \left(\int |\hat{\varphi}(\mathbf{k}) \hat{f}(\mathbf{p} - \mathbf{k})| d^3 k \right)^2 \omega_m^\alpha(\mathbf{p}) d^3 p \leq \\ &\leq C(m, \alpha) \int \left(\int |\hat{\varphi}(\mathbf{k}) \hat{f}(\mathbf{p} - \mathbf{k})| \omega_m^{|\alpha|/2}(\mathbf{k}) \omega_m^{\alpha/2}(\mathbf{p} - \mathbf{k}) d^3 k \right)^2 d^3 p = \\ &= C(m, \alpha) (\| (\omega_m^{|\alpha|/2} \hat{\varphi}) * (\omega_m^{\alpha/2} \hat{f}) \|_0)^2 = C(m, \alpha) (\| (\omega_m^{|\alpha|/2} \varphi) \cdot (\omega_m^{\alpha/2} f) \|_0)^2 \leq \\ &\leq C(m, \alpha) (\| \omega_m^{|\alpha|/2} \varphi \|_\infty)^2 \| f \|_{\alpha, m}^2. \end{aligned}$$

If $m=0$, Peetre inequality does not hold. When $\alpha=1/2$ we use the following:

$$|\mathbf{p}|^{1/2} \leq (|\mathbf{p} - \mathbf{k}|^{1/2} + 1) \chi_1(|\mathbf{k}|) + |\mathbf{k}|^{1/2} (|\mathbf{p} - \mathbf{k}|^{1/2} + 1) \chi_1^c(|\mathbf{k}|),$$

where χ_1 and χ_1^c are the characteristic functions of $[0, 1)$ and $(1, +\infty)$ respectively. With techniques similar to the preceding ones we obtain the estimate:

$$\| \varphi f \|_{1/2, 0} \leq 2\pi (\| \varphi \|_0 + \| \varphi \|_{1/2, 0} + \| \varphi \|_{1/4, 0}) \| f \|_{1/2, 0}$$

If $\alpha = 1/2$, we will use

$$\frac{1}{|\mathbf{p}|} \leq \frac{1}{|\mathbf{p}|} \chi_1(|\mathbf{p}|) + c \frac{1}{\sqrt{|\mathbf{p}|^2 + 1}} \chi_1(|\mathbf{p}|)$$

obtaining:

$$\|\varphi \cdot f\|_{-1/2, 0} \leq 2\pi(2\|\varphi\|_{1, 0} + \|\varphi\|_{-1/2, 0})\|f\|_{-1/2, 0} \quad \square$$

(A.6) LEMMA. — Let $\mathcal{O} \subset \mathbb{R}^3$ be a region with C^1 boundary. Then there exists an open cover of \mathcal{O} , $\{\mathcal{O}_j\}_{j \in \mathbb{N}}$ such that:

$$\forall j \in \mathbb{N}, \exists \mathbf{n}_j \in \mathbb{R}^3, \varepsilon_j > 0 \text{ such that } d((\mathcal{O}_j \cap \mathcal{O}) + \varepsilon_j \mathbf{n}_j, \partial \mathcal{O}) = \varepsilon_j.$$

Proof. — With each $x \in \partial \mathcal{O}$ we associate the outward unit normal \mathbf{n}_x . Because the boundary is C^1 we can find an ε_x such that:

(i) $\forall d < 3\varepsilon_x, \partial \mathcal{O} \cap B(x, d)$ is connected,

(ii) $\forall y \in \partial \mathcal{O} \cap B(x, 3\varepsilon_x)$ the angle between \mathbf{n}_x and \mathbf{n}_y is lower than $\pi/4$.

Let $\mathcal{O}_x = \mathcal{O} \cap B(x, \varepsilon_x)$. Obviously $\mathcal{O}_x - \varepsilon_x \mathbf{n}_x \subset \mathcal{O} \cap B(x, 3\varepsilon_x)$ and $d(\partial \mathcal{O}, \mathcal{O}_x - \varepsilon_x \mathbf{n}_x) = \varepsilon_x$. By $\bigcup_{x \in \mathcal{O}} B(x, \varepsilon_x) \supset \overline{\mathcal{O}}$, we have that there exist a

sequence $\{x_j\}_{j \in \mathbb{N}}$ such that $\bigcup_{j \in \mathbb{N}} B(x_j, \varepsilon_{x_j}) \supset \mathcal{O}$, obtaining $\bigcup_{j \in \mathbb{N}} \mathcal{O}_{x_j} = \mathcal{O}$. \square

We can finally obtain the proof of (A.4):

Proof (A.4). — Let $\{f_j\}_{j \in \mathbb{N}}$ be a partition of unity subordered to the covering $\{\mathcal{O}_j\}_{j \in \mathbb{N}}$ defined above. Then, by lemma (A.5), $\forall u \in H^\alpha(\mathcal{O})$,

$u_j = u f_j \in H^\alpha(\mathcal{O})$ and $\sum_{j=1}^\infty u_j$ converges to u in $H^\alpha(\mathcal{O})$.

For any $j \in \mathbb{N}$, let $\psi_j \in \mathcal{D}(\mathbb{R}^3)$ be a function such that $\text{supp } \psi_j \subset B(0, \varepsilon_j)$; if we pose $u_{\varepsilon_j, j}(\mathbf{x}) = u_j(\mathbf{x} - \varepsilon_j \mathbf{n}_j)$, then $\text{supp}(u_{\varepsilon_j, j} * \psi_j) \subset \mathcal{O}_j + \varepsilon_j \mathbf{n}_j \subset \overline{\mathcal{O}}$, which implies $u_{\varepsilon_j, j} * \psi_j \in \mathcal{D}(\overline{\mathcal{O}})$ is a function which approximate u , i.e. the result. \square

Now, observing that

$$\mathcal{D}(\mathcal{O}) \subset H^\alpha(\mathcal{O}) \subset H^{-\alpha}(\mathcal{O}^c)^0 \subset \{f \in H^\alpha(\mathbb{R}^3) / \text{supp } f \subset \overline{\mathcal{O}}\} \subset \overline{\mathcal{D}(\mathcal{O})}^{\|\cdot\|_\alpha}$$

we get the desired result:

(A.7) PROPOSITION. — For each region $\mathcal{O} \subset \mathbb{R}^3$ with C^1 boundary, $|\alpha| < 3/2$,

$$H^\alpha(\mathcal{O})^0 = H^{-\alpha}(\mathcal{O}^c).$$

The Proposition (2.4) (b) is easily proved, using the following:

(A.8) LEMMA. — Let $\alpha \geq -1/2, f \in H^\alpha(\mathbb{R}^n)$ be such that $\text{supp } f \subset \partial \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^n$ is a region with C^1 boundary, then $f \equiv 0$.

Proof. — For a point $x \in \mathbb{R}^n$ we will use the following notation: $x = (x_1, x')$, where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$. We choose $g \in \mathcal{D}'$ with

$$\text{supp } g \subset \{x_1 = 0\}$$

and $\varphi \in \mathcal{D}$ with

$$\text{supp } \varphi \cap \{x_1 = 0\} \neq \emptyset, \quad \frac{\partial \varphi}{\partial x_1} = 0 \quad (\Rightarrow \varphi(x_1, x') \equiv \varphi(x')).$$

Then φg is a distribution with compact support and $(\varphi g)^\wedge(\xi_1, \xi') = (\widehat{\varphi} * \widehat{g})(\xi') = \widehat{\varphi}(\xi')$. Using the Peetre inequality:

$$C(m, \alpha)^{-1} (1 + |\xi_1|)^{2\alpha} (1 + |\xi'|)^{-2|\alpha|} \leq \\ \leq (1 + |\xi|)^{2\alpha} \leq C(m, \alpha) (1 + |\xi_1|)^{2\alpha} (1 + |\xi'|)^{2|\alpha|}$$

we obtain

$$\text{const } A \cdot \int (1 + |\xi'|)^{-2|\alpha|} |\widehat{\varphi}(\xi')|^2 d\xi' \leq \int (1 + |\xi|)^{2\alpha} |\widehat{\varphi}(\xi)|^2 d\xi \\ \leq \text{const } A \cdot \int (1 + |\xi'|)^{2|\alpha|} |\widehat{\varphi}(\xi')|^2 d\xi',$$

where

$$A = \int (1 + |\xi_1|)^{2\alpha} d\xi_1 < +\infty \quad \Leftrightarrow \quad \alpha < -1/2,$$

therefore, for $\alpha \geq -1/2$, $\varphi g \in H^\alpha(\mathbb{R}^n)$ if and only if $\varphi \equiv 0$. Furthermore each $f \in H^\alpha(\mathbb{R}^n)$ such that $\text{supp } f \subset \{x_1 = 0\}$ can be written as $f = \varphi g$ and we have the result for functions with support in a hyperplane, and thus for functions with support in $\partial\mathcal{O}$, if $\partial\mathcal{O}$ is sufficiently regular (for ex. C^1 , see [11]). \square

We are now able to prove:

(A. 9) PROPOSITION. — $\mathcal{D}_\sigma := H^{\sigma/2}(\mathcal{O}) + H^{\sigma/2}(\mathcal{O}^c)$ is dense in $H^{\sigma/2}(\mathbb{R}^3)$ if $\sigma = \pm 1$.

Proof. — We prove the proposition only for $\sigma = +1$. The case $\sigma = -1$ has a very similar proof. Suppose that \mathcal{D}_{+1} is not dense in $H^{1/2}(\mathbb{R}^3)$. This implies that there exist $f \in H^{-1/2}(\mathbb{R}^3)$ such that

$$\langle f, g \rangle = 0, \quad \forall g \in \mathcal{D}_{+, m} = H_m^{1/2}(\mathcal{O}) + H_m^{1/2}(\mathcal{O}^c).$$

It is obvious that this condition implies that $\text{supp } f \subset \partial\mathcal{O}$, i. e., by lemma (A. 8), $f \equiv 0$. \square

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