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Schrödinger operators with form-bounded potentials in L^p-spaces

by

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ABSTRACT. — Let

$$V=V_+-V_-, \qquad 0\leqq V_\pm \in L^p_{comp}(\mathbb{R}^l),$$
$$\beta<1, \qquad V_-\leqq -\beta\Delta+C,$$
$$V_-\in L^{k,\infty}_{comp}(\mathbb{R}^l), \qquad k=(1+\sqrt{1-\beta})\Big/\left(\frac{2}{p}-1+\sqrt{1-\beta}\right)$$

($L^{p,\infty}$ is the weak $-L^p$-space). If $p\in\left(\frac{2}{1+\sqrt{1-\beta}}, \frac{2}{1-\sqrt{1-\beta}}\right)$, then the closure of the operator $(-\Delta+V+C)\restriction C_0^\infty(\mathbb{R}^l)$ is generator of the contraction C_0-semigroup in $L^p(\mathbb{R}^l)$. For $p=2$ we obtain a new theorem concerning the essential self-adjointness of the Schrödinger operator. The sharpness of the results is illustrated by the examples.

RÉSUMÉ. — $V=V_+-V_-, \qquad 0\leqq V_\pm \in L^p_{comp}(\mathbb{R}^l),$

$$\beta<1, \qquad V_-\leqq -\beta\Delta+C,$$
$$V_-\in L^{k,\infty}_{comp}(\mathbb{R}^l), \qquad k=(1+\sqrt{1-\beta})\Big/\left(\frac{2}{p}-1+\sqrt{1-\beta}\right)$$

($L^{p,\infty}$ est l'espace L^p faible). Si $p\in\left(\frac{2}{1+\sqrt{1-\beta}}, \frac{2}{1-\sqrt{1-\beta}}\right)$, la fermeture de l'opérateur $(-\Delta+V+C)\restriction C_0^\infty(\mathbb{R}^l)$ est un générateur de semigroupe C_0 de contraction dans $L^p(\mathbb{R}^l)$. Pour $p=2$ nous obtenons un nouveau théorème concernant l'auto-adjonction essentielle de l'opérateur de Schrödinger. Ces résultats sont illustrés sur des exemples.

INTRODUCTION

This article deals with the Schrödinger operators $-\Delta+V$ acting in $L^p \equiv L^p(\mathbb{R}^l, d^l x)$. We want to investigate the first spectral problem, *i. e.* the problem of *m*-accretivity. Most of the results concerning this problem contain the condition of $-\Delta$-boundedness of the operator $V_- \equiv \max\{-V, 0\}$. Our goal is to investigate the conditions of the following type:

$$\left.\begin{array}{ll} V_- \leqq -\beta\Delta + C(\beta), & \beta<1, \\ V_- \in F, & \end{array}\right\} \tag{1}$$

where F is some function space.

The first result in this direction is due to Povzner [1] [$p=2$, $\beta=1$, $F=C(\mathbb{R}^l)$]. Recently some results were also obtained. In particular the essential self-adjointness was proved provided that $F=L^k$, $k>\dfrac{2l}{l-\beta(l-2)}$ [2]; and $F=L^k$, $k=(1+\sqrt{1-\beta})/\sqrt{1-\beta}$ [3].

My purpose here is to extend the abovementioned results to the case of L^p-spaces and to choose the broader space F. Even for $p=2$ we obtain a new result which is very close to optimal.

It should be noted that our method is an extension of the Semenov's one [3].

Note that L^p-Δ-boundedness implies the condition (1) (*see* [4]) and in my opinion the condition (1) corresponds to the point of our problem. We want also to emphasize that the exact eigenfunction estimates for the operator $-\Delta+V$ and the exact results concerning the accretivity of the Schrödinger operators have been proved under the same condition (1) (*see* [5], [6]).

Some common symbols are:

$$\| f \|_p = \left(\int_{\mathbb{R}^l} | f(x) |^p d^l x \right)^{1/p};$$

$$\langle f, g \rangle = \int_{\mathbb{R}^l} f(x) \overline{g(x)} d^l x;$$

$$(f * g)(x) = \int_{\mathbb{R}^l} f(x-y) g(y) d^l y;$$

$L^p_{loc} \equiv L^p_{loc}(\mathbb{R}^l, d^l x)$ = the complex functions f on $\mathbb{R}^l$ such that $f\varphi \in L^p$, $\forall \varphi \in C_0^\infty$; meas(B) = Lebesgue measure of $B \subset \mathbb{R}^l$; $L^{p,\infty} \equiv L^{p,\infty}(\mathbb{R}^l, d^l x)$ = the complex functions f on $\mathbb{R}^l$ such that

$$\sup_{t>0} t^p \operatorname{meas} \{ x : | f(x) | > t \} < \infty$$

(note that $L^p \subset L^{p,\infty}$ and $|x|^{-l/p} \in L^{p,\infty}$). $L^{p,\infty}_{comp}$ = the set of functions in $L^{p,\infty}$ with compact support. $\Delta = \sum_{j=1}^{l} \frac{\partial^2}{\partial x_j^2}$ the Laplacian acting on $\mathscr{D}'(\mathbb{R}^l)$.

MAIN RESULTS

LEMMA 1. — *If* $V \in L^{p,\infty}_{comp}$, *then* $\| V \|_{p-\varepsilon} = O(\varepsilon^{-1/p})$, $\varepsilon \downarrow 0$.

Proof. — Let $\mu(t) = \operatorname{meas}\{x : |V(x)| > t\}$. The function $\mu(t)$ is nonincreasing and $\mu(0) = \operatorname{meas}\{\operatorname{supp} V\} < \infty$. The definition of the space $L^{p,\infty}$ implies that $\mu(t) \leqq C t^{-p}$, $\forall t > 0$, so

$$\begin{aligned} \| V \|_{p-\varepsilon}^{p-\varepsilon} &\equiv (p-\varepsilon) \int_0^\infty t^{p-\varepsilon-1} \mu(t)\, dt \\ &= (p-\varepsilon) \left(\int_0^1 t^{p-\varepsilon-1} \mu(t)\, dt + \int_1^\infty t^{p-\varepsilon-1} \mu(t)\, dt \right) \\ &\leqq (p-\varepsilon) \left(\mu(0) \int_0^1 t^{p-\varepsilon-1}\, dt + C \int_1^\infty t^{-1-\varepsilon}\, dt \right) \\ &= \mu(0) - C + \frac{Cp}{\varepsilon}. \quad \blacksquare \end{aligned}$$

DEFINITION. — Let $0 \leqq V$, $W \in L^1_{loc}$. Then $V \in PK_\beta(-\Delta + W)$ if and only if

$$\langle V\varphi, \varphi \rangle \leqq \beta (\| \vec{\nabla} \varphi \|_2^2 + \| W^{1/2} \varphi \|_2^2) + C(\beta) \| \varphi \|_2^2, \quad \forall \varphi \in C_0^\infty.$$

THEOREM 2. — *Let V be a real valued measurable function such that*

$$V=V_+-V_-,\qquad 0\leq V_+\in L^p_{comp},$$
$$0\leq V_-\in PK_\beta(-\Delta+V_+)\cap L^{k,\infty},$$
$$\beta<1,\qquad p\in\left(1,\frac{2}{1-\sqrt{1-\beta}}\right),$$
$$k=(1+\sqrt{1-\beta})\Big/\left(\frac{2}{p}-1+\sqrt{1-\beta}\right).$$

Then the range of the operator $(-\Delta+V+\lambda)\upharpoonright C_0^\infty$ *is dense in* L^p *for any* $\lambda>C(\beta)$.

Proof. — If $p\in\left(1,\frac{2}{1-\sqrt{1-\beta}}\right)$, then $1<p<k<\infty$.

Suppose that the range is not dense, then by the Hahn-Banach theorem, there exists a function $0\neq u\in L^p$ such that

$$\langle(-\Delta+V+\lambda)\varphi,u\rangle=0,\qquad\forall\varphi\in C_0^\infty.\tag{3}$$

Whithout loss we can assume $u=\operatorname{Re}u$. Let $1/t=1/p'+1/k'$, $q=p'/k'$. Let $\varepsilon>0$ be a small parameter and let us set $1/s=1/t+\varepsilon/p'$, $1/s'=(q-1-\varepsilon)/p'$. Since $V_-\in L^{k,\infty}_{comp}$ and $u\in L^{p'}$. It follows that $V_-u\in L^1\cap L^s$.

The equality (3) gives

$$(-\Delta+V_++\lambda)u=V_-u\in L^1\cap L^s\tag{4}$$

in the distributional sense; hence

$$u\in L^1\cap L^s\cap L^{p'};\tag{5}$$

(*cf.* [7]).

Moreover, $u\in L^{p'}$ implies

$$|u|^{q-1}\in L^{t'}.\tag{6}$$

Let us consider the operator H_z^+ with the domain

$$\mathscr{D}(H_z^+)=\{u\in L^z:V_+u\in L^1_{loc},\,(-\Delta+V_+)u\in L^z\},$$

where $(-\Delta+V_+)$ is taken in the distributional sense and $H_z^+u\equiv(-\Delta+V_+)u$, $\forall u\in\mathscr{D}(H_z^+)$.

H_z^+ is the generator of the strongly continuous contraction semigroup $T_z(t)\equiv\exp(-tH_z^+)$, $\forall t>0$ in L^z which has properties such as follows below

$$\left.\begin{array}{l}\|T_z(t)\varphi\|_w\leq C(t,w,y,l)\|\varphi\|_y,\\ \forall\varphi\in C_0^\infty,\quad\forall t>0,\quad\forall w\geq y\geq 1\end{array}\right\}\tag{7}$$
$$T_z(t)\upharpoonright[L^z\cap L^w]=T_w(t)\upharpoonright[L^z\cap L^w]\tag{8}$$

(*cf.* [7]).

Let $\varphi_r = T_s\left(\frac{1}{r}\right)u$. It follows from (5), (8), (4) and C_0-property that

(1) $\varphi_r \underset{L^y}{\rightarrow} u, \forall y \in [1, p']$;

(2) $(-\Delta + V_+)\varphi_r \underset{L^s}{\rightarrow} (-\Delta + V_+)u$.

According to (1) we can choose subsequence (which we shall denote by the same symbol $\{\varphi_r\}$) such that

$$\varphi_r(x) \rightarrow u(x) \quad \text{for a. e. } x \in \mathbb{R}^l.$$

Then

(3) $\varphi_r |\varphi_r|^{q-\varepsilon-2} \underset{L^{s'}}{\rightarrow} u|u|^{q-\varepsilon-2}$.

Indeed,

$$\varphi_r |\varphi_r|^{q-\varepsilon-2} \rightarrow u|u|^{q-\varepsilon-2} \quad \text{a. e. on } \mathbb{R}^l,$$

because $\varphi_r(x) \rightarrow u(x)$ a. e. Hence in order to conclude the proof of (3) it is sufficient to prove the convergence of the norms (a nice proof of this fact may be found in [8]) but

$$\| \varphi_r |\varphi_r|^{q-\varepsilon-2} \|_{s'} = \| \varphi_r \|_{p'}^{q-\varepsilon-1}$$

and the convergence $\|\varphi_r\|_{p'} \rightarrow \|u\|_{p'}$ is the consequence of (1).

In the same way we obtain

(4) $|\varphi_r|^{q-\varepsilon} \underset{L^y}{\rightarrow} |u|^{q-\varepsilon}, \forall y < k'$.

Let us set $j_m(x) = j(mx)m^{-l}$, where $0 \leq j \in C_0^\infty$, $\int_{\mathbb{R}^l} j(x) d^l x = 1$.

Let $\varphi_{mr} = j_m * \varphi_r$. By well known properties of the operation $j_m *$ (*see*, e. g., [9]), we have (1), (3), (4) for the subsequence of $\{\varphi_{mr}\}$. (7) implies $\varphi_r \in L^\infty$ ($r = 1, 2, \ldots$) so that $V_+ \varphi_{mr} \underset{L^s}{\rightarrow} V_+ \varphi_r$. This fact and the equality $\Delta(j_m * \varphi_r) = j_m * (\Delta\varphi_r)$ show that

$$(-\Delta + V_+)\varphi_{mr} \underset{L^s}{\rightarrow} (-\Delta + V_+)\varphi_r.$$

Let us choose any $\omega \in C_0^\infty$ having $\omega(0) = 1$ and let $\omega_d(x) = \omega\left(\frac{x}{d}\right)$ and $\varphi_{dmr} = \omega_d \varphi_{mr}$. It is easy to see that properties (1), (3), (4) hold for $\{\varphi_{dmr}\}$.

Using the standard diagonal process we can choose the subsequence $u_r = \varphi_{d_r m_r r}$ such that

$$u_r \underset{L^y}{\rightarrow} u, \qquad \forall y \in [1, p'], \tag{9}$$

$$(-\Delta + V_+) u_r \underset{L^s}{\rightarrow} (-\Delta + V_+) u, \tag{10}$$

$$u_r |u_r|^{q-\varepsilon-2} \underset{L^{s'}}{\rightarrow} u |u|^{q-\varepsilon-2}, \tag{11}$$

$$|u_r|^{q-\varepsilon} \underset{L^y}{\rightarrow} |u|^{q-\varepsilon}, \qquad \forall y \leqq k'. \tag{12}$$

Observe that $u_r \in C_0^\infty$, so that $u_r |u_r|^{q-\varepsilon-2}$, $|u_r|^{(q-\varepsilon)/2} \in \mathscr{D}(\vec{\nabla})$ (*see*, e. g., [10], Chapter 2), and the direct calculation yields the identity

$$\langle u_r |u_r|^{q-\varepsilon-2}, -\Delta u_r \rangle = \frac{4(q-\varepsilon-1)}{(q-\varepsilon)^2} \| \vec{\nabla} |u_r|^{(q-\varepsilon)/2} \|_2^2. \tag{13}$$

(4)-(6) imply the finiteness of the quantities

$$\langle u |u|^{q-\varepsilon-2}, (-\Delta + V_+) u \rangle; \qquad \langle u |u|^{q-\varepsilon-2}, V_- u \rangle;$$
$$\langle u |u|^{q-\varepsilon-2}, \lambda u \rangle,$$

so that the simple approximation arguments show that (3) is valid provided that $\varphi = u_r |u_r|^{q-\varepsilon-2}$ and (9)-(12) imply

$$\lim_{r\to\infty} \langle u_r |u_r|^{q-\varepsilon-2}, (-\Delta + V + \lambda) u_r \rangle$$
$$= \langle u |u|^{q-\varepsilon-2}, (-\Delta + V + \lambda) u \rangle = 0. \tag{14}$$

Using (13), (14) and the condition $V_- \in PK_\beta(-\Delta + V_+)$ we have

$$\begin{aligned}
\lambda \| u \|_{q-\varepsilon}^{q-\varepsilon} &+ \langle u |u|^{q-\varepsilon-2}, (-\Delta + V_+) u \rangle \\
&= \langle u |u|^{q-\varepsilon-2}, V_- u \rangle = \lim_{r\to\infty} \langle u_r |u_r|^{q-\varepsilon-2}, V_- u_r \rangle \\
&= \lim_{r\to\infty} \langle V_- |u_r|^{(q-\varepsilon)/2}, |u_r|^{(q-\varepsilon)/2} \rangle \\
&\leqq \lim_{r\to\infty} [\beta \| \vec{\nabla} |u_r|^{(q-\varepsilon)/2} \|_2^2 + \beta \\
&\times \| V_+^{1/2} |u_r|^{(q-\varepsilon)/2} \|_2^2 + C(\beta) \| . |u_r|^{(q-\varepsilon)/2} \|_2^2] \\
&= \lim_{r\to\infty} \left[\frac{\beta (q-\varepsilon)^2}{4(q-\varepsilon-1)} \langle u_r |u_r|^{q-\varepsilon-2}, -\Delta u_r \rangle \right. \\
&\left. + \beta \langle u_r |u_r|^{q-\varepsilon-2}, V_+ u_r \rangle + C(\beta) \| u_r \|_{q-\varepsilon}^{q-\varepsilon} \right] \equiv \lim_{r\to\infty} I_r. \qquad (15)
\end{aligned}$$

Note that $\beta<\beta(q-\varepsilon)^2/4(q-\varepsilon-1)$. Therefore,

$$\lim_{r\to\infty} \mathrm{I}_r \leqq \frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} \lim_{r\to\infty} \langle u_r|u_r|^{q-\varepsilon-2}, (-\Delta+\mathrm{V}_+)u_r\rangle$$

$$+\mathrm{C}(\beta)\|u\|_{q-\varepsilon}^{q-\varepsilon} = \frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)} \times \langle u|u|^{q-\varepsilon-2}, (-\Delta+\mathrm{V}_+)u\rangle + \mathrm{C}(\beta)\|u\|_{q-\varepsilon}^{q-\varepsilon}. \quad (16)$$

According to (15), (16),

$$(\lambda-\mathrm{C}(\beta))\|u\|_{q-\varepsilon}^{q-\varepsilon} \leqq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1\right]\langle u|u|^{q-\varepsilon-2}, (-\Delta+\mathrm{V}_+)u\rangle.$$

From (14) we have that the right side of the last inequality equals

$$\left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1\right]\langle u|u|^{q-\varepsilon-2}, (\mathrm{V}_- -\lambda)u\rangle$$

$$\leqq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1\right]\langle u|u|^{q-\varepsilon-2}, \mathrm{V}_- u\rangle.$$

So, using Hölder inequality we obtain

$$(\lambda-\mathrm{C}(\beta))\|u\|_{q-\varepsilon}^{q-\varepsilon} \leqq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1\right]\langle |u|^{q-\varepsilon}, \mathrm{V}_-\rangle$$

$$\leqq \left[\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1\right]\|u\|_{p'}^{q-\varepsilon}\|\mathrm{V}_-\|_{z(\varepsilon)}, \quad (17)$$

where $z(\varepsilon)=k/(1+\varepsilon k/p')$.

According to the lemma 1 $\|\mathrm{V}_-\|_{z(\varepsilon)}=O(\varepsilon^{-1/k})$. On the other hand

$$\frac{\beta(q-\varepsilon)^2}{4(q-\varepsilon-1)}-1=O(\varepsilon).$$

Now assuming that $\varepsilon\downarrow 0$ in (17) and using $k>1$ we obtain

$$(\lambda-\mathrm{C}(\beta))\|u\|_q^q \leqq 0.$$

Then $u\equiv 0$ as $\lambda>\mathrm{C}(\beta)$. ■

Remarks. — 1. The condition $\mathrm{V}_-\in\mathrm{PK}_\beta(-\Delta+\mathrm{V}_+)\cap \mathrm{L}^p_{\mathrm{loc}}$ is necessary in the theorem 2 but it is not sufficient as it may be seen in the example $-\Delta-\dfrac{\kappa}{|x|^2}$ (*see* [11], Chapter X). The same example shows the sharpness of the theorem 2. Inded, let $2p<m\leqq l$,

$$\mathrm{V}(x)=\kappa\chi(x)\left(\sum_{j=1}^m x_j^2\right)^{-1},$$

where $\chi(.)$ is the indicator of the unit ball in $\mathbb{R}^l$. Then the range of the operator $(-\Delta-\mathrm{V}-\lambda)\ \mathrm{C}_0^\infty$ is dense in L^p if and only if

$\kappa \leqq m(m-2p)(p-1)/p^2$ [4]. On the other hand, $V \in L_{comp}^{m/2, \infty}$, *i. e.*, $\kappa = m(m-2p)(p-1)/p^2$ implies $k=(1+\sqrt{1-\beta})\Big/\left(\frac{2}{p}-1+\sqrt{1-\beta}\right)$.

2. The condition $V_- \in PK_\beta(-\Delta+V_+)$ may be replaced by $V_- \in PK_\beta\left(-\Delta+\frac{p^2}{4(p-1)}V_+\right)$.

3. The analysis of the proof shows that the restriction $V_- \in L_{comp}^{k, \infty}$ may be replaced by the weaker condition $\| V_- \|_{k-\varepsilon} = o(\varepsilon)$.

4. The main device in the proof of the theorem 2 is the inequality $\langle -\Delta u, u|u|^{p-2}\rangle \geqq 4\frac{p-1}{p^2}\|\vec{\nabla}|u|^{p/2}\|_2^2$.

N. Th. Varopoulos [12] has proved the abstract version of this inequality for the generators of submarkovian semigroups. Thus our result may be generalized for these operators.

Corollary 3. — *Let*

$$V = V_+ - V_-, \qquad 0 \leqq V_\pm \in L^2_{comp},$$
$$\beta < 1, \qquad V_- \in PK_\beta(-\Delta+V_+) \cap L^{k, \infty}, \qquad k=(1+\sqrt{1-\beta})/\sqrt{1-\beta}.$$

Then $(-\Delta+V) \upharpoonright C_0^\infty$ *is essentially self-adjoint operator in* L^2.

Corollary 3 is proved under the restriction $V \in L^2_{comp}$, but using the results of C. Simader [13] or H. Brezis [14], we have

Theorem 4. — *Let*

$$V = V_+ - V_-, \qquad 0 \leqq V_\pm \in L^2_{loc},$$
$$\beta < 1, \qquad V_- \in PK_\beta(-\Delta+V_+) \cap L_{loc}^{k, \infty}, \qquad K=(1+\sqrt{1-\beta})/\sqrt{1-\beta}.$$

Then $(-\Delta+V)$ C_0^∞ *is essentially self-adjoint.*

Remark. — In the case $l \geqq 5$ theorem 4 is a generalization of the Kalf-Walter-Schmincke-Simon theorem (*see* [11], § X.4).

Example. — We consider the N-particle Hamiltonian

$$H = H_0 + V,$$
$$H_0 = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \frac{1}{2N}\left(\sum_{i=1}^{N}\vec{\nabla}_i\right)^2,$$
$$-\alpha\sum_{i<j}|x_i - x_j|^{-2} \leqq V \in L^2_{loc}(\mathbb{R}^{l(N-1)}),$$
$$x_j \in \mathbb{R}^l, \qquad l \geqq 5.$$

Hardy inequality implies $V_- \in PK_\beta(H_0)$, $\beta = \dfrac{2N\alpha}{(l-2)^2}$. One sees easily that $V_- \in L_{loc}^{l/2,\infty}$. If $\alpha \leqq \dfrac{(l-4)}{2N}$ it follows from theorem 4 that $H \upharpoonright C_0^\infty$ is essentially self-adjoint (*cf.* [15], [16]). Note that for the relativistic Hamiltonian

$$H_R + V_C \equiv \sum_{i=1}^{N} (-\Delta_i + m^2)^{1/2} - \alpha \sum_{i<j} |x_i - x_j|^{-1}, \qquad l=3.$$

Lieb and Thirring [17] have proved that $V_C \in PK_\beta(H_R)$ with $\beta = O(N\alpha)$ and $H_R + V_C$ is unbounded from below for $\alpha = C/N$, $C < \infty$. This correlation between β and N appears to be true for our nonrelativistic Hamiltonian as well so that the dependence $\alpha = O(1/N)$ is optimal.

To prove *m*-accretivity we need.

THEOREM 5 [6]. — *Let*

$$V = V_+ - V_-, \qquad 0 \leqq V_\pm \in L_{loc}^p,$$
$$V_- \in PK_\beta(-\Delta + V_+), \qquad \beta < 1.$$

If $p \in \left[\dfrac{2}{1+\sqrt{1-\beta}}, \dfrac{2}{1-\sqrt{1-\beta}}\right]$, *then the operator* $(-\Delta + V + C(\beta)) \upharpoonright C_0^\infty$ *is accretive operator in* L^p.

Proof. — We will show that

$$[(-\Delta + V + C(\beta))u, u] \geqq 0, \qquad \forall u \in C_0^\infty, \tag{18}$$

where $[v, w]$ is semi-inner product (*see* [11], § X.8). In the case of L^p spaces

$$[v, w] = \langle v, w|w|^{p-2}\rangle / \|w\|^{p-2},$$

so that (18) may be written in the following way

$$\langle V_- u, u|u|^{p-2}\rangle \leqq \langle(-\Delta + V_+ + C(\beta))u, u|u|^{p-2}\rangle.$$

But $V_- \in PK_\beta(-\Delta + V_+)$ so that

$$\begin{aligned}\langle V_- u, u|u|^{p-2}\rangle &= \langle V_- |u|^{p/2}, |u|^{p/2}\rangle \\ &\leqq \beta \|\vec{\nabla}|u|^{p/2}\|_2^2 + \beta\langle V_+ |u|^{p/2}, |u|^{p/2}\rangle + C(\beta)\| |u|^{p/2}\|_2^2 \\ &= \frac{\beta p^2}{4(p-1)}\langle -\Delta u, u|u|^{p-2}\rangle \\ &\qquad + \beta\langle V_+ u, u|u|^{p-2}\rangle + C(\beta)\langle u, u|u|^{p-2}\rangle.\end{aligned}$$

Since $\dfrac{\beta p^2}{4(p-1)} \leqq 1$ provided that $p \in \left[\dfrac{2}{1+\sqrt{1-\beta}}, \dfrac{2}{1-\sqrt{1-\beta}}\right]$ we have proved (18). ■

Remark. — 1. The sharpness of the result is shown in [6] by the example $-\Delta - \dfrac{\kappa}{|x|^2}$.

2. T. Kato [7] raised the problem of quasi-accretivity of the operator $-\Delta + V$ in the limiting case $p=1$ for the potentials such that

$$\lim_{\alpha \downarrow 0} \sup_x \int_{|x-y| \leq \alpha} |x-y|^{-l+2} |V(y)| d^l y = 0, \qquad l \geqq 3.$$

We will show that the answer is negative, *i. e.*, if $0 \leqq V \in L^1_{loc}$ and if $-\Delta - V$ is quasi-accretive in L^1, then $V \in L^\infty$. Indeed, the semi-inner product in L^1 is $[v, w] = \left\langle v, \dfrac{w}{|w|} \right\rangle \| w \|_1$. The quasi-accretivity implies that

$$\left\langle (-\Delta - V) u, \frac{u}{|u|} \right\rangle \geqq -C \| u \|_1,$$
$$C < \infty, \qquad \forall u \in C_0^\infty.$$

Note that $\langle \Delta u, 1 \rangle = 0$, $\forall u \in C_0^\infty$. Therefore, $\langle V, u \rangle \leqq C \| u \|_1$, *i. e.* $V \in L^\infty$.

3. If $0 \leqq V \notin L^\infty$, $V \in L^p$, $p > l/2$, $l \geqq 3$, then $-\Delta - V$ is the generator of the positivity preserving C_0-semigroup in L^1 which is not quasi-contractive (other examples may be found in [18], [6]).

Theorems 2, 5 and Lumer-Phillips theorem (*see* [11], § X.8) imply

THEOREM 6. — *Let*

$$\beta < 1, \qquad p \in \left(\frac{2}{1+\sqrt{1-\beta}}, \frac{2}{1-\sqrt{1-\beta}} \right),$$
$$V = V_+ - V_-, \qquad 0 \leqq V_\pm \in L^p_{comp},$$
$$V_- \in PK_\beta(-\Delta + V_+) \cap L^{k,\infty}$$

where $k = (1+\sqrt{1-\beta}) \Big/ \left(\dfrac{2}{p} - 1 + \sqrt{1-\beta} \right)$. *Then the closure of the operator* $(-\Delta + V + C(\beta)) \restriction C_0^\infty$ *is the generator of* C_0*-semigroup of contractions in* L^p.

Remarks. — 1. Theorem 6 is generalization (when supp V is compact) of the result stated in [4]. It is of interest to prove the L^p-version of the result of C. Simader [13] and H. Brezis [14].

2. Having completed this paper the author was informed by Yu. A. Semenov about the proof of the theorem 6 provided $0 \leqq V_\pm \in L^p_{loc}$, $p \in \left(\dfrac{2}{1+\sqrt{1-\beta}}, \dfrac{2}{1-\sqrt{1-\beta}} \right)$.

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