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The Ising transducer

by

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ABSTRACT. – The object of this article is to study the transducer which describes the fundamental state of the Ising chain with nonconstant interaction between neighbouring sites.

RÉSUMÉ. – Dans cet article nous étudions le transducteur qui décrit l'état fondamental de la chaîne d'Ising avec interactions non constantes entre plus proches voisins.

1. PHYSICAL BACKGROUND

Let us recall that the Ising chain with varying interaction is governed by the (cyclic) Hamiltonian

$$\mathscr{H}_N(\sigma) = -J \sum_{n=0}^{N-1} \varepsilon_n \sigma_n \sigma_{n+1} - H \sum_{n=0}^{N-1} \sigma_n$$

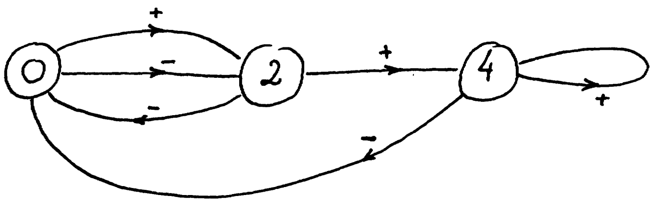
where the spins σ_n take values ± 1, $\sigma_N = \sigma_0$ and where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{N-1}$ is a given sequence of ± 1. As usual, J represents the coupling constant which we can assume to be strictly positive (the case $J=0$ is trivial and the case $J<0$ reduces to $J>0$ by changing ε to $-\varepsilon$); H is the external field.

We are interested in the fundamental state *i. e.* 0 degree.

In several earlier papers (*see* for example [1]), it is argued that as the number of sites N increases to infinity, the field δ_n induced at site n by the spins $\sigma_{n-1}, \sigma_{n-2}, \ldots$ is obtained by the induction formula

$$\delta_n = 2\,\mathrm{H}/\mathrm{J} + \varepsilon_{n-1}\,\mathrm{sgn}(\delta_{n-1}) \min\{2, |\delta_{n-1}|\}. \tag{1}$$

δ_0 is arbitrary but, as it will be shown later, there is no loss of generality in assuming $\delta_0 = 0$. The sequence $\delta = (\delta_n)$ depends on the value $\alpha = 2\,\mathrm{H}/\mathrm{J}$ and on the sequence $\varepsilon = (\varepsilon_n)$. For example, choose $\alpha = 2$ (H=J). Then formula (1) can be represented by the following transducer (in paragraph 2 we give a general definition of a transducer).



It consists of three states 0, 2, 4 which are the values taken by δ_n. Each state is linked to two states (not necessarily distinct) by two arrows $\pm$. If for example $\delta_{n-1} = 4$ and $\varepsilon_{n-1} = +1$ then $\delta_n = 4$ (follow arrow + from the state 4). If $\varepsilon_{n-1} = -1$ then $\delta_n = 0$ (follow arrow −). In other words, if at some stage we are in the state δ_{n-1} then the arrow ε_{n-1} leads to δ_n.

Actually, the sequence δ only describes half of the field at site n. One should also take in account the action of the spins $\sigma_{n+1}, \sigma_{n+2}, \ldots$ so that the induced field $\hat{\delta}_n$ at site n depends on sites $n-1$ and $n+1$.

We shall not however study this two sided dependence which has been discussed in another context by Derrida, Gardner [3] and Derrida, Vannimenus, Pomeau [4]. We shall limit our discussion to the one sided action (1) which is interesting in itself and which does describe to some extent the fundamental state of the Ising chain.

Let us fix $\alpha = 2\,\mathrm{H}/\mathrm{J}$. Formula (1) transforms the infinite sequence $\varepsilon = (\varepsilon_n) \in \{-1+1\}^{\mathbb{N}}$ into a sequence $\delta = (\delta_n)$ which can easily be seen to be defined on a finite alphabet $\mathrm{S} = \mathrm{S}_\alpha = \mathrm{T} \cap [\alpha - 2, \alpha + 2]$ where

$$\mathrm{T} = \{ n\alpha \mid n \in \mathbb{Z} \} \cup \{ n\alpha - 2 \mid n \in \mathbb{Z} \}.$$

The map $\varepsilon \to \delta$ can be represented by a transducer of the above type: the states are the values taken by δ_n, $n = 0, 1, 2, \ldots$ and the arrows are $\pm$. We propose to name it the Ising transducer. We use the notation $\mathscr{I}_\alpha$. Hence $\delta = \mathscr{I}_\alpha(\varepsilon)$.

If the Ising chain is a reasonably good model, then for large n, δ_n should be independent of the field at sites 0, 1, 2, ..., n_0 where n_0 is "small" compared with n. More precisely, two infinite sequences $\varphi = (\varphi_n)$ and

$\varphi'=(\varphi'_n)$ are said to be equivalent ($\varphi\sim\varphi'$) if for all large indices n, $\varphi_n=\varphi'_n$. We should then expect that

$$\varepsilon\sim\varepsilon' \quad\Rightarrow\quad \mathcal{I}_\alpha(\varepsilon)\sim\mathcal{I}_\alpha(\varepsilon').$$

Unfortunately this is not true as it stands. What is however true is the weaker statement according to which *for almost all* ε the above implication holds.

This result is still reasonable on physical grounds because ε can be thought of as a distribution of impurities. It is not deterministic. The "almost all" statement implies randomness.

For special improbable sequences ε, it may well happen that

$$\varepsilon\sim\varepsilon' \quad\not\Rightarrow\quad \mathcal{I}_\alpha(\varepsilon)\sim\mathcal{I}_\alpha(\varepsilon').$$

In other terms, for some highly organized sequences ε, there may well be two or more fundamental states (compare with the above mentioned authors and with Luck [5]). The situation at temperature $T=0$ could thus be quite complex. On the contrary, if ε is random, *i. e.* complex, then, as we shall see

$$\varepsilon\sim\varepsilon' \quad\Rightarrow\quad \mathcal{I}_\alpha(\varepsilon)\sim\mathcal{I}_\alpha(\varepsilon').$$

It is worthwhile underlining this fact. In our context, randomness can be thought of as simpler than order.

2. FAITHFUL TRANSDUCERS

In order to study the special case of the Ising transducer we shall establish a simple result concerning the general theory of transducers. (Computer scientists [2] usually consider more general transducers than the ones we define.)

A transducer is a triple $\mathcal{I}=(S, f, A)$ where S and A are two alphabets and where f is a map $S\times A\rightarrow S$. To avoid triviality we assume that S and A have both more than two elements. Elements of S are called states.

Let $A^*=\bigcup_{n=0}^{\infty} A^n$ be the set of words on the alphabet A.

Consider $\varepsilon=(\varepsilon_0, \varepsilon_1, \ldots)\in A^*\cup A^{\mathbb{N}}$ and let $s\in S$ be a state. Define

$$\begin{gathered}\delta_0=s\\ \delta_1=s(\varepsilon_0)=f(s, \varepsilon_0)\\ \delta_2=s(\varepsilon_0\,\varepsilon_1)=f(f(s, \varepsilon_0), \varepsilon_1)\\ \delta_3=s(\varepsilon_0\,\varepsilon_1\,\varepsilon_2)=f(f(f(s, \varepsilon_0), \varepsilon_1), \varepsilon_2)\end{gathered}$$

etc.

The sequence $\delta=(\delta_n)$, denoted $\delta=\mathscr{I}(\varepsilon, s)$, is the transduced sequence of ε. In paragraph 1 we have seen an example where $S=\{0,2,4\}$ and $A=\{+1,-1\}$. In the appendix we have drawn two other Ising transducers $\mathscr{I}_\alpha$, $\alpha=1$ and $\alpha\in]1,4/3[$.

The set $A^{\mathbb{N}}$ is endowed with a natural measure (Bernoulli measure) so that it makes sense to speak of "almost all" sequences ε. We propose to call a transudcer $\mathscr{I}$ *faithful* if for almost all $\varepsilon\in A^{\mathbb{N}}$ and for all initial states $s\in S$, $t\in S$,

$$\varepsilon'\sim\varepsilon \quad\Rightarrow\quad \mathscr{I}(\varepsilon, s)\sim\mathscr{I}(\varepsilon', t).$$

Our first result gives a characterization of faithful transducers.

THEOREM. — *A transducer $\mathscr{I}$ is faithful if and only if there exist a state $q\in S$ and a word $w\in A^*$ such that for all $s\in S$, $s(w)=q$.*

Remark. — If a couple (q, w) exists, then for all $u\in A^*$, the couple $(q(u), wu)$ serves the same purpose.

Proof of the Theorem. — Suppose there exists a state q and a word w such that for all $s\in S$, $s(w)=q$. A classic result of E. Borel asserts that almost all $\varepsilon\in A^{\mathbb{N}}$ contain the factor w infinitely many times. Let $\varepsilon'\sim\varepsilon$. Then ε' must necessarily contain factors w at the same rank as corresponding w factors in ε, and indeed infinitely often. Let n be a large number. Then $\varepsilon_0\varepsilon_1\ldots\varepsilon_n$ and $\varepsilon'_0\varepsilon'_1\ldots\varepsilon'_n$ can be written as

$$\varepsilon_0\varepsilon_1\ldots\varepsilon_n=uwv$$
$$\varepsilon'_0\varepsilon'_1\ldots\varepsilon'_n=u'wv$$

where u, u' and v are words, u and u' having the same length. Let s and s' be two states. Then

$$s(\varepsilon_0\varepsilon_1\ldots\varepsilon_n)=s(uwv)=t(wv)=t(w)(v)$$
$$s'(\varepsilon'_0\varepsilon'_1\ldots\varepsilon'_n)=s'(u'wv)=t'(wv)=t'(w)(v)$$

where t and t' are defined by $t=s(u)$, $t'=s'(u')$.

By hypothesis, $t(w)=t'(w)=q$ hence

$$s(\varepsilon_0\varepsilon_1\ldots\varepsilon_n)=s'(\varepsilon'_0\varepsilon'_1\ldots\varepsilon'_n)$$

for all large n. In other words

$$\mathscr{I}(\varepsilon, s)\sim\mathscr{I}(\varepsilon', s').$$

We now prove the converse. Assume that for at least *one* ε, and all s and s'

$$\varepsilon'\sim\varepsilon \quad\Rightarrow\quad \mathscr{I}(\varepsilon', s')\sim\mathscr{I}(\varepsilon, s).$$

By changing the origin we can replace the condition $\varepsilon'\sim\varepsilon$ by $\varepsilon'=\varepsilon$. Put

$$\mathscr{I}(\varepsilon, s)=\delta=(\delta_n)$$
$$\mathscr{I}(\varepsilon', s')=\delta'=(\delta_n(s')).$$

There exists a rank $n(s')$ such that

$$n \geqq n(s') \quad \Rightarrow \quad \delta_n(s') = \delta_n.$$

Define

$$n_0 = \max_{s' \in S} n(s').$$

The finiteness of S implies that of n_0. Consider the word

$$w = \varepsilon_0 \varepsilon_1 \dots \varepsilon_{n_0 - 1} \in A^*$$

Then

$$s'(w) = \delta_{n_0}(s') = \delta_{n_0} \text{ independent of } s'.$$

We have thus found a state $q = \delta_{n_0}$ and a word w such that

$$s'(w) = q \quad \text{for all } s' \in S.$$

Q.E.D.

COROLLARY. – *If the external field* H *is nonzero* ($\alpha \neq 0$), *then the Ising transducer* $\mathscr{I}_\alpha$ *is faithful.*

Proof. – We can always suppose $\alpha = 2H/J > 0$ with no loss of generality; indeed if $\alpha < 0$, change δ into $-\delta$. Let $s \in S$. Then $s(1^n)$ increases strictly as long as $s(1^n) < 2 + \alpha$. For some integer μ and for all $s \in S$, $s(1^\mu) = 2 + \alpha$ so that eventually $s(1^n)$ attains $2 + \alpha$ and then stabilizes. The couple $q = 2 + \alpha$ and $w = 1^\mu$ fulfills the condition of the Theorem hence $\mathscr{I}_\alpha$ is faithful.

Q.E.D.

Remark 1. – The proof of the Theorem shows that a transducer $\mathscr{I}$ is faithful if and only if there exists at least one ε such that for all states s and s'

$$\varepsilon' \sim \varepsilon \quad \Rightarrow \quad \mathscr{I}(\varepsilon', s') \sim \mathscr{I}(\delta, s).$$

Remark 2. – If $H = 0$ ($\alpha = 0$), then equation (1) reduces to $\delta_n = \varepsilon_{n-1} \delta_{n-1}$ provided $|\delta_{n-1}| \leqq 2$, hence

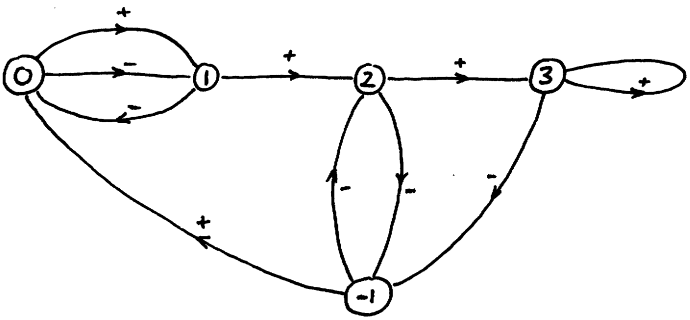
$$\delta_n = \delta_0 \varepsilon_0 \varepsilon_1 \varepsilon_2 \dots \varepsilon_{n-1}.$$

The initial value δ_0 plays here an important rôle. Two different values of δ_0 imply two different sequences δ. The condition $H \neq 0$ in the corollary is thus essential.

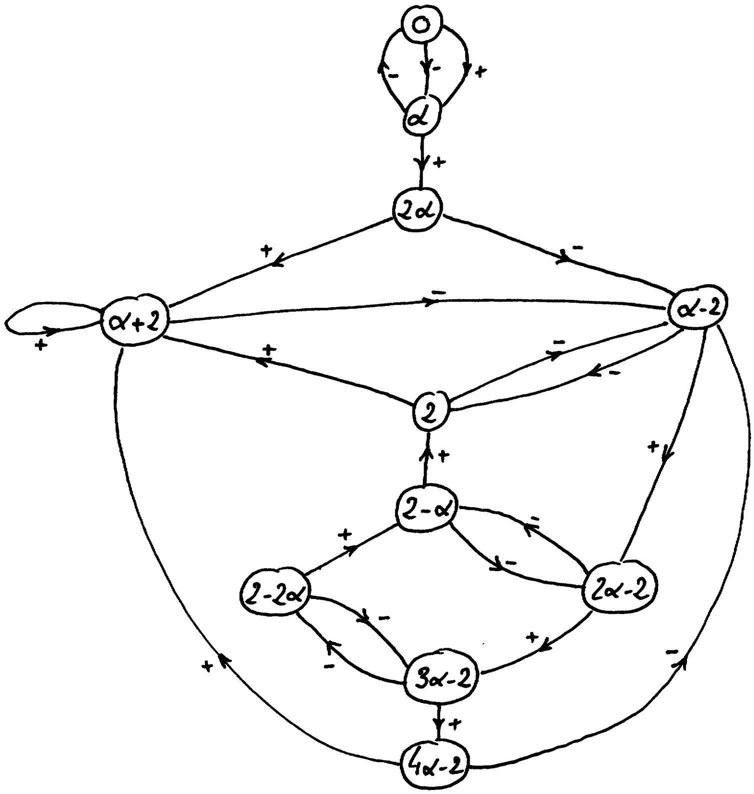
3. APPENDIX

The following graphs represent respectively the Ising transducers $\mathscr{I}_\alpha$ with $\alpha = 1$ and $1 < \alpha < 4/3$. Even though the two α's can have aproximatively the same value, the transducers are quite different.

$\alpha = 1$



$1 < \alpha < 4/3$



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