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Singularities of the scattering kernel for generic obstacles

by

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $n$ odd, be an open connected domain with $C^\infty$ smooth boundary $\partial \Omega$ and bounded complement

$$K = \mathbb{R}^n \setminus \Omega \subset \{ x : |x| \leq \rho_0 \}.$$ 

The scattering kernel $s(t, \theta, \omega)$ related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial \Omega$ has the form (see [8])

$$s(t, \theta, \omega) = C_n \int_{\partial K} \partial_t^{-2} \partial_\nu w (\langle x, \theta > - t, x; \omega) \, d\mathcal{S}_x.$$ (1.1)

Here $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, $w(\tau, x; \omega)$ is the solution of the problem

$$(\partial_t^2 - \Delta_x) w = 0 \quad \text{in } \mathbb{R} \times \Omega,$$
$$w = 0 \quad \text{on } \mathbb{R} \times \partial \Omega,$$
$$w|_{\tau = -\rho_0} = \delta (\tau - \langle x, \omega >),$$ (1.2)

$v$ is the interior unit normal to $\partial \Omega$ pointing into $\Omega$, $d\mathcal{S}_x$ is the measure induced on $\partial \Omega$, $C_n = (-1)^{(n+1)/2} 2^{-n} \pi^{(1-n)}$ and $\langle , \rangle$ is the inner product in $\mathbb{R}^n$. 

Annales de l'Institut Henri Poincaré - Physique théorique - 0246-0211
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For fixed $\omega$, $\theta$ we have $s(t, \theta, \omega) \in \mathcal{S}'(\mathbb{R})$. The analysis of the singularities of $s(t, \theta, \omega)$ for fixed $\omega$, $\theta$ is important for some inverse scattering problems.

The aim of this paper is to study $\text{sing supp } s(t, \theta, \omega)$ for general (nonconvex) obstacles.

By a reflecting $(\omega, \theta)$-ray in $\Omega$ we mean a continuous curve in $\Omega$ formed by a finite number of linear segments and two infinite linear segments – an incoming one with direction $\omega$ and an outgoing one with direction $\theta$ (cf. section 2 for a precise definition). If a reflecting $(\omega, \theta)$-ray $\gamma$ in $\Omega$ has no segments tangent to $\partial\Omega$, then $\gamma$ will be called ordinary.

By a generalized $(\omega, \theta)$-ray we mean an infinite continuous curve $\gamma$ in $\Omega$ incoming with direction $\omega$ and outgoing with direction $\theta$ which is a projection on $\Omega$ of a generalized bicharacteristic of the wave operator $\square = \partial^2_t - \Delta$ (cf. [9]) and which contains at least one gliding segment which is a geodesic on $\partial\Omega$ with respect to the standard Riemannian metric. Finally, by a $(\omega, \theta)$-ray we mean either a reflecting or a generalized $(\omega, \theta)$-ray. Throughout this paper we consider only null bicharacteristics of $\square$, i.e. bicharacteristics lying in the characteristic set $\Sigma$ of $\square$ (see [9]).

For fixed $\omega$, $\theta$ we denote by $\mathcal{L}_{\omega, \theta}$ the set of all $(\omega, \theta)$-rays. For $\gamma \in \mathcal{L}_{\omega, \theta}$ consider the sojourn time $T_\gamma$ of $\gamma$ (see section 2 for a definition). As it was suggested in [4], [11], the singularities of $s(t, \theta, \omega)$ are related to the sojourn times of the $(\omega, \theta)$-rays. In [11], [16], [17] for some special classes of obstacles all singularities of $s(t, \theta, \omega)$ have been examined.

According to the geometry of the generalized bicharacteristics of $\square$ (see [5], [22]), there could be some points on $T^*(\partial\Omega \times \mathbb{R})$ such that there are more than one generalized bicharacteristic passing through them. We shall say that a generalized bicharacteristic $\delta$ of $\square$ is uniquely extendible if for every $z \in \delta$ the only generalized bicharacteristic of $\square$ passing through $z$ is $\delta$. A $(\omega, \theta)$-ray $\gamma$ in $\Omega$ will be called uniquely extendible if $\gamma$ is a projection on $\Omega$ of a uniquely extendible bicharacteristic.

Note that if $K$ is convex or $K$ has a real analytic boundary, then every $(\omega, \theta)$-ray in $\Omega$ is uniquely extendible. The same is true if $\partial\Omega$ has no points where the curvature of $\partial\Omega$ vanishes of infinite order along some direction. Another example is the case when $K$ is a finite union of disjoint convex obstacles. We refer to [22] for an example when there exists a bicharacteristic which is not uniquely extendible.

Let $Z_1$ be a hyperplane in $\mathbb{R}^n$ orthogonal to $\omega$ and such that the open halfspace, determined by $Z_1$ and having $\omega$ as an inward normal, contains $\partial\Omega$. Given $u \in Z_1$, put $\rho_u = (-\rho_0, u, 1, -\omega) \in T^*(\mathbb{R} \times \Omega)$. Denote by $C_t(u)$ the set of those $z \in T^*(\mathbb{R} \times \Omega)$ such that there exists a generalized bicharacteristic $\gamma(\sigma)$ of $\square$ with $\gamma(-\rho_0) = \rho_u$, $\gamma(t) = z$. For $V \subset Z_1$ set

$$C_t(V) = \bigcup_{u \in V} C_t(u).$$
Our first result is the following.

**Theorem 1.** Let \( \theta \neq \omega \) be fixed. Assume that every \((\omega, \theta)\)-ray in \( \Omega \) is uniquely extendible. Then

\[
\text{sing supp } s(t, \theta, \omega) \subset \{ -T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta} \}.
\]  

**Remark 1.1.** The assumption of Theorem 1 concerns only the \((\omega, \theta)\)-rays. Thus for fixed \( \omega \) the relation (1.3) shows that if \( K \) is connected, then the shadow of \( K \) with respect to \( \omega \) does not contribute to \( \text{sing supp } s(t, \theta, \omega) \). If we make some observations with rays incoming with direction \( \omega \), then note that some bicharacteristics of \( \Box \) which are not related to \((\omega, \theta)\)-rays can be not uniquely extendible.

**Remark 1.2.** The assumption of Theorem 1 is satisfied also for \((-\theta, -\omega)\)-rays. This agrees with the relation \( s(t, -\omega, -\theta) = s(t, \theta, \omega) \).

**Remark 1.3.** Under stronger assumptions concerning the rays incoming with directions \( \pm \omega \), the relation (1.3) was examined in [11]. The inclusion (1.3) is similar to the Poisson relation for the distribution

\[
\sigma(t) = \sum_{j=1}^{\infty} \cos \lambda_j t, \text{ where } \{ \lambda_j^2 \}_{j=1}^{\infty} \text{ is the spectrum of the Laplace operator in a bounded domain with smooth boundary (see [1], [13]).}
\]

From physical point of view it is more interesting to study the obstacles for which (1.3) becomes an equality. This makes it possible to recover all singularities of \( s(t, \theta, \omega) \) and to consider them as scattering data (see [16] for a result in this direction). One way to attack this problem is to fix \( \theta \neq \omega \) and to consider generic obstacles. We follow this way in the present paper and show that generically for some ordinary \((\omega, \theta)\)-rays \( \gamma \) we have

\[
- T_\gamma \in \text{sing supp } s(t, \theta, \omega). 
\]  

(1.4)

Recently, one of the authors [21] proved that for generic obstacles in \( \mathbb{R}^3 \), (1.4) holds for any \((\omega, \theta)\)-ray \( \gamma \). The proof of this result is based on Theorem 2 stated below and the fact that for fixed \( \omega \neq \theta \) and generic obstacles \( K \) in \( \mathbb{R}^3 \) there are no generalized \((\omega, \theta)\)-rays in the complement of \( K \).

Another way to study (1.3) is to fix \( K \) and \( \omega \) and to consider generic directions \( \theta \). For some obstacles \( K \) it is known (see [16], [12]) that for every fixed \( \omega \in S^{n-1} \) there exists a residual subset \( \mathcal{R}(\omega) \) of \( S^{n-1} \) such that for every \( \theta \in \mathcal{R}(\omega) \) all \((\omega, \theta)\)-rays in \( \mathbb{R}^n \setminus K \) are ordinary. For such directions we can apply Theorem 1 and obtain (1.4) for all \((\omega, \theta)\)-rays. We conjecture that for each obstacle and each fixed \( \omega \) it is possible to find a residual subset \( \mathcal{R}(\omega) \) with the properties mentioned above.

To state our second result we need some notations.

Let \( X = \partial \mathbb{R}^n \) and let \( C^\infty(X, \mathbb{R}^n) \) be the space of all \( C^\infty \) maps of \( X \) into \( \mathbb{R}^n \) endowed with the Whitney \( C^\infty \) topology (cf. [3], ch. II). The subspace
\( C^\infty_{emb}(X, \mathbb{R}^n) \) of all \( C^\infty \) embeddings is open in \( C^\infty(X, \mathbb{R}^n) \), hence it is Baire space. A subset \( \mathcal{R} \) of a topological space \( Z \) is called residual if \( \mathcal{R} \) is a countable intersection of open dense subsets of \( Z \).

Given \( f \in C^\infty_{emb}(X, \mathbb{R}^n) \), denote by \( \Omega_f \) the unbounded domain with boundary \( f(X) \) and by \( \mathcal{L}_{\omega, \theta, f} \) the set of all \((\omega, \theta)\)-rays in \( \Omega_f \). Let \( \mathcal{L}_{\omega, \theta, f}^R \) (resp. \( \mathcal{L}_{\omega, \theta, f}^{g} \)) be the set of all ordinary (resp. generalized) \((\omega, \theta)\)-rays in \( \Omega_f \). The results of section 4, combined with those in [14], [15], imply the existence of a residual subset \( \mathcal{R} \) of \( C^\infty_{emb}(X, \mathbb{R}^n) \) such that for each \( f \in \mathcal{R} \) we have

\[
\mathcal{L}_{\omega, \theta, f} = \mathcal{L}_{\omega, \theta, f}^R \cup \mathcal{L}_{\omega, \theta, f}^{g}.
\]

In particular, if \( \mathcal{L}_{\omega, \theta, f}^R = \emptyset \), then every \((\omega, \theta)\)-ray is an ordinary one.

If \( \gamma \) is an ordinary \((\omega, \theta)\)-ray, we denote by \( x_\gamma \) (resp. \( y_\gamma \)) the first (resp. the last) reflection point of \( \gamma \). Let \( m_\gamma \) be the number of reflections of \( \gamma \) and let \( dJ_\gamma(u_\gamma) \) be the differential of the map \( J_\gamma \) introduced in section 2. Here \( u_\gamma \) is the orthogonal projection of \( x_\gamma \) on \( Z_1 \). Finally, set

\[
\mathcal{G}_f = \{ T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta, f}^R \}.
\]

Our second result is the following.

**Theorem 2.** Let \( \theta \neq \omega \) be fixed. Then there exists a residual subset \( \mathcal{A} \) of \( C^\infty_{emb}(X, \mathbb{R}) \) such that for each \( f \in \mathcal{A} \)

\[
\{ -T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta, f}, T_\gamma \notin \mathcal{G}_f \} \subset \text{sing supp } s_f(t, \theta, \omega)
\]

holds, where \( s_f(t, \theta, \omega) \) is the scattering kernel related to \( \Omega_f \). Moreover, for \( t \) sufficiently close to \(-T_\gamma \) with \( \gamma \in \mathcal{L}_{\omega, \theta, f}, T_\gamma \notin \mathcal{G}_f \), we have

\[
s_f(t, \theta, \omega) = C \left| \frac{\det dJ_\gamma(u_\gamma)}{\langle v(x_\gamma), \omega \rangle} \right|^{-1/2} \delta^{(n-1)/2}(t + T_\gamma)
\]

+ smoother terms,

where \( C = (2\pi)^{(1-n)/2}(-1)^{m_\gamma-1} \pi^{n-1} \) and \( \sigma_\gamma \in \mathbb{N} \) is related to a Maslov index.

For the proof of Theorem 1 we use the results in [9] for propagation of \( C^\infty \) singularities. The crucial point is the application of Proposition 3.1, where we generalize an idea used previously in [11].

Given \( \rho(t + t_0) \in C^\infty_0(\mathbb{R}^n) \) with support in a small neighbourhood of \(-t_0\), we need to examine the asymptotic of

\[
I(\lambda) = (s(t, \theta, \omega), \rho(t + t_0) e^{-i\lambda t}).
\]

The results for propagation of singularities of the solution of (1.2) are not sufficient since some critical points of the phase of \( I(\lambda) \) make contributions which must be cancelled from physical point of view. Thus we are going to use a stationary approach connected with the \((i\lambda)\)-outgoing Green function.

The proof of Theorem 2 is based essentially on some generic properties of \((\omega, \theta)\)-rays with linear segments. These properties are obtained in section 4 following the approach in [13], [19]. Some of these properties have been
previously announced in [20], [15]. The formula (1.6) has been obtained in [11].

The paper is organized as follows. In section 2 we collect some notations and definitions. Theorem 1 is proved in section 3. In section 4 we consider several generic properties of reflecting \((\omega, \theta)\)-rays and prove Theorem 2.

2. PRELIMINARIES

2.1. By a segment in \(\mathbb{R}^n\) we mean either a finite segment \([x, y]\) or an infinite one, that is a straightline ray starting at some point and having a given direction.

Let \(X\) be a smooth compact \((n-1)\)-dimensional submanifold of \(\mathbb{R}^n\), \(n \geq 2\). If \(l_1\) and \(l_2\) are two segments in \(\mathbb{R}^n\) with a common end \(x \in X\), we say that \(l_1\) and \(l_2\) satisfy the law of reflection at \(x\) (with respect to \(X\)) if \(l_1\) and \(l_2\) make equal acute angles with a normal vector \(v_x \neq 0\) to \(X\) at \(x\) and \(l_1\), \(l_2\) and \(v_x\) lie in a common two-dimensional plane.

2.2. DEFINITION. - Let \(\omega\) and \(\theta\) be two fixed unit vectors in \(\mathbb{R}^n\).

Consider a curve \(y = \bigcup_{i=0}^k l_i\), where \(l_i = [x_i, x_{i+1}]\) are finite segments for \(i = 1, \ldots, k-1\) \((k \geq 1\), \(x_i \in X\) for all \(i\), \(l_0\) (resp. \(l_k\)) is the infinite segment starting at \(x_1\) (resp. \(x_k\)) and having direction \(-\omega\) (resp. \(\theta\)). Then the curve \(y\) is called a reflecting \((\omega, \theta)\)-ray on \(X\) if the following conditions are satisfied:

(i) the open segments \(l_i\) do not intersect transversally \(X\);
(ii) either \(l_i \cap l_{i+1} = \{x_{i+1}\}\) for every \(i = 0, 1, \ldots, k-1\) or \(k = 2m + 1\) \((m = 0, 1, \ldots)\), \(l_i \cap l_{i+1} = \{x_{i+1}\}\) for \(i = 0, 1, \ldots, m\) and \(l_{m-i} = l_{m+1-i}\) for \(i = 0, 1, \ldots, m\);
(iii) for every \(i\) the segments \(l_i\) and \(l_{i+1}\) satisfy the law of reflection at \(x_{i+1}\) with respect to \(X\).

The points \(x_1, \ldots, x_k\) will be called reflection points of \(y\). If \(y\) is of the same form and has the above properties except (i) for \(i = k\), we shall say that \(y\) is a \((\omega, \theta)\)-trajectory on \(X\). Note that every reflecting \((\omega, \theta)\)-ray is a \((\omega, \theta)\)-trajectory, but the converse is not true in general since the last segment (which is infinite and has direction \(\theta\)) of a \((\omega, \theta)\)-trajectory could intersect \(X\). Mention also that the second part of (ii) is only possible for \(\theta = -\omega\).

2.3. Suppose \(\mathcal{R} \subset C^\infty_{\text{emb}}(X, \mathbb{R}^n)\) and \((U_k)_{k=1}^\infty\) is a sequence of open subsets of \(\mathbb{R}^n\) with \(\bigcup_k U_k = \mathbb{R}^n\) and \(U_k \supset X\) for every \(k\). Assume in addition that \(\mathcal{R}\) contains a residual subset of \(C^\infty_{\text{emb}}(X, U_k)\) for every \(k\). Then it is easily seen that \(\mathcal{R}\) contains a residual subset of \(C^\infty_{\text{emb}}(X, \mathbb{R}^n)\).
2.4. Let $\omega, \theta \in S^{n-1}$ be fixed and $U_0$ be an open ball with radii $a > 0$ containing $X$. Let $Z_1$ and $Z_2$ be the hyperplanes tangent to $U_0$ such that $Z_1$ (resp. $Z_2$) is orthogonal to $\omega$ (resp. $\theta$) and the halfspace $H_1$ (resp. $H_2$), determined by $Z_1$ and $\omega$ (resp. by $Z_2$ and $-\theta$) contains $U_0$. Given a reflecting $(\omega, \theta)$-ray $\gamma$ on $X$ with successive reflection points $x_1, \ldots, x_k$, the sojourn time $T_{\gamma}$ of $\gamma$ (cf. Guillemin [4]) is defined by

$$T_{\gamma} = \|\pi_1(x_1) - x_1\| + \sum_{i=1}^{k-1} \|x_i - x_{i+1}\| + \|x_k - \pi_2(x_k)\| - 2a,$$

where $\pi_i : \mathbb{R}^n \to Z_i$ are the orthogonal projections. Clearly, $T_{\gamma} + 2a$ is the length of this part of $\gamma$ which lies in $H_1 \cap H_2$. We define $T_{\gamma}$ when $\gamma$ is a $(\omega, \theta)$-trajectory or a generalized $(\omega, \theta)$-ray so that $T_{\gamma} + 2a$ is the length of this part of $\gamma$ which lies in $H_1 \cap H_2$. It is known [4] that the definition of $T_{\gamma}$ does not depend on the choice of the ball $U_0$. Set $u_\gamma = \pi_1(x_1)$ and assume that $\gamma$ is a $(\omega, \theta)$-trajectory which has no segments tangent to $X$. Then there exists a neighbourhood $W_{\gamma}$ of $u_\gamma$ in $Z_1$ such that for every $u \in W_{\gamma}$ there are unique $\theta(u) \in S^{n-1}$ and points $x_1(u), \ldots, x_k(u) \in X$ which are the successive reflection points of a $(\omega, \theta(u))$-trajectory on $X$ with $\pi_1(x_1(u)) = u$. We set $J_{\gamma}(u) = \theta(u)$, thus obtaining a map

$$J_{\gamma} : W_{\gamma} \to S^{n-1}.$$ This map was also introduced by Guillemin [4].

Given a set $A$ and an integer $s \geq 2$, we set

$$A^{(s)} = \{(a_1, \ldots, a_s) \in A^s : a_i \neq a_j \text{ whenever } i \neq j\}.$$ If $f : X \to Y$ is a map, by $f^s : X^s \to Y^s$ we denote the map given by $f^s(x_1, \ldots, x_s) = (f(x_1), \ldots, f(x_s))$.

### 3. SINGULARITIES OF THE SCATTERING KERNEL

Let $\rho(t) \in C_0^\infty(\mathbb{R})$, $\text{supp} \rho \subset (-1, 1)$, $\rho(t) = 1$ for $|t| \leq 1/2$. Set $\rho_\delta(t) = \rho(t/\delta)$, $0 < \delta \leq 1.1$ Let $v \in \mathcal{D}'(\mathbb{R} \times \Omega)$ be the solution of the problem

$$\square v = F \quad \text{in } \mathbb{R} \times \Omega,$$

$$v = h \quad \text{on } \mathbb{R} \times \partial \Omega,$$

$$v|_{t = 0} = 0,$$

where $\tau < -\rho_0$ is fixed. Here $F \in C^\infty(\mathbb{R}^s \times \mathbb{R}_x^s)$, $h \in H^s_{\text{loc}}(\mathbb{R} \times \partial \Omega)$ with some $s < 0$ and $F = 0$, $h = 0$ for $t < \tau$. By $\mathcal{D}'(\mathbb{R} \times \Omega)$ we denote the space of all distributions in $\mathbb{R} \times \Omega$ admitting extensions as distributions on $\mathbb{R} \times \mathbb{R}_x$. 

*Annales de l'Institut Henri Poincaré - Physique théorique*
Then the traces \( \frac{\partial^j v}{\partial v^j} \bigg|_{\mathbb{R} \times \partial \Omega} \), \( j = 0, 1 \), exist since \( \mathbb{R} \times \partial \Omega \) is non-characteristic for \( \square \) (see [5]). Let

\[ T_1 = \sup \left\{ t : t \leq \rho_0 + |t_0| + \delta, \text{ there exists } y \in \partial K \text{ with} \right\} \]

\( (t, y) \in (\text{sing supp } h) \cup (\text{sing supp } \left( \frac{\partial v}{\partial v} \bigg|_{\mathbb{R} \times \partial K} \right)) \).

Consider the integral

\[ I(\lambda) = \int_{\mathbb{R}} \int_{\partial K} e^{\lambda (t - \langle y, \theta \rangle)} \rho_0 (\langle y, \theta \rangle - t + t_0) \left( \frac{\partial}{\partial v} - \langle v, \theta \rangle \frac{\partial}{\partial \theta} \right) v \, dt \, dS_y. \]

For the proof of Theorem 1 we need the following

**Proposition 3.1.** \( \) Assume that for some \( \varepsilon, 0 < \varepsilon \leq 1 \), we have

\[ \text{WF} (\tau) \cap \left\{ (t, y, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : T_1 + \varepsilon \leq t \leq T_1 + 2 \varepsilon, \right\}
\]

\[ |y| \leq \tau_1 + T_1 + 2 \varepsilon = \emptyset, \quad (3.1) \]

where \( \tau_1 = \rho_0 - \tau \). Then

\[ I(\lambda) = O (|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N}. \]

**Proof.** \( \) Choose two functions \( \alpha(t) \in C_0^\infty (\mathbb{R}) \), \( \beta(x) \in C_0^\infty (\mathbb{R}^n) \) such that:

\[ \alpha(t) = \begin{cases} 1 & \text{for } t \leq T_1 + \varepsilon, \\ 0 & \text{for } t \geq T_1 + 2 \varepsilon, \end{cases} \]

\[ \beta(x) = \begin{cases} 1 & \text{for } |x| \leq \tau_1 + T_1 + 2 \varepsilon, \\ 0 & \text{for } |x| \geq \tau_1 + T_1 + 3 \varepsilon. \end{cases} \]

For the distribution \( \widetilde{v}(t, x) = \alpha(t) \beta(x) v(t, x) \) we obtain the problem

\[ \square \widetilde{v} = \tilde{F} \text{ in } \mathbb{R} \times \Omega, \]

\[ \widetilde{v} = \alpha \beta h \text{ on } \mathbb{R} \times \partial \Omega, \]

\[ \tilde{v}_{1 < t} = 0 \]

with

\[ \tilde{F} = 2 \alpha \beta v_t + \alpha_n \beta v - 2 \alpha \langle \nabla \beta, \nabla v \rangle - \alpha (\Delta \beta) v + \alpha \beta F. \]

By a finite speed of propagation argument we conclude that \( v \in C^\infty \) for \( t \leq T_1 + 2 \varepsilon, \right\} |x| \leq \tau_1 + T_1 + 2 \varepsilon. \) This shows that \( \tilde{F} \) is singular only for \( T_1 + \varepsilon \leq t \leq T_1 + 2 \varepsilon. \) Then the assumption (3.1) implies

\[ \text{WF} (\tilde{F}) \cap \left\{ (t, y, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) \right\} = \emptyset. \quad (3.2) \]

Since

\[ \text{WF} (v_{1 | \mathbb{R} \times \Omega}) = \{ (t, x, \tau, \xi) \in T^* (\mathbb{R} \times \Omega) \setminus \{ 0 \} : \tau^2 = |\xi|^2 \} \]

Vol. 53, n° 4-1990.
by a standard argument we deduce that for each $m > 0$ there exists $s(m) < 0$ so that

$$v \in H^s_{loc}(\mathbb{R}^n; H^m_{loc}(\Omega)).$$

We can take the partial Fourier transformation with respect to $t$ of $\tilde{v}$ and $\tilde{F}$. Put

$$V(x, \lambda) = (\tilde{v}(t, x), e^{-it\lambda}),$$
$$f(x, \lambda) = (\tilde{F}(t, x), e^{-it\lambda}),$$
$$g(x, \lambda) = (\alpha \beta h(t, x), e^{-it\lambda}).$$

The existence of the Fourier transformation of $h(t, x)$ follows from the fact that $WF(v_{|t|} \cdot \pi_x^{\mathbb{R}})$ is contained in the set of hyperbolic and glancing points of $\Box$ (see [5], [9]). We obtain the problem

$$(\Delta + \lambda^2) V(t, x) = -f(x, \lambda) \quad \text{in } \Omega,$$
$$V = g \quad \text{on } \partial \Omega,$$
$$V \text{ is a } i\lambda-\text{outgoing solution.}$$

The latter condition means that for $|x| \to \infty$ we have the representation

$$V(x, \lambda) = \int_{\partial \Omega} \left[ \frac{\partial V}{\partial \nu}(y, \lambda) G^+_\lambda(x - y) - V(y, \lambda) \frac{\partial}{\partial \nu} G^+_\lambda(x - y) \right] dS_y - \int_{\Omega} G^+_\lambda(x - y) f(y, \lambda) dy. \quad (3.3)$$

Here the integrals are taken in the sense of distributions and $G^+_\lambda(x)$ is the $(i\lambda)$-outgoing Green function of the operator $\Delta + \lambda^2$ (cf. [7]). More precisely,

$$G^+_\lambda(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{(n-1)/2}} \left( \frac{1}{r} \right)^{(n-3)/2} e^{-i\lambda/r}, \quad r = |x|.$$ 

Notice that for $|x| \to \infty$ we have

$$G^+_\lambda(x) = \text{const. } \lambda^{(n-3)/2} e^{-i\lambda |x|/|x|^{(n-1)/2}} + O(1/|x|^{(n+1)/2}).$$

We set in (3.3) $x = r\theta$, $r = |x|$, and multiply (3.3) by $r^{(n-1)/2} e^{i\theta r}$. Taking the limit as $r \to \infty$, we get

$$\int_{\partial \Omega} e^{i\lambda \langle \theta, \theta \rangle} \left[ \frac{\partial V}{\partial \nu}(y, \lambda) - i \lambda \langle \nu, \theta \rangle V(y, \lambda) \right] dS_y = \int_{\Omega} \int_{\mathbb{R}} e^{-i\lambda \langle t - \langle \nu, \theta \rangle \rangle} \tilde{F}(t, y) dt dy, \quad (3.4)$$

where the integrals are taken in the sense of distributions. The condition (3.2) shows that the right-hand side of (3.4) can be estimated by $O(\lambda^{-m})$. 

Annales de l’Institut Henri Poincaré - Physique théorique
for all $m \in \mathbb{N}$. Thus we deduce

$$(2 \pi)^{-1} \int_{\mathbb{R}} \left( \int_{\partial \mathcal{K}} e^{\lambda \langle y, \theta \rangle} \left[ \frac{\partial V}{\partial V} (y, \lambda) - i \lambda \langle y, \theta \rangle V(y, \lambda) \right] dS_y \right) e^{i\alpha} d\lambda$$

$$= \int_{\partial \mathcal{K}} \left( \frac{\partial \tilde{\nu}}{\partial V} - \langle y, \theta \rangle \frac{\partial \tilde{\nu}}{\partial t} \right) (t + \langle y, \theta \rangle, y) dS_y \in C^0_0(\mathbb{R}).$$

Next,

$$\int_{-\infty}^{\infty} \left( \int_{\partial \mathcal{K}} \left( \frac{\partial \tilde{\nu}}{\partial V} - \langle y, \theta \rangle \frac{\partial \tilde{\nu}}{\partial t} \right) (t + \langle y, \theta \rangle, y) dS_y \right) e^{i\alpha} \rho_\delta (-t + t_0) dt$$

$$= \int_{-\infty}^{\infty} \left( \int_{\partial \mathcal{K}} e^{\lambda (t - \langle y, \theta \rangle)} \rho_\delta (\langle y, \theta \rangle - t + t_0) \left( \frac{\partial \tilde{\nu}}{\partial V} - \langle y, \theta \rangle \frac{\partial \tilde{\nu}}{\partial t} \right) dt dS_y \right)$$

$$= I(\lambda) + O(\lambda^{-m}) \text{ for all } m \in \mathbb{N}.$$

The left-hand side can be estimated by $O(\lambda^{-m})$ and this completes the proof of Proposition 3.1.

**Proof of Theorem 1.** — We shall recall some properties of the generalized Hamiltonian flow established by Melrose and Sjöstrand [9]. Our assumption implies that if there exists a $(\omega, \theta)$-ray $\gamma$ passing through $\rho_\omega$, then $C_\gamma (u) = \gamma(t)$, where $\gamma(t)$ is the generalized bicharacteristic the projection of which on $\Omega$ is $\gamma$.

Consider the map $Z_1 \times \mathbb{R} \ni (u, t) \rightarrow C_t (u)$. Melrose and Sjöstrand proved (cf. Theorem 3.22 in [9], II) that $C_t (u)$ is continuous with respect to the metric $D(\rho, \mu)$ (cf. section 3 in [9], II for the definition of $D(\rho, \mu)$). In particular, for fixed $\varepsilon > 0$ and $T > 0$ there exists a neighbourhood $U$ of $u_0$ in $Z_1$ such that for each $u \in U$ and each $t \in [-\rho_0, T]$ we have

$$\max \left\{ D(\rho, \mu) : \rho \in C_t (u), \mu \in C_t (u_0) \right\} < \varepsilon.$$

Let $-t_0$ be fixed so that

$$-t_0 \notin \left\{ -T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta} \right\}.$$

Choose $T > 0$ with $|t_0| < T$. Since the set

$$\left\{ T_\gamma : |T_\gamma| \leq T, \gamma \in \mathcal{L}_{\omega, \theta} \right\}$$

is closed, we can find $\varepsilon_0 > 0$ such that

$$T_\gamma \notin [t_0 - \varepsilon_0, t_0 + \varepsilon_0] \text{ for all } \gamma \in \mathcal{L}_{\omega, \theta}. \quad (3.5)$$

We shall study $s(t, \theta, \omega)$ for $|t| \leq T$ and fixed $\theta \neq \omega$. Let $0 < \delta \leq \varepsilon_0/2$, then

$$(s(t, \theta, \omega), \rho_\delta (t + t_0) e^{-i\lambda \delta}) = I(\lambda)$$

$$= \sum_{k=0}^{n-2} c_k (-i\lambda)^{n-2-k} \int_{\partial \mathcal{K}} e^{i\lambda (t - \langle y, \theta \rangle)} \rho_\delta^{(k)} (\langle y, \theta \rangle - t + t_0) \frac{\partial \nu}{\partial V} (t, y, \omega) dt dS_y$$

Vol. 53, n° 4-1990.
with \( c_k = \text{Const.}, \quad c_0 = C_n, \quad \rho^{(k)} = \frac{d^k \rho}{dt^k} \). We shall examine the integral for \( k = 0 \); the analysis of the others is completely analogous.

Obviously, we have to study the singularities of \( w \) for \( |t| \leq \rho_0 + T + \delta \). Without loss of generality we may assume that \( \omega = (0, \ldots, 0, 1) \). Consider the hyperplane

\[
Z_1 = \{ x \in \mathbb{R}^n : x_n = \tau \},
\]

where \( \tau < -\rho_0 \) is fixed. For \( \varphi_j(x') \in C^\infty_0 (\mathbb{R}^{n-1}), \ x' = (x_1, \ldots, x_{n-1}) \), consider the Cauchy problem

\[
\begin{align*}
\Box v_j &= 0 \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}^n_{x_n}, \\
\left. v_j \right|_{t=\tau} &= \varphi_j(x') \delta (\tau - x_n), \\
\left. \frac{\partial v_j}{\partial t} \right|_{t=\tau} &= \varphi_j(x') \delta' (\tau - x_n),
\end{align*}
\tag{3.6}
\]

where \( \mathbb{R}_t^+ = \{ t \in \mathbb{R} : t > \tau \} \), and the mixed problem

\[
\begin{align*}
\Box W_j &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
W_j &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
\left. W_j \right|_{t=\tau} &= \varphi_j(x') \delta (\tau - x_n), \\
\left. \frac{\partial W_j}{\partial t} \right|_{t=\tau} &= \varphi_j(x') \delta' (\tau - x_n).
\end{align*}
\]

Clearly, there exists a compact set \( F_0' \subset \mathbb{R}^{n-1} \) such that if \( \supp \varphi_j \cap F_0' = \emptyset \), then

\[
WF \left( \left. \frac{\partial W_j}{\partial v} \right|_{\mathbb{R} \times \partial K} \right) \cap \{ (t, y, 1, -\theta |_{T_y (\partial K)}) : y \in \partial K \} = \emptyset. \tag{3.7}
\]

Then we obtain

\[
\int_{\mathbb{R} \times K} \int_{\partial K} e^{\frac{2\pi}{\lambda} t - \langle x, \theta \rangle} \rho_0 (\langle y, \theta \rangle - t + t_0) \left. \frac{\partial W_j}{\partial v} \right| dt dS_y = O (|\lambda|^{-m}), \quad m \in \mathbb{N}. \tag{3.8}
\]

Set \( F_0 = \{ x \in \mathbb{R}^n : x' \in F_0', x_n = \tau \} \). For \( u_0 \in F_0 \) denote by \( l(u_0) \) the straight-line ray issued from \( u_0 \) in direction \( \omega \). Let \( l(u_0) \) has a direction \( \omega \) for \( 0 \leq t \leq T \). Assume that

\[
\emptyset \neq l(u_0) \cap K \subset \partial K,
\]

that is \( l(u_0) \) meets \( \partial K \) only at points, where \( l(u_0) \) is tangent to \( \partial K \). Then \( l(u_0) \) is the projection on \( \hat{\Omega} \) of a uniquely extendible bicharacteristic \( \gamma_0 (t) \) of \( \Box \) which is determined uniquely by the Hamiltonian flow of \( \Box \). Consequently, \( C_t (u_0) = \gamma_0 (t) \). Choosing a small neighbourhood \( U (u_0) \) of \( u_0 \) and \( \varphi_j \) with \( \supp \varphi_j \subset \partial U (u_0) \), the results on propagation of singularities [9] and the continuity of the \( C_t (u) \), discussed above, imply \( (3.7) \) for \( |t| \leq T \). Thus for such \( W_j \) we have \( (3.8) \).

Annales de l'Institut Henri Poincaré - Physique théorique
If the case described above does not occur, then \( l(u_0) \) has common points with the interior of \( K \). Denote by \( x_1(u_0) \) the point on \( l(u_0) \) such that the segment \([u_0, x_1(u_0)]\) is the maximal one which has no common points with the interior of \( K \). There are two possibilities:

1. \( l(u_0) \) meets transversally \( \partial K \) at \( x_1(u_0) \);
2. \( l(u_0) \) is tangential to \( \partial K \) at \( x_1(u_0) \) and \( \omega \) is an asymptotic direction for \( \partial K \) at \( x_1(u_0) \).

Let \( t_1(u_0) = |u_0 - x_1(u_0)| \). It is easy to show that

\[
WF(v_j) = \{ (t, x, \pm \sigma, \pm \sigma \omega) \in T^* (\mathbb{R}^{n+1}) \setminus \{ 0 \} : \sigma > 0, x \in Z_1, x^* \in \text{supp} \phi_j \text{ and } s \geq 0 \text{ with } t = \tau \pm s, x = \hat{x} \pm s \omega \}.
\]

In the case (1) we modify \( v_j \) in the interior of \( K \) in a small neighbourhood of \( x_1(u_0) \), provided \( \text{supp} \phi_j \) is sufficiently small. We denote the modified \( v_j \) by \( \tilde{v}_j \) and arrange \( \tilde{v}_j = 0 \) for \( t > t_1 + \varepsilon_1 \), where \( t_1 = \max \{ t_1(u) : u \in \Theta(u_0) \} \), while \( \Theta(u_0) \) and \( \varepsilon_1 \) are chosen sufficiently small. In the case (2) we repeat the same procedure modifying \( v_j \) in the interior of \( K \). This is possible since \( l(u_0) \) enters the interior of \( K \).

Clearly, \( h_j = v_j |_{\mathbb{R}^n_+ \times \partial K} = 0 \) for \( t \) sufficiently close to \( \tau \). Extending \( h_j \) as 0 for \( t < \tau \), denote by \( w_j \) the solution of the problem

\[
\begin{align*}
\Box w_j &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
w_j + h_j &= 0 \quad \text{on } \mathbb{R} \times \partial \Omega, \\
w_j |_{t = \tau} &= 0.
\end{align*}
\]

Since \( \frac{\partial}{\partial t} (w_j + \tilde{v}_j)|_{\mathbb{R}^n_+ \times \partial K} = 0 \), we have to study the integrals

\[
I_{j, \delta}(\lambda) = \int_{\mathbb{R} \times \partial K} e^{\lambda(t - \langle y, \theta \rangle)} \rho_\delta(\langle y, \theta \rangle - t + t_0) \left( \frac{\partial}{\partial v} - \langle v, \theta \rangle \frac{\partial}{\partial t} \right) w_j dt dS_y,
\]

\[
J_{j, \delta}(\lambda) = \int_{\mathbb{R} \times \partial K} e^{\lambda(t - \langle y, \theta \rangle)} \rho_\delta(\langle y, \theta \rangle - t + t_0) \left( \frac{\partial}{\partial v} - \langle v, \theta \rangle \frac{\partial}{\partial t} \right) \tilde{v}_j dt dS_y.
\]

It is easy to see that

\[
J_{j, \delta}(\lambda) = O(\lambda^{m}) \quad \text{for all } m \in \mathbb{N}. \tag{3.10}
\]

Indeed, observe that for small \( \varepsilon > 0 \) we have \( v_j = \tilde{v}_j \) for \( t \leq t \leq \tau + \varepsilon < -\rho_0 \). Then \( \theta \neq \omega \) yields

\[
WF(\tilde{v}_j) \cap \{ (t, y, 1, -\theta) \in T^* (\mathbb{R}^{n+1}) : t \leq t \leq \tau + \varepsilon \} = \emptyset.
\]

Choose a function \( \alpha_1(t) \in C^\infty(\mathbb{R}) \) such that

\[
\alpha_1(t) = \begin{cases} 
0 & \text{for } t \leq \tau + \varepsilon/2, \\
1 & \text{for } t \geq \tau + \varepsilon.
\end{cases}
\]

Then we obtain (3.10) applying the argument of the proof of Proposition 3.1 for \( \alpha_1(t) \tilde{v}_j(t, x) \).

Vol. 53, n° 4-1990.
Thus it remains to study $I_{j,s}(\lambda)$. Next, for each $u_0 \in F_0$, satisfying (1) or (2), we introduce a sufficiently small neighbourhood $\mathcal{O}(u_0) \subset Z_1$, and we take $\text{supp } \varphi_j \subset \mathcal{O}(u_0)$. Thus the singularities of $w_j$ are localized along the generalized rays $\gamma(u_0)$ issued from $u_0 \in F_0$ in direction $\omega$.

There are two cases.

**Case A.** For all $\sigma > \rho_0 + T + 1$ we have

$$C_\sigma(u_0) \cap \{ (\sigma, x, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : \rho_0 \leq |x| \leq \tau_1 + \sigma + 1 \} = \emptyset. \quad (3.11)$$

Then for all $\tau \geq \tau$ we obtain

$$C_\tau(u_0) \cap \{ (\tau, x, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : |x| \geq \rho_0 \} = \emptyset.$$

Indeed, assume that for some $\tau \leq \tau$ we can find a generalized bicharacteristic $\gamma(\tau; u_0) \subset C_\tau(u_0)$ such that

$$(\tau, \dot{x}, 1, -\theta) \in \gamma(\tau; u_0) \quad \text{with } |\dot{x}| \geq \rho_0.$$

Then $\gamma(\sigma; u_0)$ has direction $\theta$ for all $\sigma \geq \tau$, and we obtain a contradiction with (3.11).

By using the continuity of $C_\sigma(u_0)$ with respect to $t$ and $u_0$, we can find small neighbourhood so that for all $u \in \mathcal{O}(u_0)$ and all $t \in [\tau, \rho_0 + T + 2]$ we have

$$C_t(u_0) \cap \{ (t, x, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : \rho_0 \leq |x| \leq \rho_0 + 2 \} = \emptyset. \quad (3.12)$$

Now let $\beta(x) \in C_\infty^\infty(\mathbb{R}^n)$ be a function such that

$$\beta(x) = \begin{cases} 
1 & \text{for } |x| \leq \rho_0, \\
0 & \text{for } |x| \geq \rho_0 + 1.
\end{cases}$$

For $\text{supp } \varphi_j \subset \mathcal{O}(u_0)$ and for $w_j$ we obtain

$$\Box (w_j) = -2 \langle \nabla x \beta, \nabla x w_j \rangle - (\Delta \beta) w_j = F_j.$$

Applying the results for propagation of singularities and (3.12), we conclude that

$$WF(F_j) \cap \{ (t, x, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : \tau \leq t \leq \rho_0 + T + 2 \} = \emptyset. \quad (3.13)$$

It is easy to see that the Fourier transform

$$\tilde{w}_j(x, \lambda) = F_{t \to \lambda}(\beta w_j)$$

exists. To check this it is sufficient to use the $(i\lambda)$-outgoing condition and to prove that the solution of the problem

$$(\Delta + \lambda^2)W_j = 0 \quad \text{in } \Omega,$$

$W_j = -F_{t \to \lambda}(h_j) \quad \text{on } \partial\Omega,$

$W_j$ is $(i\lambda)$-outgoing,

is a tempered distribution with respect to $\lambda$. 

---

*Années de l'Institut Henri Poincaré - Physique théorique*
Setting $F_j(x, \lambda) = \mathcal{F}_{t \to \lambda} (F_j)$, as in the proof of Proposition 3.1 we obtain

$$\int_{\partial \mathbf{K}} e^{i \langle \gamma, \theta \rangle} \left( \frac{\partial \tilde{w}_j}{\partial \gamma} (y, \lambda) - i \lambda \langle \gamma, \theta \rangle \tilde{w}_j (y, \lambda) \right) dS_y = \int_{\Omega} e^{i \langle \gamma, \theta \rangle} F_j (y, \lambda) dy. $$

Taking the inverse Fourier transform, we deduce

$$\int_{\partial \mathbf{K}} \left( \frac{\partial \tilde{w}_j}{\partial \gamma} - \langle \gamma, \theta \rangle \frac{\partial \tilde{w}_j}{\partial t} \right) (t + \langle \gamma, \theta \rangle, y) dS_y = \int_{\Omega} F_j (t + \langle \gamma, \theta \rangle, y) dy. $$

Then the relation (3.13) leads to

$$I_{j, \delta} (\lambda) = \int_{\mathbb{R}} \int_{\Omega} e^{i \langle t - \langle \gamma, \theta \rangle \rangle} \rho_\delta (\langle \gamma, \theta \rangle) - t + t_0 \rangle F_j (t, y) dt dy$$

$$= O (|\lambda|^{-m}) \text{ for all } m \in \mathbb{N}. $$

**Case B.** For some $\sigma > \rho_0 + T + 1$ we have

$$C_\sigma (u_0) \cap \{ (\sigma, x, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : \rho_0 \leq |x| \leq \tau_1 + \sigma + 1 \} \neq \emptyset. $$

Then there exists a generalized bicharacteristic $\gamma (t; u_0)$ issued from $u_0$ in direction $\omega$ passing through some point $y$ for $t = \sigma$, $|y| \leq \rho_0$, with direction $\theta$. The projection of $\gamma (t; u_0)$ on $\Omega$ is a $(\omega, \theta)$-ray $\gamma$, and our assumption yields $C_\sigma (u_0) = \gamma (t; u_0)$. Let $T_\gamma$ be the sojourn time of $\gamma$ and let

$$\gamma (t; u_0) = (t, x(t), 1, -\xi(t)) \in T^* (\mathbb{R} \times \Omega), |\xi(t)| = 1, t \geq \tau. $$

Introduce the numbers

$$T_2 = \inf \{ \sigma : \sigma \geq \tau, \xi(t) = \theta \text{ for } t \geq \sigma \},$$

$$T_3 = \inf \{ \sigma : \sigma \geq \tau, x(t) \notin \partial \mathbf{K} \text{ for } t > \sigma \}. $$

Notice that $T_2 \leq T_3$. Then

$$I_{j, \delta} (\lambda) = \int_{-\infty}^{\infty} \int_{\partial \mathbf{K}} \int_{\partial \mathbf{K}} I_{j, \delta} (\lambda) + I'_{j, \delta} (\lambda), $$

where $s < T_2$ will be chosen below. A simple geometrical argument yields $t - \langle x(t), \theta \rangle = T_1$ for $T_2 \leq t \leq T_3$. By (3.5) we obtain

$$|\langle x(t), \theta \rangle - t + t_0| \geq \varepsilon_0 \quad (T_2 \leq t \leq T_3). $$

For small $\partial (u_0)$, $\text{supp } \varphi_j \subset \partial (u_0)$ and $|t| \leq T_3$ the singularities of $w_j$ are contained in a small neighbourhood of $\gamma (t; u_0)$. This makes it possible to choose $\partial (u_0)$ and $T_2 - s$ so small that

$$\xi(s) \neq \theta, \quad (3.14)$$

$$|\langle y, \theta \rangle + t_0 - t| \geq \varepsilon_0 / 2 \quad \text{for } t \geq s \quad (3.15)$$

and

$$(t, y) \in \text{sing supp } w_j |_{\mathbb{R} \times \partial \mathbf{K}} \cup \text{sing supp } \left( \frac{\partial \tilde{w}_j}{\partial \gamma} |_{\mathbb{R} \times \partial \mathbf{K}} \right).$$

Vol. 53, n° 4-1990.
Moreover, we take \( s < T_2 \) so that either \( x(s) \notin \partial K \) or \( x(s) \in \partial K \) and \( \gamma(s; u_0) \) is a glancing point for \( \square \). In the latter case (3.14) implies

\[
\xi(s) \neq \theta |_{\tau_x(s) (\partial K)}.
\]  

(3.16)

Fixing \( s \), we conclude that

\[
I''_{j, \delta}(\lambda) = O(|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N},
\]

(3.17)

since \( \varepsilon_0/2 \geq \delta \) and \( \rho_{\delta}(\langle y, \theta \rangle - t + t_0) = 0 \) for \((t, y)\) satisfying (3.15).

To deal with \( I'_{j, \delta}(\lambda) \), we take \( C(u_0) \) sufficiently small and arrange

\[
WF(w_j) \cap \{ (s, y, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : |y| \leq \tau_1 + s + 1 \} = \emptyset.
\]

To do this, we exploit (3.14) and the continuity of \( C_s(u) \) for \( u \in C(u_0) \).

Since \( WF(w_j) \) is closed, we can choose \( \varepsilon > 0 \) so that

\[
WF(w_j) \cap \{ (t, y, 1, -\theta) \in T^* (\mathbb{R} \times \Omega) : s \leq t \leq s + \varepsilon, |y| \leq \tau_1 + s + 1 \} = \emptyset. \quad (3.18)
\]

Similarly, we use (3.16) to arrange

\[
\left( WF(w_j |_{\mathbb{R} \times \partial K}) \cup WF \left( \frac{\partial w_j}{\partial \nu} \bigg|_{\mathbb{R} \times \partial K} \right) \right)
\]

\[
\cap \{ (t, y, 1, -\theta |_{T_y (\partial K)}) : s \leq t \leq s + \varepsilon, y \in \partial K \} = \emptyset. \quad (3.19)
\]

Next, we take a function \( \alpha_2(t) \in C^\infty(\mathbb{R}) \) such that

\[
\alpha_2(t) = \begin{cases} 
1 & \text{for } t \leq T_2 - s, \\
0 & \text{for } t \geq T_2 - s + \varepsilon.
\end{cases}
\]

By applying (3.19), for \( \tilde{w}_j = \alpha_2(t) w_j(t, x) \) we get

\[
I_{j, \delta}(\lambda) = \int_{-\infty}^{\infty} \int_{\partial K} e^{i \lambda (t - \langle y, \theta \rangle)}
\]

\[
\times \rho_{\delta}(\langle y, \theta \rangle - t + t_0) \left( \frac{\partial}{\partial \nu} - \langle v, \theta \rangle \frac{\partial}{\partial t} \right) \tilde{w}_j dt dS_y
\]

\[
= I'_{j, \delta}(\lambda) + O(|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N}.
\]

On the other hand, for \( \tilde{w}_j \) we can apply the arguments of the proof of Proposition 3.1, since \( \square \ w_j = \tilde{F}_j \) satisfies (3.2) as a consequence of (3.18) and the finite speed of propagation of singularities. Finally, we conclude that

\[
I_{j, \delta}(\lambda) = O(|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N}. \quad (3.20)
\]

In this way for each \( u_0 \in F_0 \) we have chosen a neighbourhood \( C(u_0) \).

We obtain a covering \( \{ C(u_0) : u_0 \in F_0 \} \) of \( F_0 \), and we may assume

\[
F_0 \subset \bigcup_{j=1}^{M} C(u_0^j).
\]
Let for $j=1, \ldots, N$, $N \leq M$, the points $u_0^{(p)} \in F_0$ satisfy the assumptions in (1) or (2). Choose a partition of unity $\{\varphi_j(x')\}_{j=1}^{\infty}$ of $Z_1$ so that $\text{supp} \varphi_j \subset \varnothing (u_0^{(p)})$ for $j=1, \ldots, N$ and $(\text{supp} \varphi_j) \cap F_0 = \varnothing$ for $j > M$. Set
\[
\tilde{w} = \sum_{j=1}^{N} (w_j + \tilde{v}_j) + \sum_{j>N} W_j.
\]
Then
\[
\nabla \tilde{w} = 0 \quad \text{in } \mathbb{R}_+^* \times \Omega, \quad \tilde{w} = 0 \quad \text{on } \mathbb{R}_+^* \times \partial \Omega, \quad \tilde{w}_{t=t} = \delta(\tau - x_n), \quad \frac{\partial \tilde{w}}{\partial t} \bigg|_{t=\tau} = \delta'(\tau - x_n).
\]
Consequently, $w = \tilde{w}$ in $\mathbb{R}_+^* \times \Omega$ and we can replace $w$ by $\tilde{w}$ in $J(\lambda)$. Then by (3.8), (3.10), (3.17), (3.20) we conclude that $-t_0 \notin \text{sing} \text{supp} s(t, \theta, \omega)$. This completes the proof of Theorem 1.

4. SOME GENERIC PROPERTIES OF $(\omega, \theta)$-TRAJECTORIES

In this section we will use several times the following result of [15].

**Theorem 4.1.** Let $n \geq 2$, $s \geq 2$, $p$ and $q$ be natural numbers and let $U$ be an open subset of $(\mathbb{R}^q)^{(s)}$. Let

$$
H = (H_1, \ldots, H_p): \quad U \rightarrow \mathbb{R}^p
$$

be a smooth map such that for every $i=1, \ldots, s$ there exists $r_i$, $1 \leq r_i \leq p$, with $\text{grad}_y H_{r_i}(y) \neq 0$ for all $y \in U$, $y = (y_1, \ldots, y_s)$. Let $L: U \rightarrow \mathbb{R}^q$ be a smooth map such that $dL(y) \neq 0$ for every $y \in U$ with $L(y) = 0$. Denote by $T$ the set of those $f \in C_{\text{emb}}^{\infty}(X, \mathbb{R}^p)$ such that for every critical point $x$ of $H \circ f^s$ with $f^s(x) \in U$ we have $L(f^s(x)) \neq 0$. Then $T$ contains a residual subset of $C_{\text{emb}}^{\infty}(X, \mathbb{R}^p)$.

This is Theorem 3.1 (B) of [15], where the assumption for $L$ is stronger, namely, it is required that $dL(y) \neq 0$ for every $y$ in $U$. However, the proof in [15] holds without any changes if we assume $dL(y) \neq 0$ only for those $y \in U$ with $L(y) = 0$.

Let $\gamma = \bigcup_{i=0}^{k} L_i$ be a $(\omega, \theta)$-trajectory on $X$ with $k \geq 2$. Then $l_0$ and $l_k$ cannot be orthogonal to $X$ at their end points. If in addition for every $i=1, \ldots, k-1$, $L_i = [x_i, x_{i+1}]$ is not orthogonal to $X$ at $x_i$ and $x_{i+1}$, then $\gamma$ will be called a non-symmetric $(\omega, \theta)$-trajectory on $X$. In this case we set $d(\gamma) = k - s$ (the defect of $\gamma$), where $s$ is the number of all different reflection points of $\gamma$. If some $L_i$ is orthogonal to $X$ at $x_i$ or $x_{i+1}$, then we must
have $\theta = -\omega$, the second part of (ii) in 2.2 is satisfied, and $\gamma = \bigcup_{i=0}^{m} l_{i}$, where $l_{m}$ is orthogonal to $X$ at $x_{m+1}$. In this case $\gamma$ is a reflecting $(\omega, \theta)$-ray, it will be called a symmetric $\omega$-ray on $X$, and we set $d(\gamma) = m - s + 1$. Note that if $\gamma$ is a non-symmetric $(\omega, \theta)$-trajectory, then $d(\gamma) = 0$ means that $\gamma$ passes only once through each of its reflection points. For symmetric $\gamma$, $d(\gamma) = 0$ means that $\gamma$ passes exactly twice through each of its reflection points excluding that of them at which $\gamma$ is orthogonal to $X$.

The first main result in this section is the following.

**Theorem 4.2.** Let $\mathcal{D}$ be the set of those $f \in C_{\text{emb}}^{\infty}(X, \mathbb{R}^{n})$ such that every $(\omega, \theta)$-trajectory on $f(X)$ has zero defect. Then $\mathcal{D}$ contains a residual subset of $C_{\text{emb}}^{\infty}(X, \mathbb{R}^{n})$.

This theorem can be proved using arguments similar to those in the proof of Theorem A in [19]. Here we proceed in a different way applying Theorem 4.1 above. This way is simpler and shorter, and can also be used to simplify the proofs in [19] and [15].

We begin with a combinatorical classification of $(\omega, \theta)$-trajectories, similar to that used in [13], [19] for periodic reflecting rays.

Let $k \geq s \geq 2$ be integers and let

$$\alpha: \{1, \ldots, k\} \rightarrow \{1, \ldots, s\}$$

be a map with

$$\alpha(i) \neq \alpha(i+1) \quad (i=1, \ldots, k-1).$$

If

$$\{\alpha(i), \alpha(i+1)\} \neq \{\alpha(j), \alpha(j+1)\}$$

holds whenever $1 \leq i < j \leq k-1$, then $\alpha$ will be called a $ns$-map. If $k = 2m+1$, (4.2) holds for $1 \leq i < j \leq k-1$, then $\alpha$ will be called a $s$-map.

In this section we will always assume that $\alpha$ is a $ns$-map or a $s$-map, and by definition we set

$$\alpha(0) = 0, \quad \alpha(k+1) = s + 1. \quad (4.3)$$

So $\alpha$ will be considered as a map

$$\alpha: \{0, 1, \ldots, k+1\} \rightarrow \{0, 1, \ldots, s+1\}.$$  

As in [13], [19] we will use the notation

$$I_{i}(\alpha) = \{j: \text{there is } t=0, 1, \ldots, k \text{ with } \{i, j\} = \{\alpha(t), \alpha(t+1)\}\}$$

for $i = 1, 2, \ldots, s$. 

*Annales de l'Institut Henri Poincaré - Physique théorique*
Fix an open ball \( U_0 \) in \( \mathbb{R}^n \) containing \( X \), and let \( Z_i \) and \( \pi_i \) be as in subsection 2.4. For \( y = (y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)} \) we set \( y_0 = \pi_1(y_1) \) and \( y_{s+1} = \pi_2(y_{(s)}) \). Denote by \( U_\alpha \) the set of those \( y \in U_0^{(s)} \) which satisfy the following two conditions:

\[
y_i \notin \text{convex hull} \{ y_j : j \in I_1(\alpha) \} \quad (i = 1, \ldots, s),
\]

and

for every \( i = 1, \ldots, s \) if \( m, j, r, t \) are distinct elements of \( I_1(\alpha) \), then either \( y_i, y_m, y_j, y_r, y_t \) are not collinear. Then \( U_\alpha \) is an open subset of \( U_0^{(s)} \), and the map

\[
F = F_\alpha : U_\alpha \to \mathbb{R},
\]

defined by

\[
F(y) = \sum_{i=0}^{k} \| y_{\alpha(i)} - y_{\alpha(i+1)} \|,
\]

is smooth. If \( y_1, \ldots, y_s \) are all different reflection points of a \((\omega, \theta)\)-trajectory \( \gamma \) on \( X \) such that \( y_{\alpha(1)}, \ldots, y_{\alpha(s)} \) are the successive reflection points of \( \gamma \), then \( \gamma \) will be called a \((\omega, \theta)\)-trajectory of type \( \alpha \). In this case we have \( y = (y_1, \ldots, y_s) \in U_\alpha \) and \( F(y) \) is just the length of this part of \( \gamma \) which lies in \( H_1 \cap H_2 \). Moreover, \( y \) is a critical point of the map

\[
F|_{X^s} : X^s \to \mathbb{R}.
\]

It is also clear that for every \((\omega, \theta)\)-trajectory \( \gamma \) there exists a surjective map \( \alpha \) which is either a \( ns \)-map or a \( s \)-map such that \( \gamma \) is of type \( \alpha \).

Proof of Theorem 4.2. — Fix an arbitrary surjective \( ns \)-map \((4.1)\) extended by \((4.3)\), and suppose \( k > s \). Denote by \( D_\alpha \) the set of those \( f \in C^\infty_{\text{emb}}(X, U_0) \) such that there are no \((\omega, \theta)\)-trajectories of type \( \alpha \) on \( f(X) \). We are going to prove that \( D_\alpha \) contains a residual subset of \( C^\infty_{\text{emb}}(X, U_0) \). To this end we will use Theorem 4.1 for \( U = U_\alpha, p = 1, \) and \( H = F : U_\alpha \to \mathbb{R} \). As in the proof of Lemma 4.3 in [13], one can easily verify that for every \( y \in U_\alpha \) and every \( i = 1, \ldots, s \) there exists \( j = 1, \ldots, n \) such that

\[
\frac{\partial F}{\partial y_i^{(\alpha)}}(y) \neq 0.
\]

Here \( y_i^{(\alpha)} \) are the components of the vector \( y_i \in \mathbb{R}^n \).

Since \( k > s \), there exists \( i = 1, \ldots, s \) such that \( |\alpha^{-1}(i)| > 1 \). Take two distinct elements \( j_1, j_2 \) of \( \alpha^{-1}(i) \). Then \( m = \alpha(j_1 - 1), j = \alpha(j_1 + 1), r = \alpha(j_2 - 1), t = \alpha(j_2 + 1) \) are distinct elements of \( I_1(\alpha) \). Clearly, \( \{m, j\} \neq \{0, s+1\} \), so either \( m \) or \( j \) is not contained in \( \{0, s+1\} \). We may assume \( m \notin \{0, s+1\} \) (otherwise we can exchange the notation: \( m = \alpha(j_1 + 1), j = \alpha(j_1 - 1) \)). Similarly, we may assume \( r \notin \{0, s+1\} \). Set

\[
L_u(y) = \frac{y_m - y_i}{\|y_m - y_i\|} + (1 - u) \frac{y_j - y_i}{\|y_j - y_i\|} + \frac{y_r - y_i}{\|y_r - y_i\|} - (1 - u) \frac{y_t - y_i}{\|y_t - y_i\|}
\]

(4.6)}
We have to check that if \( L(y) = 0 \) for some \( y \in U_a \), then \( dL(y) \neq 0 \).

Suppose \( y \in U_a \) and \( L(y) = 0 \). If \( \frac{\partial L}{\partial y^{(l)}}(y) = 0 \) for every \( l = 1, \ldots, n \), by direct calculations we find that \( y_m - y_i \) is collinear with \( y_{m} - y_{i} \). Note that \( y \in U_a \) implies \( v \neq 0 \). Since \( L_1(y) = 0 \) and \( y_m - y_i \) and \( y_j - y_i \) are unit vectors, we obtain that \( y_j - y_i \) is also collinear with \( v \). Therefore the points \( y_i, y_m \) and \( y_j \) are collinear.

Suppose also that \( y_1, \ldots, y_n \) are the reflection points of a \((\omega, \theta)\)-trajectory of type \( \alpha \), then for \( y = (y_1, \ldots, y_{s}) \in U_a \) we have \( L(y) = 0 \). Now, applying Theorem 4.1, we find that \( D_{a} \) contains a residual subset of \( C_{\text{emb}}^{\infty}(X, U_0) \).

If \( \theta = -\omega \) and \( \alpha \) is a surjective \( s \)-map (4.1) with \( k > 2s - 1 \), the argument above with minor changes shows that \( D_{a} \) again contains a residual subset of \( C_{\text{emb}}^{\infty}(X, U_0) \). We omit the details in this case.

Finally, mention that \( D = \bigcap_{a} D_{a} \), where \( \alpha \) runs over the surjective maps (4.1) which are either \( ns \)-maps with \( k > s \) or \( s \)-maps with \( k > 2s - 1 \). Therefore \( D \) contains a residual subset of \( C_{\text{emb}}^{\infty}(X, U_0) \) which proves the theorem.

**Theorem 4.3.** — Let \( \mathcal{C} \) be the set of those \( f \in C_{\text{emb}}^{\infty}(X, \mathbb{R}^n) \) such that every two different \((\omega, \theta)\)-trajectories on \( f(X) \) have no common reflection points. Then \( \mathcal{C} \) contains a residual subset of \( C_{\text{emb}}^{\infty}(X, \mathbb{R}^n) \).

**Proof.** — We have to consider pairs of \( ns \)- or \( s \)-maps. We deal in details only with the case of two \( ns \)-maps. The other cases are quite similar.

Let \( U_0, Z_i \) and \( \pi_i (i = 1, 2) \) be as above. For a given \( Y = f(X) \), \( f \in C_{\text{emb}}^{\infty}(X, U_0) \), suppose \( \gamma_1 \) and \( \gamma_2 \) are two different non-symmetric \((\omega, \theta)\)-trajectories on \( Y \), and let \( y_{1}, \ldots, y_{s} \) be all reflection points of \( \gamma_1 \) and \( \gamma_2 \) taken together. Then there exist integers \( k, l \geq 1 \) and \( ns \)-maps (4.1) and 

\[
\beta: \{1, \ldots, l\} \to \{1, \ldots, s\} \tag{4.7}
\]

such that

\[
\text{Im} \; \alpha \cup \text{Im} \; \beta = \{1, \ldots, s\}, \tag{4.8}
\]

\[
\{ \alpha(i), \alpha(i + 1) \} \neq \{ \beta(j), \beta(j + 1) \} \quad (1 \leq i \leq k, 1 \leq j \leq l). \tag{4.9}
\]
$y_{\alpha(1)}, \ldots, y_{\alpha(k)}$ are the successive reflection points of $\gamma_1$ and $y_{\beta(1)}, \ldots, y_{\beta(l)}$ are the successive reflection points of $\gamma_2$. In this case we will say that $(\gamma_1, \gamma_2)$ is a pair of type $(\alpha, \beta)$. Set $\beta(0) = -1$ and $\beta(l+1) = s+2$, thus extending $\beta$ to a map

$$\beta : \{0, 1, \ldots, l, l+1\} \to \{-1, 1, \ldots, s, s+2\}.$$ 

We will use the notation $y_{-1} = \pi_1(y_{\beta(1)})$, $y_{s+2} = \pi_2(y_{\beta(l)})$. Define $F$ by (4.4) and (4.5) and $G : U_\beta \to \mathbb{R}$ by

$$G(y) = \sum_{i=0}^{l} ||y_{\beta(i)} - y_{\beta(i+1)}||.$$ 

Then $y = (y_1, \ldots, y_s) \in U = U_\alpha \cap U_\beta$ and $y$ is a critical point for both $F \circ f^s$ and $G \circ f^s$.

Let $(\alpha, \beta)$ be a pair of maps (4.1) and (4.7) with (4.8), (4.9) and

$$\text{Im } \alpha \cap \text{Im } \beta \neq \emptyset. \quad (4.10)$$

Denote by $\mathscr{C}_{\alpha, \beta}$ the set of those $f \in C^\infty_{\text{emb}}(X, U_\alpha)$ for which there is no pair $(\gamma_1, \gamma_2)$ of $(\alpha, \theta)$-trajectories on $f(X)$ of type $(\alpha, \beta)$. To prove that $\mathscr{C}_{\alpha, \beta}$ contains a residual subset of $C^\infty_{\text{emb}}(X, U_\alpha)$, we proceed exactly as in the proof of Theorem 4.2. We omit the details.

Denote by $\mathscr{I}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that $T_\gamma \neq T_\delta$ for every two different $(\alpha, \theta)$-trajectories $\gamma$ and $\delta$ on $f(X)$, and by $\mathscr{D}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that if $\gamma$ is a non-symmetric $(\alpha, \theta)$-trajectory on $f(X)$, then any two different segments of $\gamma$ are not parallel, and if $\gamma$ is a symmetric $(\alpha, \theta)$-trajectory on $f(X)$, then there are no different parallel segments among the first half of the segments of $\gamma$.

The following generic properties of $(\alpha, \theta)$-trajectories will be important.

**Theorem 4.4.** - Each of the sets $\mathscr{I}$ and $\mathscr{D}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

**Proof.** - We deal again with the intersections of $\mathscr{I}$ and $\mathscr{D}$ with $C^\infty_{\text{emb}}(X, U_\alpha)$, where $U_\alpha$ is a fixed open ball containing $X$.

If $T_\gamma \neq T_\delta$ for two different $(\alpha, \theta)$-trajectories $\gamma$ and $\delta$ on $Y = f(X)$, $f \in \mathscr{C} \cap \mathscr{D}$, there exist different elements $y_1, \ldots, y_s$ of $Y$ such that $y_1, \ldots, y_k$ are the successive reflection points of $\gamma$ for some $k < s$, while $y_{k+1}, \ldots, y_s$ are the successive reflection points of $\delta$. Moreover, $F(y) = G(y)$, where $F$, $G : U \to \mathbb{R}$ are defined by

$$F(y) = ||\pi_1(y_1) - y_1|| + \sum_{i=1}^{k-1} ||y_i - y_{i+1}|| + ||y_k - \pi_2(y_k)||, \quad (4.11)$$

$$G(y) = ||\pi_1(y_{k+1}) - y_{k+1}|| + \sum_{i=k+1}^{s-1} ||y_i - y_{i+1}|| + ||y_s - \pi_2(y_s)||. \quad (4.12)$$
Here $U$ is the set of those $y \in (\mathbb{R}^n)^0$ such that $y_i \notin [y_{i-1}, y_{i+1}]$ for all $i = 2, \ldots, k-1$ and $i = k+1, \ldots, s-1$, $y_1 \notin \pi_1(y_2)$, $y_2 \notin \pi_2(y_3)$, $y_{k+1} \notin \pi_1(y_{k+2})$, $y_{k+2} \notin \pi_2(y_3)$, and $y_s \notin \pi_1(y_{s-1})$. Applying Theorem 3.1 for $H = (F, G)$ and $L: U \to \mathbb{R}$, $L(y) = F(y) - G(y)$, we obtain that
\[ S^{k,s}_{x,s} = \{ f \in C^\infty_{emb}(X, U_0): \text{if } \nabla_x f^G(x) = 0, \text{then } L(f^G(x)) \neq 0 \} \]
contains a residual subset of $C^\infty_{emb}(X, U_0)$. Since
\[ \bigcap_{k<s} S^{k,s}_{x,s} \bigcap C \bigcap D \subset S, \]
we deduce that $S$ contains a residual subset of $C^\infty_{emb}(X, U_0)$.

To deal with $\mathcal{P}$ we define $F$ by (4.11) with $k = s$, exchanging $U$ suitably. For fixed $i$ and $j$ with $1 \leq i < j \leq s$ we use the function $L: U \to \mathbb{R}$, where and to express the fact that $y_{i+1}$ and $y_j$ are parallel. We omit the details.

**Proof of Theorem 2.** Denote by $\mathcal{T}$ the set of those $f \in C^\infty_{emb}(X, \mathbb{R}^n)$ such that every $(\omega, \theta)$-trajectory of $f(X)$ has no segments tangent to $f(X)$ and $\det dJ_y \neq 0$ (cf. subsection 2.4). It follows by [14], [15] that if we define $\mathcal{T}'$ in the same way by means of reflecting $(\omega, \theta)$-rays instead of $(\omega, \theta)$-trajectories, then $\mathcal{T}'$ contains a residual subset of $C^\infty_{emb}(X, \mathbb{R}^n)$. The same argument shows that $\mathcal{T}$ has this property, too.

Next, denote by $\mathcal{H}$ the set of those $f \in C^\infty_{emb}(X, \mathbb{R}^n)$ such that for every $y \in f(X)$ there are no directions $v \in T_y f(X) \setminus \{0\}$ such that the curvature of $f(X)$ at $y$ with respect to $v$ vanishes of order $2n-3$. It can be derived from the results of Landis [6] that $\mathcal{H}$ contains a residual subset of $C^\infty_{emb}(X, \mathbb{R}^n)$. Then $A = \mathcal{T} \cap \mathcal{P} \cap \mathcal{T} \cap \mathcal{H}$ contains a residual subset of $C^\infty_{emb}(X, \mathbb{R}^n)$. We will show that the inclusion (1.4) holds for $\Omega_f$, provided $f \in \mathcal{A}$.

Denote by $L_{\omega, \theta}^\infty(\Omega_f)$ the set of all $(\omega, \theta)$-ray in $\Omega_f$. Note that the set $\mathcal{B}_f$ is closed. Instead, assume that $\gamma_m \in L_{\omega, \theta}^\infty f$ for every $m \in \mathbb{N}$ and $T_{\gamma_m} \to T_0$. By a standard argument we deduce the existence of a $(\omega, \theta)$-ray $\gamma_0$ with sojourn time $T_0$. Moreover, the starting point $z_0 \in Z_1$ of $\gamma_0$ is a limit point of the set of starting points $\{ z_m : m \in \mathbb{N} \}$ of the rays $\gamma_m$. If $\gamma_0$ is formed only by linear segments, then all these segments are not tangent to $f(X)$, since $f \in \mathcal{T}$. On the other hand, if $\gamma_0$ is ordinary, then $f \in \mathcal{T}$ shows that the rays starting in a small neighbourhood of $z_0$ in $Z_1$ with direction $\omega$ are not $(\omega, \theta)$-rays. Thus $\gamma_0 \in L_{\omega, \theta}^\infty f$ and $\mathcal{B}_f$ is closed.

Let $\gamma \in L_{\omega, \theta}^\infty(\Omega_f)$ be an ordinary reflecting $(\omega, \theta)$-ray with sojourn time $T_\gamma$. Since $f \in \mathcal{T} \cap \mathcal{P} \cap \mathcal{T}$ and $T_\gamma \notin \mathcal{B}_f$, a continuity argument implies that for some $\varepsilon_0 > 0$ we have $T_\delta \notin [T_\gamma - \varepsilon_0, T_\gamma + \varepsilon_0]$ for all $\delta \in L_{\omega, \theta}(\Omega_f) \setminus \{ \gamma \}$.\[\]
Then we can repeat the localization procedure in the proof of Theorem 1. This procedure shows that the singularities of $s(t, \theta, \omega)$ in a small neighbourhood of $-T$, depend only on the ray $\gamma$. Since $\gamma$ is an ordinary $(\omega, \theta)$-ray with a non-vanishing differential cross section, we can repeat the arguments in [11], [16] to finish the proof of Theorem 2.

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