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# The four positive vortices problem: regions of chaotic behavior and the non-integrability (*)

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ABSTRACT. — We consider the problem of the planar motion of four point vortices with intensities $(1, 1, 1, \varepsilon)$, in a Eulerian incompressible fluid, as a perturbation of the problem of three unit vortices. The unperturbed problem is reduced to a planar autonomous Hamiltonian system which admits saddle connections. For $\varepsilon>0$ and sufficiently small, we also reduce, in a neighborhood of the above saddle connections, the problem to a planar Hamiltonian system, which is no longer autonomous but periodically time dependent. The Poincaré-map of the perturbed problem presents transversal intersections between stable and unstable manifolds of two

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hyperbolic points; this implies that there are new regions of chaotic behavior, different from the ones previously found by Ziglin. In particular our result yields a new proof of the non-analytic integrability of the four positive vortices problem.

RÉSUMÉ. — Nous considérons le problème du mouvement plan de quatre vortices avec intensités $(1, 1, 1, \varepsilon)$, dans un fluide parfait incompressible, comme une perturbation du problème de trois vortices unitaires. Le problème non perturbé est réduit à un système Hamiltonian autonome plan qui a des connections de selles. Pour $\varepsilon > 0$ suffisamment petit, dans un voisinage de ces connections, le problème complet peut être aussi réduit à un système Hamiltonian plan qui toutefois n'est pas autonome mais périodique par rapport au temps. La transformation de Poincaré du problème perturbé a des intersections transversales de variétés stable et instable de points hyperboliques; cela entraîne l'existence de régions de comportement chaotique différent de la région obtenue précédemment par Ziglin. En particulier notre résultat fournit une preuve nouvelle de la non intégrabilité analytique du problème des quatre vortices avec intensités positives.

## 1. INTRODUCTION AND THE GENERAL PROBLEM OF N VORTICES

Many detailed presentations about the planar vortex model in fluid mechanics are available in the literature. We refer the reader to Chorin and Marsden [C-M], Marchioro and Pulvirenti [M-P] and also to [O1]. In the present paper we are dealing with the vortex model in $\mathbf{R}^2$ (that is, with no boundary). The vorticity is assumed to be concentrated in N point-vortices, $\vec{x}_j = (x_j, y_j) \in \mathbf{R}^2$, $j = 1, \ldots, N$, and have constant intensities (circulations) $K_1, \ldots, K_N$, respectively. The velocity $\vec{u}_j(\vec{x}, t)$ at $\vec{x} = (x, y) \in \mathbf{R}^2$ due to the $j$-th vortex is given by

$$\vec{u}_j(\vec{x}, t) = \left( -\frac{K_j(y - y_j)}{2\pi |\vec{x} - \vec{x}_j|^2}, \frac{K_j(x - x_j)}{2\pi |\vec{x} - \vec{x}_j|^2} \right),$$

provided that we ignore the other vortices. When all the vortices are moving, they produce at $\vec{x}$ the velocity field $\vec{u}(\vec{x}, t) = \sum_{j=1}^{N} \vec{u}_j(\vec{x}, t)$. Each vortex ought to move as it was carried by the net velocity field of the other vortices, that is each $\vec{x}_j$, $j=1, \ldots, N$, moves according to the equations

$$\dot{x}_j = -\frac{1}{2\pi}\sum_{i\neq j}\frac{K_i(y_j-y_i)}{r_{ij}^2}, \qquad \dot{y}_j = \frac{1}{2\pi}\sum_{i\neq j}\frac{K_i(x_j-x_i)}{r_{ij}^2}, \qquad r_{ij}=|\vec{x}_i-\vec{x}_j|,$$

or, equivalently, for $i, j=1, \ldots, N$:

$$K_j\dot{x}_j=\frac{\partial H}{\partial y_j}, \qquad K_j\dot{y}_j=-\frac{\partial H}{\partial x_j}, \qquad H=-\frac{1}{4\pi}\sum_{i\neq j}K_iK_j\log|\vec{x}_i-\vec{x}_j|. \quad (1.1)$$

Because of the symmetries of the function H, system (1.1) above has the four first integrals

$$I_1=H, \qquad I_2=\sum_{\alpha=1}^{N}K_\alpha x_\alpha, \qquad I_3=\sum_{\alpha=1}^{N}K_\alpha y_\alpha, \qquad I_4=\sum_{\alpha=1}^{N}K_\alpha(x_\alpha^2+y_\alpha^2).$$

The construction of the velocity field $\vec{u}(\vec{x}, t)$ produces formal solutions of the Euler's equation in $\mathbf{R}^2$ and has the property that the classical circulation theorems are satisfied (*see* op. cit. above).

The general problem of N vortices (1.1) is defined in an open and dense set of $\mathbf{R}^{2N}$, since $r_{ij}\neq 0$ (collisions of vortices are not allowed) and becomes a Hamiltonian system presenting three first integrals independent and in involution with respect to the symplectic 2-form $\omega=\sum_{\alpha=1}^{N} dx'_\alpha \wedge dy'_\alpha$, where the canonical coordinates $(x'_\alpha, y'_\alpha)$ are given by:

$$x'_\alpha=\sqrt{|K_\alpha|}\,x_\alpha, \qquad y'_\alpha=\sqrt{|K_\alpha|}\,\mathrm{sign}(K_\alpha)\,y_\alpha, \qquad \alpha=1, \ldots, N.$$

Indeed, as was observed by Aref and Pomphrey [A-P],

$$\{I_2^2+I_3^2, I_4\}=0, \quad \{I_2, I_3\}=\sum_{\alpha=1}^{N}K_\alpha, \quad \{I_2, I_4\}=2I_3, \quad \{I_3, I_4\}=-2I_2,$$

where $\{\ ,\ \}$ denotes the Poisson bracket. Therefore, the vortex system for $N=3$ is Liouville analytically integrable. The motion of three vortices was completely analysed by Synge [Sy].

In the case of positive intensities ($K_\alpha>0$, $\alpha=1, \ldots, N$) all the solutions in the phase space are bounded (since $I_4=$Const. defines a compact set) and defined for all time (since $H=$Const.); in particular, when $N=2$ or 3 the phase space has regions foliated by invariant tori. Using carefully KAM theory, Khanin [K] showed that in the phase space of any system with an arbitrary number of vortices there exists a set of initial conditions of positive measure for which the motions of vortices are quasi-periodic.

On the other side, Ziglin [Z] considered the *restricted* problem of four vortices, that is, three unit vortices and a fourth vortex with zero intensity (that is, a simple particle of fluid). Let $a_i$, $i=1, 2, 3$ be the sides of the triangle determined by the three unit vortices, and $A_i$, $i=1, 2, 3$, be their opposite angles. Then the *relative* problem of the three vortices has the following equations derived from (1.1):

$$\left.\begin{aligned} \dot{a}_1 &= \frac{1}{2\pi}\left(\frac{\sin A_3}{a_2} - \frac{\sin A_2}{a_3}\right) \\ \dot{a}_2 &= \frac{1}{2\pi}\left(\frac{\sin A_1}{a_3} - \frac{\sin A_3}{a_1}\right) \\ \dot{a}_3 &= \frac{1}{2\pi}\left(\frac{\sin A_2}{a_1} - \frac{\sin A_1}{a_2}\right). \end{aligned}\right\} \tag{1.2}$$

System (1.2) admits two independent first integrals: $a_1 a_2 a_3 = c_1^3$ and $a_1^2 + a_2^2 + a_3^2 = c_2^2$. Substituting $a_3 = \dfrac{c_1^3}{a_1 a_2}$ into (1.2) Ziglin obtained the system $\dot{a} = F(a, c_2)$, $a = (a_1, a_2)$, with a center $a_0 = (c_1, c_1)$, that corresponds to an equilateral triangular configuration; then, he took the periodic solutions close to this elliptical fixed point which are given by a one-parameter family of periodic functions; choosing properly a small parameter $\nu$ for this family, he substituted these periodic functions into the two equations of motion of the fourth vortex. In this way he obtained a periodically time dependent Hamiltonian system

$$\frac{d\xi}{d\tau} = \frac{\partial F}{\partial \eta}, \qquad \frac{d\eta}{d\tau} = -\frac{\partial F}{\partial \xi}, \tag{1.3}$$

where $F = F(\xi, \eta, \tau, \nu) = F_0(\xi, \eta) + \nu F_1(\xi, \eta, \tau) + \ldots$ The unperturbed Hamiltonian system ($\nu = 0$) is defined by $F_0(\xi, \eta)$ and the corresponding phase portrait has a hyperbolic homoclinic fixed point. As usual (*see* Holmes [H]), for $\nu \neq 0$ it is necessary to examine system (1.3) in the extended phase space $\{\xi, \eta, \tau\,(\mathrm{mod}\, 2\pi)\}$ and consider the Poincaré map of the plane $\{\tau\,(\mathrm{mod}\, 2\pi) = \tau_0\}$ to itself, given by the cylindrical phase-flow. If, for $\nu \neq 0$, the homoclinic orbit of that hyperbolic fixed point splits into the unstable and the stable manifolds which intersect transversally in a (nondegenerate) homoclinic point, then the perturbed system presents a chaotic behavior (*see* Moser [Mo] and Smale [S]) since a *horseshoe* appears; in particular, no domain containing the closure of the trajectory of that homoclinic point admits an analytic first integral. The existence of such nondegenerate homoclinic point is assured, if the so-called Melnikov [M] integral has a simple zero. Ziglin reduced the proof of this condition to the nonvanishing of an improper integral and he succeeded in showing this, by evaluating the integral by computer. In [K] (appendix), Ziglin, by using only continuity arguments, extended the previous non-integrability result to the problem of four-vortices with positive intensities ($K_1$, $K_2$, $K_3$, $K_4$) sufficiently close to the intensities (1, 1, 1, 0) of the restricted case.

Many discussions appeared, after Ziglin result was published, but no other proof was presented. We decided to come back again to the question of the chaotic behaviour and the non-integrability of the four vortices problem. Our approach is to consider the problem of four vortices with intensities (1, 1, 1, $\varepsilon$) as a perturbation of the problem of motion of three unit vortices. This last problem admits saddle connections, and we reduce it, in a neighborhood of a saddle connection, to the integration of a planar Hamiltonian autonomous system. For $\varepsilon>0$ small enough, we also reduce the problem of the four vortices with intensities (1, 1, 1, $\varepsilon$) to a planar Hamiltonian system, which is no more autonomous but periodically time dependent. The Poincaré map related to this system has still two saddle points; the existence of a transversal intersection of the stable manifold of the first one with the unstable manifold of the other one is proved (by using the Melnikov method) by showing that a certain integral is different from zero. This integral has been evaluated by numerical methods and the accuracy of the result is assured by the boundedness of the integrand function. Our result still implies that there are new regions of chaotic behavior in the problem of four vortices with positive intensities (1, 1, 1, $\varepsilon$) and, in particular, gives another proof to the analytic non-integrability.

In [O2] one of the authors of the present paper reproduced the content of a survey talk which dealed briefly with the subject of this paper, by that time in preparation.

In [K-C], Koiller and Carvalho presented an analytical proof of the non-integrability of the four vortices problem, but in the case of two opposite strong vortices and two advected weak ones.

## 2. THE CASE OF FOUR VORTICES WITH POSITIVE INTENSITIES AND THE INTEGRABLE CASE OF THREE VORTICES

**2.1.** Let us consider three vortices $P_i=(x_i, y_i)$, $i=1, 2, 3$, with unit intensities and a vortex $P_4=(x_4, y_4)$ with intensity $\varepsilon>0$. Let $M_0$ and $M_1$ be the center of mass of $P_1 P_2 P_3 P_4$ and $P_1 P_2 P_3$, respectively (the masses are the intensities of the vortices). Then the following equalities hold:

$$\begin{cases} (P_1-M_1)+(P_2-M_1)+(P_3-M_1)=0 \\ 3(M_1-M_0)+\varepsilon(P_4-M_0)=0 \end{cases}$$

and in particular

$$(M_1 - M_0) = -\frac{\varepsilon}{3+\varepsilon}(P_4 - M_1).$$

Using these equalities, one can easily get:

$$P_1 - M_0 = 1/2\,[(P_1 - P_2) + (M_1 - P_3)] + \frac{\varepsilon}{3+\varepsilon}(M_1 - P_4)$$
$$P_2 - M_0 = -1/2\,[(P_1 - P_2) - (M_1 - P_3)] + \frac{\varepsilon}{3+\varepsilon}(M_1 - P_4)$$
$$P_3 - M_0 = -(M_1 - P_3) + \frac{\varepsilon}{3+\varepsilon}(M_1 - P_4)$$
$$P_4 - M_0 = -\frac{3}{3+\varepsilon}(M_1 - P_4).$$

Let us set:

$$\begin{gathered} M_0 = (\eta\, x_0,\ \eta\, y_0) \\ P_1 - P_2 = \alpha\sqrt{\tilde{p}_1}\, e^{i\theta_1} \\ M_1 - P_3 = \beta\sqrt{\tilde{p}_2}\, e^{i\theta_2} \\ M_1 - P_4 = \gamma\sqrt{\tilde{p}_3}\, e^{i\theta_3}, \end{gathered}$$

and let us determine positive numbers $\eta$, $\alpha$, $\beta$, $\gamma$, such that the transformation which takes $((x_1, y_1), (x_2, y_2), (x_3, y_3), \sqrt{\varepsilon}(x_4, y_4))$ into $(x_0, y_0, \tilde{p}_1, \theta_1, \tilde{p}_2, \theta_2, \tilde{p}_3, \theta_3)$ is a canonical one.

Then, one necessarily has:

$$\eta = \sqrt{\frac{1}{3+\varepsilon}}, \qquad \alpha = 2, \qquad \beta = \frac{2\sqrt{3}}{3}, \qquad \gamma = \sqrt{\frac{2(3+\varepsilon)}{3\varepsilon}},$$

and the transformation is given by:

$$x_1 = \frac{1}{\sqrt{3+\varepsilon}}x_0 + \left[\sqrt{\tilde{p}_1}\cos\theta_1 + \frac{\sqrt{3}}{3}\sqrt{\tilde{p}_2}\cos\theta_2\right] + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\cos\theta_3$$
$$y_1 = \frac{1}{\sqrt{3+\varepsilon}}y_0 + \left[\sqrt{\tilde{p}_1}\sin\theta_1 + \frac{\sqrt{3}}{3}\sqrt{\tilde{p}_2}\sin\theta_2\right] + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\sin\theta_3$$
$$x_2 = \frac{1}{\sqrt{3+\varepsilon}}x_0 - \left[\sqrt{\tilde{p}_1}\cos\theta_1 - \frac{\sqrt{3}}{3}\sqrt{\tilde{p}_2}\cos\theta_2\right] + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\cos\theta_3$$

$$
\begin{aligned}
y_2 &= \frac{1}{\sqrt{3+\varepsilon}} y_0 - \left[\sqrt{\tilde{p}_1}\sin\theta_1 - \frac{\sqrt{3}}{3}\sqrt{\tilde{p}_2}\sin\theta_2\right] + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\sin\theta_3 \\
x_3 &= \frac{1}{\sqrt{3+\varepsilon}} x_0 \qquad - \frac{2\sqrt{3}}{3}\sqrt{\tilde{p}_2}\cos\theta_2 + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\cos\theta_3 \\
y_3 &= \frac{1}{\sqrt{3+\varepsilon}} y_0 \qquad - \frac{2\sqrt{3}}{3}\sqrt{\tilde{p}_2}\sin\theta_2 + \sqrt{\frac{2}{3}}\sqrt{\frac{\varepsilon}{3+\varepsilon}}\sqrt{\tilde{p}_3}\sin\theta_3 \\
\bar{x}_4 &:= \sqrt{\varepsilon}\, x_4 = \sqrt{\frac{\varepsilon}{3+\varepsilon}} x_0 \qquad - \sqrt{2}\sqrt{\frac{3}{3+\varepsilon}}\sqrt{\tilde{p}_3}\cos\theta_3 \\
\bar{y}_4 &:= \sqrt{\varepsilon}\, y_4 = \sqrt{\frac{\varepsilon}{3+\varepsilon}} y_0 \qquad - \sqrt{2}\sqrt{\frac{3}{3+\varepsilon}}\sqrt{\tilde{p}_3}\sin\theta_3.
\end{aligned}
$$

If one makes $\varepsilon=0$, we have $M_1=M_0$, and the transformation above is reduced to the canonical transformation which takes $x_0$, $y_0$, $\tilde{p}_1$, $\theta_1$, $\tilde{p}_2$, $\theta_2$ to the cartesian coordinates of the three vortices $P_1$, $P_2$, $P_3$ and to the transformation

$$\bar{x}_4 = -\sqrt{2}\sqrt{\tilde{p}_3}\cos\theta_3, \qquad \bar{y}_4 = -\sqrt{2}\sqrt{\tilde{p}_3}\sin\theta_3.$$

The Hamiltonian function H of the system is given by:

$$
\begin{aligned}
H = -\frac{1}{4\pi}[(\log r_{12}^2 + \log r_{13}^2 + \log r_{23}^2) + \varepsilon(\log r_{14}^2 + \log r_{24}^2 + \log r_{34}^2)] \\
= H_0 + \varepsilon H_1,
\end{aligned}
$$

$H_0$ being the Hamiltonian function of the three unit vortices problem.

**2.2.** The squares of the distances between the three vortices will be expressed in the new coordinates as follows:

$$
\begin{aligned}
r_{12}^2 &= 4\tilde{p}_1, \\
r_{13}^2 &= \tilde{p}_1 + 3\tilde{p}_2 + 2\sqrt{3}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2), \\
r_{23}^2 &= \tilde{p}_1 + 3\tilde{p}_2 - 2\sqrt{3}\sqrt{\tilde{p}_1\tilde{p}_2}\cos(\theta_1-\theta_2).
\end{aligned}
$$

Therefore:

$$-4\pi H_0 = \log[\tilde{p}_1(\tilde{p}_1+3\tilde{p}_2)^2 - 12\tilde{p}_1^2\tilde{p}_2\cos^2(\theta_1-\theta_2)].$$

We remark that:

(*a*) $H_0$ does not depend on $x_0$, $y_0$ and therefore $x_0$ and $y_0$ are first integrals of the three vortices problem;

(*b*) $H_0$ depends on $\theta_1$ and $\theta_2$ by their difference only; consequently $\tilde{p}_1+\tilde{p}_2$ is a first integral of the system of three vortices.

Let $(p_1, q_1, p_2, q_2)$ be new coordinates defined by the canonical transformation:

$$\left.\begin{array}{ll} p_1=\tilde{p}_1 & q_1=\theta_1-\theta_2 \\ p_2=\tilde{p}_1+\tilde{p}_2 & q_2=\theta_2. \end{array}\right\} \tag{2.1}$$

The Hamiltonian function of the three unit vortices is expressed in the new coordinates by:

$$H_0=-\frac{1}{4\pi}\log[p_1(p_1+3(p_2-p_1))^2-12p_1^2(p_2-p_1)\cos^2 q_1)],$$

and the equations of the motion of three vortices are written as:

$$\left.\begin{array}{ll} \dot{p}_1=\dfrac{\partial H_0}{\partial q_1} & \dot{q}_1=-\dfrac{\partial H_0}{\partial p_1} \\ \dot{p}_2=0 & \dot{q}_2=-\dfrac{\partial H_0}{\partial p_2}. \end{array}\right\} \tag{2.2}$$

By defining V as:

$$V=-[p_1(p_1+3(p_2-p_1))^2-12p_1^2(p_2-p_1)\cos^2 q_1],$$

and introducing the new time:

$$\tau=\frac{1}{4\pi}e^{4\pi H_0}t,$$

[$H_0$ is constant along the solutions of (2.2)], system (2.2) turns into:

$$\left.\begin{array}{ll} \dfrac{dp_1}{d\tau}=\dfrac{\partial V}{\partial q_1} & \dfrac{dp_2}{d\tau}=0 \\ \dfrac{dq_1}{d\tau}=-\dfrac{\partial V}{\partial p_1} & \dfrac{dq_2}{d\tau}=-\dfrac{\partial V}{\partial p_2}. \end{array}\right\} \tag{2.3}$$

Due to the definition of $\tilde{p}_1$, $\tilde{p}_2$ and to (2.1), we will consider the function V restricted to the set:

$$\{(p_1, q_1, p_2): 0<p_1<p_2\}. \tag{2.4}$$

## 3. THE REDUCTION OF THE THREE UNIT VORTICES PROBLEM TO A PLANAR HAMILTONIAN SYSTEM

As $p_2$ is constant along the solutions of (2.3), the integration of system (2.3) is equivalent to the integration of the system:

$$\left.\begin{aligned}\frac{dp_1}{d\tau} &= -24 p_1^2 (\mu - p_1) \cos q_1 \sin q_1\\ \frac{dq_1}{d\tau} &= \cos^2 q_1 [36 p_1^2 - 24 \mu p_1] + (3\mu - 2p_1)(3\mu - 6p_1),\end{aligned}\right\} \tag{3.1}$$

with $\mu$ positive parameter. The critical points of (3.1), satisfying $0 < p_1 < \mu$, are:

(I) $p_1 = \frac{1}{2}\mu, \cos q_1 = 0$ equilateral triangle configurations;

(II) $p_1 = \frac{3}{4}\mu$, $\sin q_1 = 0$ collisions of $P_2$ and $P_3$ or $P_1$ and $P_3$;

(III) $p_1 = \frac{1}{4}\mu$, $\sin q_1 = 0$ collinear configurations $P_1\, P_2\, P_3$ or $P_3\, P_1\, P_2$;

The points (I) and (II) are centers and the points (III) are saddles. The function V assumes the value $-\mu^3$ at the positions (III). Therefore, the saddle connections are on the energy level $V(p_1, q_1, \mu) = -\mu^3$. As we have:

$$V(p_1, q_1, \mu) + \mu^3 = (p_1 - \mu)\left(p_1 - \frac{\mu}{2(2+\sqrt{3}\sin q_1)}\right)\left(p_1 - \frac{\mu}{2(2-\sqrt{3}\sin q_1)}\right),$$

the curve:

$$p_1 = \frac{\mu}{2(2+\sqrt{3}\sin q_1)}, \qquad 0 < q_1 < \pi, \quad p_2 = \mu, \tag{3.2}$$

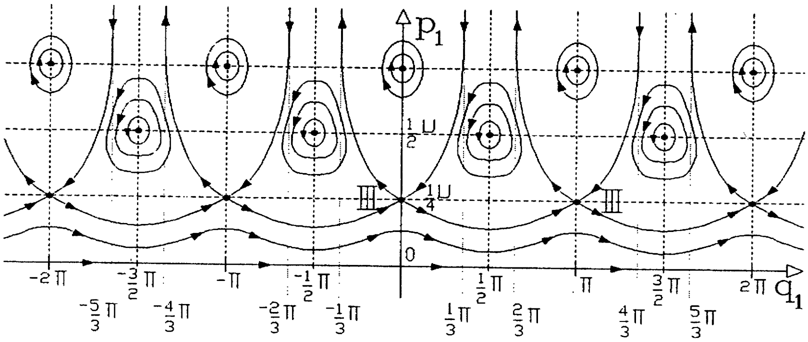
is a saddle connection of (3.1) contained in the set (2.4). The phase portrait of (3.1) is illustrated in the picture (*see* p. 108).

Now, we are interested in considering the solutions of (2.3) belonging to a preassigned energy level:

$$V(p_1, q_1, p_2) = -\mu^3 < 0. \tag{3.3}$$

Equation (3.3) can be explicitely solved with respect to $p_2$ and we have:

$$p_2 = \frac{2}{3} p_1 (1 + \cos^2 q_1) \pm \frac{1}{3}\sqrt{\frac{\mu^3 - p_1^3 \sin^2(2q_1)}{p_1}}, \tag{3.4}$$




with the right hand side defined for $0<p_1\leqq\mu$. The branch of (3.4) containing the curve (3.2) is:

$$p_2=\frac{2p_1(1+\cos^2 q_1)}{3}+\frac{1}{3}\sqrt{\frac{\mu^3-p_1^3\sin^2(2q_1)}{p_1}}=:h_0(p_1,\, q_1,\, \mu). \quad (3.5)$$

As, in a neighborhood of the separatrix (3.2), we have $\dfrac{\partial \mathrm{V}}{\partial p_2}\neq 0$, the solutions of (2.3) which satisfy (3.3) and whose orbits are near to (3.2) can be parametrized by means of $q_2$ and satisfy (3.5) and the system:

$$\left.\begin{aligned}\frac{dp_1}{dq_2}&=\frac{\partial h_0}{\partial q_1}(p_1,\, q_1,\, \mu)\\ \frac{dq_1}{dq_2}&=-\frac{\partial h_0}{\partial p_1}(p_1,\, q_1,\, \mu).\end{aligned}\right\} \quad (3.6)$$

The solution of (3.6), having as orbit the curve (3.2), is obtained by integrating the equation:

$$\begin{aligned}\frac{dq_1}{dq_2}&=-\frac{\partial h_0}{\partial p_1}\left(p_1=\frac{\mu}{2(2+\sqrt{3}\sin q_1)},\, q_1\cdot\, \mu\right)\\ &=\frac{(\partial \mathrm{V}/\partial p_1)(p_1=\mu/(2(2+\sqrt{3}\sin q_1)),q_1,\, \mu)}{(\partial \mathrm{V}/\partial p_2)(p_1=\mu/(2(2+\sqrt{3}\sin q_1)),\, q_1,\, \mu)}\\ &=\frac{2\sin q_1(\sqrt{3}+2\sin q_1)(2+\sqrt{3}\sin q_1)}{4+\sin^2 q_1+3\sqrt{3}\sin q_1}.\end{aligned}$$

We have:

$$q_2(q_1)-q_2^0=\int_{q_1^0}^{q_1}\frac{4+\sin^2 t+3\sqrt{3}\sin t}{(2\sin t)(\sqrt{3}+2\sin t)(2+\sqrt{3}\sin t)}dt$$
$$=\frac{1}{2\sqrt{3}}\log\left\{\frac{\operatorname{tg}^2(q_1/2)\sqrt{3}+\operatorname{tg}(q_1/2)(1/\sqrt{3})+\operatorname{tg}(q_1^0/2)}{\operatorname{tg}^2(q_1^0/2)\sqrt{3}+\operatorname{tg}(q_1^0/2)(1/\sqrt{3})+\operatorname{tg}(q_1/2)}\right\}$$
$$-\operatorname{arc\,tg}\left(\sqrt{3}+2\operatorname{tg}\frac{q_1}{2}\right)+\operatorname{arc\,tg}\left(\sqrt{3}+2\operatorname{tg}\frac{q_1^0}{2}\right).$$

Let us set:

$$q_1=x+\pi/2 \qquad -\pi/2<x<\pi/2$$
$$q_1^0=x_0+\pi/2 \qquad -\pi/2<x_0<\pi/2;$$

then we have:

$$q_2(x,\,x_0,\,q_2^0):=q_2(x+\pi/2)=q_2^0+\frac{1}{2\sqrt{3}}\log\frac{F(x)}{F(x_0)}$$
$$-\operatorname{arc\,tg}\left[\frac{(2+\sqrt{3})+(2-\sqrt{3})\operatorname{tg}(x/2)}{1-\operatorname{tg}(x/2)}\right]$$
$$+\operatorname{arc\,tg}\left[\frac{(2+\sqrt{3})+(2-\sqrt{3})\operatorname{tg}(x_0/2)}{1-\operatorname{tg}(x_0/2)}\right]$$
$$=q_2^0+\frac{1}{2\sqrt{3}}\log\frac{F(x)}{F(x_0)}\operatorname{arc\,tg}\left[(2-\sqrt{3})\operatorname{tg}\frac{x}{2}\right]$$
$$+\operatorname{arc\,tg}\left[(2-\sqrt{3})\operatorname{tg}\frac{x_0}{2}\right]=q_2^0+s(x)-s(x_0), \qquad (3.7)$$

with

$$F(x)=\frac{(1+\operatorname{tg}(x/2))^2}{(1-\operatorname{tg}(x/2))^2}\,\frac{1+\sqrt{3}-(\sqrt{3}-1)\operatorname{tg}(x/2)}{1+\sqrt{3}+(\sqrt{3}-1)\operatorname{tg}(x/2)}, \qquad (3.8)$$

and

$$s(x)=\frac{1}{2\sqrt{3}}\log F(x)-\operatorname{arc\,tg}\left[(2-\sqrt{3})\operatorname{tg}\frac{x}{2}\right]. \qquad (3.9)$$

We observe that $F(-x)=(F(x))^{-1}$, and, therefore $s(-x)=-s(x)$.

## 4. NEW REGIONS OF CHAOTIC BEHAVIOUR IN THE PROBLEM OF FOUR VORTICES

Let us consider the Hamiltonian function of the four vortices as function of the coordinates $(\tilde{p}_1, \theta_1; \tilde{p}_2, \theta_2; \tilde{p}_3, \theta_3)$:

$$H = H(\tilde{p}_1, \theta_1, \tilde{p}_2, \theta_2, \tilde{p}_3, \theta_3) = H_0 + \varepsilon H_1,$$

with:

$$H_0 = -\frac{1}{4\pi}\log[-V]\Bigg|_{\substack{p_1=\tilde{p}_1\\ q_1=\theta_1-\theta_2\\ p_2=\tilde{p}_1+\tilde{p}_2}}, \qquad H_1 = -\frac{1}{4\pi}\log(r_{14}^2 . r_{24}^2 . r_{34}^2).$$

It is easy to check that:

$$r_{14}^2 . r_{24}^2 = \frac{1}{\varepsilon^2}\left\{4\tilde{p}_3^2 + \sqrt{2(3+\varepsilon)}\sqrt{\varepsilon}\left[\frac{8}{3}\tilde{p}_3\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_3-\theta_2)\right] + \varepsilon A + \varepsilon^2 B + \sqrt{2(3+\varepsilon)}\,\varepsilon\sqrt{\varepsilon}\,C\right\},$$

with A, B and C defined as follows:

$$A = \frac{8}{3}\tilde{p}_3^2 + 4\tilde{p}_1\tilde{p}_3 + \frac{4}{3}\tilde{p}_2\tilde{p}_3 + \frac{8}{3}\tilde{p}_2\tilde{p}_3\cos^2(\theta_3-\theta_2) - 8\tilde{p}_1\tilde{p}_3\cos^2(\theta_3-\theta_1),$$

$$B = \tilde{p}_1^2 + \frac{1}{9}\tilde{p}_2^2 + \frac{4}{9}\tilde{p}_3^2 + \frac{4}{3}\tilde{p}_1\tilde{p}_3 + \frac{2}{3}\tilde{p}_1\tilde{p}_2 + \frac{4}{9}\tilde{p}_2\tilde{p}_3 + \frac{8}{9}\tilde{p}_2\tilde{p}_3\cos^2(\theta_3-\theta_2) - \frac{4}{3}\tilde{p}_1\tilde{p}_2\cos^2(\theta_2-\theta_1) - \frac{8}{3}\tilde{p}_1\tilde{p}_3\cos^2(\theta_3-\theta_1),$$

$$C = \sqrt{\tilde{p}_2\tilde{p}_3}\left[\cos(\theta_3-\theta_2)\left(\frac{4}{3}\tilde{p}_1 + \frac{8}{9}\tilde{p}_3 + \frac{4}{9}\tilde{p}_2\right) - \frac{8}{3}\tilde{p}_1\cos(\theta_1-\theta_3)\cos(\theta_1-\theta_2)\right].$$

Now it is possible to evaluate $r_{14}^2 . r_{24}^2 . r_{34}^2$, and one obtains:

$$r_{14}^2 . r_{24}^2 . r_{34}^2 = \frac{1}{\varepsilon^3}\{8\tilde{p}_3^3 + \varepsilon\phi_1 + \varepsilon^{3/2}\phi_2\},$$

with $\phi_1$ and $\phi_2$ defined by:

$$\phi_1 = -16\tilde{p}_2\tilde{p}_3^2\cos^2(\theta_3-\theta_2) + 8\tilde{p}_3^3 + 8\tilde{p}_2\tilde{p}_3^2 + 8\tilde{p}_1\tilde{p}_3^2 - 16\tilde{p}_1\tilde{p}_3^2\cos(\theta_1-\theta_3);$$

$$\phi_2 = \varepsilon^{1/2}\left[-\frac{64}{9}\tilde{p}_3^2\tilde{p}_2\cos^2(\theta_3-\theta_2) + A\frac{4}{3}\tilde{p}_2 + A\frac{2}{3}\tilde{p}_3 + B\,2\tilde{p}_3 - 8(3+\varepsilon)C\sqrt{\tilde{p}_3\tilde{p}_2}\cos(\theta_3-\theta_2)\right]$$
$$+\varepsilon\left[-\sqrt{2(3+\varepsilon)}B\sqrt{\tilde{p}_3\tilde{p}_2}\cos(\theta_3-\theta_2)\right.$$

$$+\sqrt{2(3+\varepsilon)}\,C\frac{4}{3}\tilde{p}_2-8(3+\varepsilon)\,C\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_3-\theta_2)\Big]$$
$$+\sqrt{2(3+\varepsilon)}\frac{32}{9}\tilde{p}_2\tilde{p}_3\sqrt{\tilde{p}_2\tilde{p}_3}\,.\cos(\theta_3-\theta_2)$$
$$+\sqrt{2(3+\varepsilon)}\frac{16}{9}\tilde{p}_3^2\sqrt{\tilde{p}_2\tilde{p}_3}\cos(\theta_3-\theta_2)$$
$$-\frac{4}{3}\sqrt{2(3+\varepsilon)}\,A\sqrt{\tilde{p}_3\tilde{p}_2}\cos(\theta_3-\theta_2)+\sqrt{2(3+\varepsilon)}\,C\,2\tilde{p}_3.$$

Let us denote by $\sigma(\varepsilon)$ the product $\varepsilon^{3/2}\,\phi_2$. We have:

$$H=-\frac{1}{4\pi}\log\{[-V]_{\substack{p_1=\tilde{p}_1\\ q_1=\theta_1-\theta_2\\ p_2=\tilde{p}_1+\tilde{p}_2}}[8\tilde{p}_3^3+\varepsilon\phi_1+\sigma(\varepsilon)]^{\varepsilon}\}+\varepsilon\log\frac{1}{\varepsilon^3}.$$

By means of the canonical transformation:

$$\begin{aligned} p_1&=\tilde{p}_1 & q_1&=\theta_1-\theta_2\\ p_2&=\tilde{p}_1+\tilde{p}_2 & q_2&=\theta_2-\theta_3\\ p_3&=\tilde{p}_1+\tilde{p}_2+\tilde{p}_3 & q_3&=\theta_3,\end{aligned}$$

the Hamiltonian function H turns into:

$$H=-\frac{1}{4\pi}\log\{-W\},$$

where W is defined by:

$$W(p_1,q_1,p_2,q_2,p_3)=[V(p_1,q_1,p_2)][8(p_3-p_2)^3+\varepsilon\phi_1+\sigma(\varepsilon)]^{\varepsilon},$$

with $\phi_1$ and $\sigma(\varepsilon)$ expressed by means of the new coordinates. In particular:

$$\phi_1=-16(p_3-p_2)^2(p_2-p_1)\cos^2 q_2-8(p_3-p_2)^3+8(p_3-p_2)^2(p_2-p_1)$$
$$+8p_1(p_3-p_2)^2-16p_1(p_3-p_2)^2\cos^2(q_1+q_2).$$

The function W is defined for $p_3>p_2$, it is $2\pi$-periodic in $q_2$ and it is independent of $q_3$. We have:

$$W=V(p_1,q_1,p_2)\Big\{1+3\varepsilon\log[2(p_3-p_2)]$$
$$+\frac{\varepsilon^2}{2}\Big[9\log^2[2(p_3-p_2)]+\frac{\phi_1}{4(p_2-p_3)^3}\Big]\Big\}+o(\varepsilon^2).$$

By using the new time $\tau$ defined by:

$$\frac{dt}{d\tau}=4\pi e^{-4\pi H},$$

the equations of motion are written as:

$$\left.\begin{array}{lll}\dfrac{dp_1}{d\tau}=\dfrac{\partial W}{\partial q_1} & \dfrac{dp_2}{d\tau}=\dfrac{\partial W}{\partial q_2} & \dfrac{dp_3}{d\tau}=0\\[2ex] \dfrac{dq_1}{d\tau}=-\dfrac{\partial W}{\partial p_1} & \dfrac{dq_2}{d\tau}=-\dfrac{\partial W}{\partial p_2} & \dfrac{dq_3}{d\tau}=-\dfrac{\partial W}{\partial p_3}.\end{array}\right\} \tag{4.1}$$

System (4.1) has the two first integrals:

$$p_3=\text{Const.} \qquad W(p_1, q_1, p_2, q_2, p_3, \varepsilon)=\text{Const.}$$

As, for a fixed $\mu>0$, $\left(\dfrac{\partial W}{\partial p_2}\right)_{\varepsilon=0}=\dfrac{\partial V}{\partial p_2}\neq 0$ along the curve (3.2), then the equation:

$$W(p_1, q_1, p_2, q_2, \mu+\alpha, \varepsilon)=-\mu^3, \tag{4.2}$$

is solvable with respect to $p_2$ for $\alpha>0$, $\varepsilon>0$, $\varepsilon$ small, and in a suitable neighborhood of the curve (3.2). We can assume that the solution of (4.2) takes its values in $\left\{p_2: |p_2-\mu|<\dfrac{\alpha}{2}\right\}$, and it can be written as:

$$p_2(p_1, q_1, q_2, \alpha, \varepsilon, \mu)=h_0(p_1, q_1, \mu)+\varepsilon\chi(p_1, q_1, q_2, \alpha, \varepsilon, \mu), \tag{4.3}$$

where $\chi$ is $2\pi$-periodic in $q_2$. As (4.3) solves (4.2), we get:

$$\chi_0:=(\chi)_{\varepsilon=0}=\frac{3\mu^3\log[2(\mu+\alpha-h_0(p_1, q_1, \mu))]}{(\partial V/\partial p_2)_{p_2=h_0(p_1, q_1, \mu)}}, \tag{4.4}$$

$$\begin{aligned}\chi_0':=\left(\frac{\partial\chi}{\partial\varepsilon}\right)_{\varepsilon=0}=-\frac{1}{(\partial V/\partial p_2)_{p_2=h_0}}\Bigg[&-\frac{9}{2}\mu^3\log^2[2(\mu+\alpha-h_0)]\\ &-\frac{\mu^3\phi_1}{8(\mu+\alpha-h_0)^3}+9\mu^6\frac{\log[2(\mu+\alpha-h_0)]}{(\mu+\alpha-h_0)(\partial V/\partial p_2)_{p_2=h_0}}\\ &+3\{\log[2(\mu+\alpha-h_0)]\}\left(\frac{\partial V}{\partial p_2}\right)_{p_2=h_0}+\frac{1}{2}\left(\frac{\partial^2 V}{\partial p_2^2}\right)_{p_2=h_0}\chi_0^2\Bigg],\end{aligned} \tag{4.5}$$

and:

$$p_2(p_1, q_1, q_2, \alpha, \varepsilon, \mu)=h_0(p_1, q_1, \mu)+\varepsilon\chi_0+\varepsilon^2\chi_0'+o(\varepsilon^2).$$

The solutions of (4.1), which are near to the curve (3.2) and in the energy level (4.2), satisfy the system:

$$\left.\begin{aligned}\frac{dp_1}{dq_2}&=\frac{\partial h_0}{\partial q_1}+\varepsilon\left[\frac{\partial\chi_0}{\partial q_1}+\varepsilon\frac{\partial\chi_0'}{\partial q_1}+o(\varepsilon)\right]\\ \frac{dq_1}{dq_2}&=-\frac{\partial h_0}{\partial p_1}-\varepsilon\left[\frac{\partial\chi_0}{\partial p_1}+\varepsilon\frac{\partial\chi_0'}{\partial p_1}+o(\varepsilon)\right].\end{aligned}\right\} \tag{4.6}$$

System (4.6) reduces to system (3.6), which describes the motion of three unit vortices, if we make $\varepsilon=0$.

The Melnikov integral (*see* [M] and [H]), related with the solution of (3.6), having as orbit the saddle connection (3.2), and the perturbed system (4.6) is written as:

$$\mathrm{I}(q_2^0)=-\int_{-\infty}^{+\infty}\left[\frac{\partial h_0}{\partial p_1}\frac{\partial \chi}{\partial q_1}-\frac{\partial h_0}{\partial q_1}\frac{\partial \chi}{\partial p_1}\right]_{\substack{p_1=p_1(q_2-q_2^0)\\ q_1=q_1(q_2-q_2^0)}} dq_2,$$

where $p_1=p_1(q_2-q_2^0)$, $q_1=q_1(q_2-q_2^0)$ represent the solution of (3.6), defined by (3.2) and (3.7) with $x_0=0$, that is:

$$\left.\begin{aligned} q_2-q_2^0&=s(x)\\ p_1&=\frac{\mu}{2(2+\sqrt{3}\cos x)},\qquad x\in(-\pi/2,\pi/2),\end{aligned}\right\}\tag{4.7}$$

with $q_1=x+\pi/2$ and $s(x)$ defined by (3.9).

Let us denote the pair $(p_1, q_1)$ by $z$; then one has:

$$\begin{aligned}\mathrm{I}(q_2^0)&=\int_{-\infty}^{+\infty}\frac{d}{dq_2}[\chi(z(q_2-q_2^0), q_2, \alpha, \varepsilon, \mu)]\,dq_2\\ &\quad-\int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial q_2}\chi(z, q_2, \alpha, \varepsilon, \mu)\right\}_{z=z(q_2-q_2^0)} dq_2\\ &=\int_{-\infty}^{q_2^0}\frac{\partial}{\partial q_2}\chi(\bar{z}, q_2, \alpha, \varepsilon, \mu)\,dq_2+\int_{q_2^0}^{+\infty}\frac{\partial}{\partial q_2}\chi(\bar{\bar{z}}, q_2, \alpha, \varepsilon, \mu)\,dq_2\\ &\quad-\int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial q_2}\chi(z, q_2, \alpha, \varepsilon, \mu)\right\}_{z=z(q_2-q_2^0)} dq_2,\end{aligned}$$

with $\bar{z}=\left(\frac{\mu}{4}, 0\right)$, $\bar{\bar{z}}=\left(\frac{\mu}{4}, \pi\right)$. By (4.4) and (4.5), it follows that:

$$\begin{aligned}\frac{\partial \chi}{\partial q_2}&=\varepsilon\frac{\partial}{\partial q_2}\chi_0'(p_1, q_1, q_2, \alpha, \mu)+o(\varepsilon)\\ &=\frac{1}{8}\frac{1}{(\partial \mathrm{V}/\partial p_2)_{p_2=h_0(p_1, q_1, \mu)}}\mu^3\varepsilon\frac{1}{[\mu+\alpha-h_0(p_1, q_1, \mu)]^3}\\ &\quad\times\left(\frac{\partial \phi_1^{(\mathrm{R})}}{\partial q_2}\right)_{p_2=h_0(p_1, q_1, \mu)}+o(\varepsilon),\end{aligned}$$

where

$$\phi_1^{(\mathrm{R})}=-16(\mu+\alpha-p_2)^2(p_2-p_1)\cos^2 q_2-16p_1(\mu+\alpha-p_2)^2\cos^2(q_1+q_2).$$

Along the motion (4.7) one has $h_0(p_1, q_1, \mu)=\mu$, and therefore:

$$\left[\frac{\partial}{\partial q_2}\chi\right]_{z=z(q_2-q_2^0)}=\frac{1}{8}\,\frac{1}{(\partial V/\partial p_2)_{p_2=\mu}}\mu^3\,\varepsilon\frac{1}{\alpha^3}\,\frac{\partial \phi_1^{(R)}}{\partial q_2}(p_1, q_1, q_2, \alpha, \mu)+o(\varepsilon)$$

with

$$\phi_1^{(R)}(p_1, q_1, q_2, \alpha, \mu)=-16\,\alpha^2\,[(\mu-p_1)\cos^2 q_2+p_1\cos^2(q_1+q_2)].$$

Now fix $\alpha=\frac{1}{2}$. Then one has:

$$\begin{aligned}
\mathrm{I}(q_2^0)=&-4\,\mu^3\,\varepsilon\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{\partial V/\partial p_2}[(\mu-p_1)\sin 2q_2\right.\right.\\
&\left.\left.+p_1\sin 2(q_1+q_2)]\right]_{\substack{p_1=p_1(q_2-q_2^0)\\ q_1=q_1(q_2-q_2^0)\\ p_2=\mu}}+\frac{1}{3}\mu\sin 2q_2\right\}dq_2+o(\varepsilon)\\
=&-4\,\mu^3\int_{-\pi/2}^{\pi/2}\left\{\frac{1}{(\partial V/\partial p_1)}\left[\left(\mu-\frac{\mu}{2}\,\frac{1}{2+\sqrt{3}\cos x}\right)\sin(2q_2^0+2s(x))\right.\right.\\
&\left.+\frac{\mu}{2}\,\frac{1}{2+\sqrt{3}\cos x}\sin(2q_2^0+x+\pi+2s(x))\right]\\
&\left.+\frac{1}{3\mu}\sin(2q_2^0+2s(x))\left(\frac{\partial V}{\partial p_2}\right)\Big/\left(\frac{\partial V}{\partial p_1}\right)\right\}_{\substack{p_1=\mu/2\,[1/2+\sqrt{3}\cos x]\\ q_1=x+\pi/2\\ p_2=\mu}}dx+o(\varepsilon)\\
=&-4\,\mu^3\,\varepsilon\int_{-\pi/2}^{\pi/2}\left\{-\frac{2+\sqrt{3}\cos x}{12\,\mu\cos x(\sqrt{3}+2\cos x)}\right.\\
&\times\left[\left(2-\frac{1}{2+\sqrt{3}\cos x}\right)\sin(2q_2^0+2s(x))\right.\\
&\left.-\frac{1}{2+\sqrt{3}\cos x}\sin(2q_2^0+2x+2s(x))\right]\\
&+\frac{1}{3\mu}\sin(2q_2^0+2s(x))\\
&\left.\times\frac{4+\cos^2 x+3\sqrt{3}\cos x}{2\cos x(\sqrt{3}+2\cos x)(2+\sqrt{3}\cos x)}\right\}dx+o(\varepsilon).
\end{aligned}$$

Finally, by grouping the factors of $\sin 2q_2^0$ and $\cos 2q_2^0$, we have:

$$\mathrm{I}(q_2^0)=\frac{2}{3}\mu^2\,\varepsilon\,[\mathrm{I}_1\sin 2q_2^0+\mathrm{I}_2\cos 2q_2^0]+o(\varepsilon).$$