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On essential self-adjointness of the relativistic hamiltonian of a spinless particle in a negative scalar potential

by

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ABSTRACT. – The relativistic quantum hamiltonian H describing a spinless particle in an electromagnetic field is considered, where H is associated with the classical hamiltonian $c\{m_0^2 c^2 + |p - A(x)|^2\}^{1/2} + V(x)$ via the Weyl correspondence. We show that if $V(x)$ is bounded below by a polynomial, H is essentially self-adjoint on $C_0^\infty(R^n)$. This result is quite different from that on the non-relativistic hamiltonian, *i. e.* the Schrödinger operator, and is close to that on the Dirac equation. Our proof is done by using the commutator theorem in [6].

RÉSUMÉ. – L'hamiltonien relativiste quantique H décrivant une particule sans spin dans un champ électromagnétique est considéré, où H est associé à l'hamiltonien classique $c\{m_0^2 c^2 + |p - A(x)|^2\}^{1/2} + V(x)$ via la correspondance de Weyl. Nous démontrons que si $V(x)$ est borné inférieurement par un polynôme, H est essentiellement auto-adjoint sur $C_0^\infty(R^n)$. Ce résultat est tout à fait différent de celui sur l'hamiltonien non-relativiste, c'est-à-dire l'opérateur de Schrödinger, et est voisin de celui sur l'opérateur de Dirac. La preuve est faite en utilisant le théorème du commutateur dans [6].

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1. INTRODUCTION

In the present paper we study the problem of essential self-adjointness of the operator

$$\mathrm{H} f(x)=\mathrm{H}_{\mathrm{A}} f(x)+\mathrm{V}(x) f(x) \equiv(2 \pi)^{-n} \mathrm{Os}-\iint e^{i(x-x') \cdot \xi} \times h_{\mathrm{A}}\left(\frac{x+x'}{2}, \xi\right) f(x') \, dx' \, d\xi+\mathrm{V}(x) f(x) \quad (1.1)$$

as an operator in the Hilbert space $\mathrm{L}^2(\mathrm{R}^n)$, where

$$\left.\begin{array}{l} h_{\mathrm{A}}(x, \xi)=c\{m_0^2 c^2+|\xi-\mathrm{A}(x)|^2\}^{1/2}, \\ \mathrm{A}(x)=(a_1(x), \ldots, a_n(x)), \end{array}\right\} \quad (1.2)$$

$\mathrm{V}(x)$ is a real valued function and c, m_0 are positive constants. $\mathrm{Os}-\iint \ldots dx' d\xi$ means the oscillatory integral (e.g. chapter 1 in [11]). $\mathrm{L}^2=\mathrm{L}^2(\mathrm{R}^n)$ is the space of all square integrable functions on R^n. H_{A} is called the Weyl quantized hamiltonian with a classical hamiltonian $h_{\mathrm{A}}(x, \xi)$. When $n=3$, this operator H can be considered as the hamiltonian describing a relativistic spinless particle with charge one and rest mass m_0 in an electromagnetic field whose scalar and vector potentials are given by $\mathrm{V}(x)$ and $\mathrm{A}(x)$ respectively. There c denotes the velocity of light ([16], [7], [4], [8] and etc.).

Let $\mathrm{C}_0^\infty(\mathrm{R}^n)$ be the space of all infinitely differentiable functions with compact support. We denote H_{A} where $\mathrm{A}(x)=(0, \ldots, 0)$ by H_0. Essential self-adjointness and spectral properties of $\mathrm{H}_0+\mathrm{V}(x)$ where $\mathrm{V}(x)$ is the Coulomb potential, a Yukawa-type potential and their sum have been studied in [16], [7] and [4]. On the other hand as for general H_{A}, essential self-adjointness of $\mathrm{H}=\mathrm{H}_{\mathrm{A}}+\mathrm{V}(x)$ has been studied in [12], [8] and [9] under the assumption that $\mathrm{V}(x)$ is bounded from below. Recently the author proved self-adjointness of H with domain $\{f(x) \in \mathrm{L}^2;\ \mathrm{H} f(x) \in \mathrm{L}^2\}$ as one of results in [10] under the assumptions (1.3) and (1.4) below.

$\left(\frac{\partial}{\partial x}\right)^\alpha a_j(x) \equiv\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} a_j(x)\ (j=1, 2, \ldots, n)$ are bounded on R^n for all multi-indices $\alpha=(\alpha_1, \ldots, \alpha_n)$ such that

$$|\alpha| \equiv \alpha_1+\ldots+\alpha_n \neq 0. \quad (1.3)$$

There exists a constant $m \geqq 0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \mathrm{V}(x)\right| \leqq \mathrm{C}_\alpha(1+|x|)^m \quad \text{on } \mathrm{R}^n$$

are valid for all multi-indices α with constants C_α. (1.4)

Our aim in the present paper is to show that the above assumption (1.4) can be replaced by a much weaker one for essential self-adjointness of H with domain $C_0^\infty(R^n)$. For example, we can obtain the following results. We denote by $L^2_{loc} \equiv L^2_{loc}(R^n)$ the space of all locally square integrable functions. Let $V(x)$ be a real valued function in L^2_{loc} such that

$$-C(1+|x|)^m \leqq V(x) \quad \text{on } R^n \tag{1.5}$$

is valid for non-negative constants C and m. Let Z be a constant less than $(n-2)c/2$. Then both $H_A + V(x)$ and $H_0 - \dfrac{Z}{|x|} + V(x)$ with domain $C_0^\infty(R^n)$ are essentially self-adjoint under a slightly weaker assumption than (1.3) (Theorem 2.2 and Corollary 2.4 in the present paper). $n \geqq 3$ is assumed for the latter operator. The assumption (1.3) is not so limited, because we need such an assumption to define H_A by (1.1). But we must note that a more general definition of H_A is proposed in [8].

As for the Schrödinger operators $-\dfrac{1}{2m_0}\Delta + V_S(x)$, we know that we need for their essential self-adjointness the limitation on the decreasing rate at infinity of negative part of $V_S(x)$ (e.g. Theorem 2 in [5] and page 157 in [1]). On the other hand as for the Dirac operator, we know from Theorem 2.1 in [3] that such a limitation is not necessary at all for its essential self-adjointness. Hence our decreasing rate (1.5) for essential self-adjointness of H lies between those of the Schrödinger and the Dirac operators.

Our proof in the present paper is quite different from that in [10]. In [10] we studied the theory of pseudo-differential operators with basic weight functions and applied it. In the present paper we use the commutator theorem in [6].

The plan of the present paper is as follows. In section 2 we will state all results. Some of results will be proved there. Sections 3 and 4 will be devoted to the proofs of main results.

2. THEOREMS

Let $k(x, \xi)$ be a C^∞-function on R^{2n}. We suppose that for any multi-indices $\alpha \neq (0, \ldots, 0)$ and β there exists a constant $C_{\alpha,\beta}$ satisfying

$$|k^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\langle x \rangle \quad \text{on } R^{2n}, \tag{2.1}$$

where $\langle x \rangle = \{1+|x|^2\}^{1/2}$ and $k^{(\alpha)}_{(\beta)}(x, \xi) = \left(\dfrac{\partial}{\partial \xi}\right)^\alpha \left(\dfrac{1}{i}\right)^{|\beta|} \left(\dfrac{\partial}{\partial x}\right)^\beta k(x, \xi)$. It follows from the mean value theorem that

$$|k_{(\beta)}(x, \xi) - k_{(\beta)}(x, 0)| \leqq C_\beta \langle x \rangle \langle \xi \rangle$$

are valid for all β with constants C_β. Hence by analogy with arguments in chapter 2 of [11] and chapter 4 of [15] we can define the pseudo-differential operator $K(X, D_x)$ with symbol $k(x, \xi)$ by

$$K(X, D_x) f(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} k(x, \xi) \hat{f}(\xi) \, d\xi, \tag{2.2}$$

for $f(x) \in \mathscr{S}$. $\hat{f}(\xi)$ denotes the Fourier transformation $\int e^{-ix \cdot \xi} f(x) \, dx$ and $\mathscr{S}$ the space of all rapidly decreasing functions on R^n. It is easy to show that $K(X, D_x)$ makes a continuous operator from $\mathscr{S}$ into $\mathscr{S}$.

THEOREM 2.1. – *Let* $\Phi(x)$ *be a real valued function in* $L^2_{loc}(R^n)$. *Assume that* $K(X, D_x)$ *defined above is symmetric on* $C_0^\infty(R^n)$ *and that*

$$K(X, D_x) + \Phi(x) \geqq 0 \quad \text{on } C_0^\infty(R^n). \tag{2.3}$$

The quadratic form inequality (2.3) means that

$(\{K(X, D_x) + \Phi(x)\} f(x), f(x)) \geqq 0$ for all $f(x) \in C_0^\infty(R^n)$. Moreover we assume that for all $W(x)$ being in L^2_{loc} with $W(x) \geqq 0$ almost everywhere (a. e.) $K(X, D_x) + \Phi(x) + W(x)$ with domain $C_0^\infty(R^n)$ is essentially self-adjoint. Then if $V(x) \in L^2_{loc}$ satisfies (1.5) for non-negative constants C and m, then $K(X, D_x) + \Phi(x) + V(x)$ with domain $C_0^\infty(R^n)$ is also essentially self-adjoint.

Theorem 2.1 will be proved in section 3. We will prove the following theorem from Theorem 2.1 by using the results obtained in [8].

THEOREM 2.2. – *Consider* H *defined by* (1.1) *with domain* $C_0^\infty(R^n)$. We assume

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha a_j(x) \right| \leqq C_\alpha \log \{ \langle x \rangle \} \tag{2.4}$$

for all $\alpha \neq (0, \ldots, 0)$ with constants C_α. Let $V(x)$ be the same function as in Theorem 2.1. Then H is essentially self-adjoint.

Remark 2.1. – As was stated in introduction, H defined by (1.1) with domain $\{ f(x) \in L^2; H f(x) \in L^2 \}$ is self-adjoint under the assumptions (1.3) and (1.4). We note that this H is also self-adjoint even if (1.3) is replaced by (2.4) there. This result follows from Theorem 1 in [10] at once.

Proof of Theorem 2.2. – We can easily have from the assumption

$$| h_{A(\beta)}^{(\alpha)}(x, \xi) | \leqq C'_{\alpha, \beta} \{ \langle x \rangle^2 + \langle \xi \rangle \}$$

for all α and β with constants $C'_{\alpha, \beta}$. So it follows from the analogy with arguments in section 2 of chapter 2 in [11] that H_A makes a continuous operator from $\mathscr{S}$ to $\mathscr{S}$ and H_A is symmetric on $\mathscr{S}$. We note that the assertion in Lemma 2.2 in [8] remains valid under our weaker assumption (2.4) than that in [8]. So Theorem 5.1 in [8] indicates $H_A \geqq 0$ on $C_0^\infty(R^n)$ and essential self-adjointness of $H_A + W(x)$ with domain $C_0^\infty(R^n)$ for any $W(x) \in L^2_{loc}$ such that $W(x) \geqq 0$ a.e.

We set

$$p(x, \xi) = (2\pi)^{-n} \operatorname{Os} - \iint e^{-iy\cdot\eta} h_A(x + y/2, \xi + \eta)\, dy\, d\eta.$$

Then

$$P(X, D_x) = H_A \quad \text{on } C_0^\infty(R^n) \tag{2.5}$$

follows from analogy of Theorem 2.5 in [11]. Let l be an even integer such that $l > n + 1$. Then taking the integration by parts, we have

$$p_{(\beta)}^{(\alpha)}(x, \xi) = (2\pi)^{-n} \operatorname{Os} - \iint e^{-iy\cdot\eta} \langle y \rangle^{-l} (1 - \Delta_\eta)^{l/2} \times \{ \langle \eta \rangle^{-l} (1 - \Delta_y)^{l/2} h_{A(\beta)}^{(\alpha)}(x + y/2, \xi + \eta) \}\, dy\, d\eta$$

for any α and β. We note that $h_A(x, \xi)$ satisfies the same inequalities as (2.1) for all α and β such that $|\alpha + \beta| \neq 0$ with another constants $C_{\alpha,\beta}$ under the assumption (2.4). So using $\langle x + y \rangle^\sigma \leqq \sqrt{2} \langle x \rangle^\sigma \langle y \rangle$ ($\sigma = 1$ and -1, $x, y \in R^n$), we can see that

$$| p_{(\beta)}^{(\alpha)}(x, \xi) | \leqq C''_{\alpha,\beta} \langle x \rangle \tag{2.6}$$

are valid for all α and β such that $|\alpha + \beta| \neq 0$ with constants $C''_{\alpha,\beta}$. Hence we can easily see from (2.5) and (2.6) that we can apply Theorem 2.1 to $H_A + V(x)$ as $K(X, D_x) = H_A = P(X, D_x)$ and $\Phi(x) = 0$. So Theorem 2.2 can be proved.

Q.E.D.

Remark 2.2. — As will be noted in Remark 3.1 in the present paper, the assumption in Theorem 2.1 that (2.1) must hold for all $\alpha \neq (0, \ldots, 0)$ and β can be weaken. The assertion of Theorem 2.1 remains valid even if we replace this assumption by a weaker one that (2.1) holds for all $\alpha \neq (0, \ldots, 0)$ and β satisfying $|\alpha| \leqq J$ and $|\beta| \leqq J$, where J is an integer determined from n and m. So the assumption on $\{a_j(x)\}_{j=1}^n$ in Theorem 2.2 can be similarly replaced by a weaker one that

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha a_j(x) \right| \leqq C_\alpha \langle x \rangle^\varepsilon$$

are valid for all $0 < |\alpha| \leqq J$, where $\varepsilon > 0$ is a sufficiently small constant and J is a sufficiently large integer. ε and J are determined from n and m.

THEOREM 2.3. — *Let* H_0 *be the operator defined in introduction with domain* $C_0^\infty(R^n)$. *Suppose that* $\Phi(x)$ *is a real valued function in* L^2_{loc} *and a* H_0-*bounded multiplication operator with relative bound less than one. Let*

$V(x)$ be the same function as in Theorem 2.1. *Then* $H_0+\Phi(x)+V(x)$ *with domain* $C_0^\infty(R^n)$ *is essentially self-adjoint.*

Theorem 2.3 will be proved in section 4.

COROLLARY 2.4. — *Let* $n\geqq 3$ *and* Z *be a constant less than* $(n-2)c/2$. *Let* $V(x)$ *be the same function as in Theorem* 2.1. *Then* $H_0-\dfrac{Z}{|x|}+V(x)$ *with domain* $C_0^\infty(R^n)$ *is essentially self-adjoint.*

Proof of Corollary 2.4. — When $Z\leqq 0$, essential self-adjointness of $H_0-\dfrac{Z}{|x|}+V(x)$ follows from Theorem 2.2 at once. Let $0<Z<\left(\dfrac{n-2}{2}\right)c$. We denote L^2-norm by $\|\,.\,\|$. We know the Hardy inequality

$$\left(\frac{n-2}{2}\right)^2\left\|\frac{\psi(x)}{|x|}\right\|^2\leqq\sum_{j=1}^n\left\|\frac{\partial\psi}{\partial x_j}(x)\right\|^2$$

for $\psi(x)\in C_0^\infty(R^n)$ (e.g. page 169 in [13] and (2.9) in [7]). So

$$\begin{aligned}\left(\frac{n-2}{2}\right)^2\left\|\frac{\psi(x)}{|x|}\right\|^2&\leqq(2\pi)^{-n}\int|\xi|^2|\hat{\psi}(\xi)|^2\,d\xi\\&\leqq(2\pi)^{-n}\int|\{m_0^2c^2+|\xi|^2\}^{1/2}\hat{\psi}(\xi)|^2\,d\xi\\&=c^{-2}\|H_0\psi(x)\|^2\end{aligned}$$

holds for $\psi(x)\in C_0^\infty(R^n)$. Consequently $-\dfrac{Z}{|x|}$ is H_0-bounded with relative bound less than one. Hence Corollary 2.4 follows from Theorem 2.3 at once.

Q.E.D.

3. PROOF OF THEOREM 2.1

LEMMA 3.1. — *Suppose that* $k(x,\xi)$ *satisfies* (2.1) *for all* $\alpha\neq(0,\ldots,0)$ *and* β. *Let* ζ *be a non-negative constant. Then there exists a positive constant* $d=d(\zeta)$ *such that*

$$\|[K(X,D_x),\langle x\rangle^{\zeta/2}]f(x)\|\leqq d\|\langle x\rangle^{\zeta/2}f(x)\|\qquad(3.1)$$

are valid for all $f(x)\in\mathcal{S}$. $[K(X,D_x),\langle x\rangle^{\zeta/2}]$ *denotes the commutator of operators* $K(X,D_x)$ *and* $\langle x\rangle^{\zeta/2}$.

Proof. – We set

$$q(x, \xi) = (2\pi)^{-n}\,\mathrm{Os}-\iint e^{-iy\cdot\eta}\,k(x, \xi+\eta)\,\langle x+y\rangle^{\zeta/2}\,dy\,d\eta - \langle x\rangle^{\zeta/2}\,k(x, \xi). \quad (3.2)$$

Then we get by analogy with arguments in chapter 2 of [11]

$$Q(X, D_x) = [K(X, D_x), \langle x\rangle^{\zeta/2}] \quad \text{on } \mathscr{S}. \quad (3.3)$$

It is easy to see

$$q(x, \xi) = (2\pi)^{-n}\int_0^1 d\theta \sum_{|\alpha|=1} \mathrm{Os}-\iint e^{-iy\cdot\eta}\,k^{(\alpha)}(x, \xi+\theta\eta)\,D_x^{\alpha}\langle x+y\rangle^{\zeta/2}\,dy\,d\eta,$$

where $D_x^{\alpha} = \left(\frac{1}{i}\right)^{|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha}$. Let l_1 and l_2 be integers such that $l_1 > n + \left|\frac{\zeta}{2} - 1\right|$ and $l_2 > n$. Then taking the integration by parts,

$$|q(x, \xi)| \leqq (2\pi)^{-n}\int_0^1 d\theta \sum_{|\alpha|=1}\iint |\langle y\rangle^{-l_1}(1-\Delta_\eta)^{l_1/2}\{\langle\eta\rangle^{-l_2}(1-\Delta_y)^{l_2/2}\,k^{(\alpha)}(x, \xi+\theta\eta)\,D_x^{\alpha}\langle x+y\rangle^{\zeta/2}\}|\,dy\,d\eta$$

holds. So we get

$$|q(x, \xi)| \leqq C_0\langle x\rangle^{\zeta/2}$$

with a constant C_0 from the assumption (2.1) in the same way to the proof of (2.6). Similarly we obtain

$$|q_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha,\beta}\langle x\rangle^{\zeta/2} \quad (3.4)$$

for all α and β with constants $C_{\alpha,\beta}$.

Next we set

$$r(x, \xi) = (2\pi)^{-n}\,\mathrm{Os}-\iint e^{-iy\cdot\eta}\,q(x, \xi+\eta)\,\langle x+y\rangle^{-\zeta/2}\,dy\,d\eta. \quad (3.5)$$

Then we have

$$R(X, D_x) = Q(X, D_x)\circ\langle x\rangle^{-\zeta/2}. \quad (3.6)$$

$\cdot\circ\cdot$ denotes the product of operators. Then we obtain from (3.4)

$$|r_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C'_{\alpha,\beta} \quad (3.7)$$

for all α and β with constants $C'_{\alpha,\beta}$ in the same way to the proof of (3.4).

We note that

$$[K(X, D_x), \langle x\rangle^{\zeta/2}] = R(X, D_x)\circ\langle x\rangle^{\zeta/2} \quad \text{on } \mathscr{S}$$

holds from (3.3) and (3.6). So applying the Calderón-Vaillancourt theorem in [2] to $R(X, D_x)$, we get Lemma 3.1.

Q.E.D.

Proof of Theorem 2.1. — For the sake of simplicity we denote $C_0^\infty(R^n)$ by $\mathscr{E}$. Let $d = d(m)$ be the constant determined in Lemma 3.1. We can choose a constant $M > 0$ satisfying

$$M \geqq 2\, d(m) \qquad \text{and} \qquad V(x) + M\langle x\rangle^m \geqq 0 \quad \text{a.e.} \tag{3.8}$$

because of the assumption (1.5). We fix this M. Set

$$T = K(X, D_x) + \Phi(x) + V(x) \tag{3.9}$$

with domain $\mathscr{E}$. It follows from the assumptions in Theorem 2.1 and (3.8) that $T + 3M\langle x\rangle^m \geqq 2M\langle x\rangle^m$ on $\mathscr{E}$ holds and $T + 3M\langle x\rangle^m$ with domain $\mathscr{E}$ is essentially self-adjoint. Let N be the self-adjoint operator defined by the closure of $T + 3M\langle x\rangle^m$. Then

$$N \geqq 2M\langle x\rangle^m > 0 \quad \text{on } \mathscr{E} \tag{3.10}$$

is valid and $\mathscr{E}$ is a core for N.

We will prove

$$\| T f(x) \| \leqq \| N f(x) \| \qquad [f(x) \in \mathscr{E}] \tag{3.11}$$

and

$$\pm i\{(Tf, Nf) - (Nf, Tf)\} \leqq 3\, d(Nf, f) \qquad [f(x) \in \mathscr{E}]. \tag{3.12}$$

$(.,.)$ implies the inner product in $L^2(R^n)$. Then Corollary 1.1 in [6] shows that T is essentially self-adjoint, which completes the proof.

We will first prove (3.11). Let $f(x) \in \mathscr{E}$. Since each $\Phi(x)$ and $V(x)$ is in L^2_{loc}, we can easily have

$$\begin{aligned}(Tf, \langle x\rangle^m f) = {} & (T \circ \langle x\rangle^{m/2} f, \langle x\rangle^{m/2} f) \\ & - ([K(X, D_x), \langle x\rangle^{m/2}] f, \langle x\rangle^{m/2} f).\end{aligned} \tag{3.13}$$

We denote by Re(.) and Im(.) the real part and the imaginary part of complex number respectively. Then noting $Nf = Tf + 3M\langle x\rangle^m f$, we get by (3.13)

$$\begin{aligned}\| Nf \|^2 &= \| Tf \|^2 + 6M\, \mathrm{Re}(Tf, \langle x\rangle^m f) + 9M^2 \| \langle x\rangle^m f \|^2 \\ &= \| Tf \|^2 + 6M\, \mathrm{Re}(\{T + M\langle x\rangle^m\} \circ \langle x\rangle^{m/2} f, \langle x\rangle^{m/2} f) \\ &\quad + 3M^2 \| \langle x\rangle^m f \|^2 - 6M\, \mathrm{Re}([K(X, D_x), \langle x\rangle^{m/2}] f, \langle x\rangle^{m/2} f).\end{aligned} \tag{3.14}$$

It is easy to see from the assumption (2.3) and (3.8)

$$T + M\langle x\rangle^m \geqq 0 \quad \text{on } \mathscr{E}. \tag{3.15}$$

Hence applying Lemma 3.1 to (3.14), we obtain by (3.8)

$$\begin{aligned}
&\|\mathrm{N} f\|^2 \\
&\geqq \|\mathrm{T} f\|^2 + 3\,\mathrm{M}^2 \|\langle x\rangle^m f\|^2 - 6\,\mathrm{M}\,d \|\langle x\rangle^{m/2} f\|^2 \\
&\geqq \|\mathrm{T} f\|^2 + 3\,\mathrm{M}(\mathrm{M} - 2\,d) \|\langle x\rangle^m f\|^2 \\
&\geqq \|\mathrm{T} f\|^2,
\end{aligned}$$

which shows (3.11).

Next we will prove (3.12). Let $f(x)\in\mathscr{E}$. Using $\mathrm{N} f = \mathrm{T} f + 3\,\mathrm{M}\langle x\rangle^m f$ and $\Phi(x)$, $\mathrm{V}(x)\in\mathrm{L}^2_{\mathrm{loc}}$, we have

$$\begin{aligned}
&(\mathrm{T} f, \mathrm{N} f) - (\mathrm{N} f, \mathrm{T} f) \\
&= (\mathrm{T} f, 3\,\mathrm{M}\langle x\rangle^m f) - (3\,\mathrm{M}\langle x\rangle^m f, \mathrm{T} f) \\
&= 3\,\mathrm{M}\{(\mathrm{K}(\mathrm{X}, \mathrm{D}_x) f, \langle x\rangle^m f) - (\langle x\rangle^m f, \mathrm{K}(\mathrm{X}, \mathrm{D}_x) f)\} \\
&= 6\,\mathrm{M}\, i\, \mathrm{Im}(\mathrm{K}(\mathrm{X}, \mathrm{D}_x) f, \langle x\rangle^m f).
\end{aligned}$$

Apply the equality

$$\begin{aligned}
(\mathrm{K}(\mathrm{X}, \mathrm{D}_x) f, \langle x\rangle^m f) = {}& (\mathrm{K}(\mathrm{X}, \mathrm{D}_x)\circ\langle x\rangle^{m/2} f, \langle x\rangle^{m/2} f) \\
& - ([\mathrm{K}(\mathrm{X}, \mathrm{D}_x), \langle x\rangle^{m/2}] f, \langle x\rangle^{m/2} f)
\end{aligned}$$

to the above. Then since $\mathrm{K}(\mathrm{X}, \mathrm{D}_x)$ is assumed to be symmetric on $\mathscr{E}$,

$$\begin{aligned}
&(\mathrm{T} f, \mathrm{N} f) - (\mathrm{N} f, \mathrm{T} f) \\
&\qquad = -6\,\mathrm{M}\, i\, \mathrm{Im}([\mathrm{K}(\mathrm{X}, \mathrm{D}_x), \langle x\rangle^{m/2}] f, \langle x\rangle^{m/2} f) \quad (3.16)
\end{aligned}$$

is valid. Hence we obtain by Lemma 3.1

$$\begin{aligned}
&\pm i\{(\mathrm{T} f, \mathrm{N} f) - (\mathrm{N} f, \mathrm{T} f)\} \\
&\quad \leqq 6\,\mathrm{M}\, d \|\langle x\rangle^{m/2} f\|^2 \\
&\quad = 6\,\mathrm{M}\, d (\langle x\rangle^m f, f) \\
&\quad \leqq 3\, d(\mathrm{N} f, f).
\end{aligned}$$

Here we used (3.10) for the last inequality. Thus (3.12) could be proved. This completes the proof.

Q.E.D.

Remark 3.1. — We can easily see in the proof of Theorem 2.1 from the Calderón-Vaillancourt theorem that if (3.7) holds for $|\alpha|\leqq 3\,n$ and $|\beta|\leqq 3\,n$, (3.1) is valid. Hence as was stated in Remark 2.2, we can weaken the assumption in Theorem 2.1 that (2.1) hold for all $\alpha\neq(0, \ldots, 0)$ and β. This can be easily verified by following the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.3

We denote $C_0^\infty(R^n)$ by $\mathscr{E}$ as in section 3. It is easy to see $H_0 \geqq m_0 c^2 > 0$ on $\mathscr{S}$. $\Phi(x)$ was assumed to be H_0-bounded with relative bound less than one. So it follows from Theorem X.18 in [13] that $\Phi(x)$ is form-bounded with the same relative bound with respect to H_0. That is, there exists a constant $b \geqq 0$ such that

$$|(\Phi(x) f, f)| < (H_0 f, f) + b(f, f)$$

are valid for all $f(x) \in \mathscr{E}$. Hence we see

$$\{H_0 + \Phi(x) + b\} \geqq 0 \quad \text{on } \mathscr{E}. \tag{4.1}$$

We will show that $H_0 + \Phi(x) + b + W(x)$ with domain $\mathscr{E}$ are essentially self-adjoint for all $W(x)$ being in L^2_{loc} with $W(x) \geqq 0$ a.e. Then the proof of Theorem 2.3 can be completed by Theorem 2.1. We will prove essential self-adjointness of $H_0 + \Phi(x) + b + W(x)$ by analogy with arguments in the proof of Theorem X.29 in [13] where Schrödinger operators are studied. There we will use the Kato-type inequality obtained in [8].

Let $W(x) \geqq 0$ a.e. be in L^2_{loc}. Noting (4.1), it follows from Theorem X.26 in [13] that iff $H_0 + \Phi(x) + b + W(x)$ with domain $\mathscr{E}$ is essentially self-adjoint, the range of $\lambda + H_0 + \Phi(x) + b + W(x)$ is dense in L^2 for a constant $\lambda > 0$.

We may assume $b = 0$ without the loss of generality. Let $\lambda > 0$ be a constant and $u(x)$ be in L^2 such that

$$(u(x), \{\lambda + H_0 + \Phi(x) + W(x)\} f(x)) = 0 \tag{4.2}$$

hold for all $f(x) \in \mathscr{E}$. (4.2) indicates that

$$(\lambda + H_0 + \Phi + W) u(x) = 0 \tag{4.2$'$}$$

holds in a distribution sense. Since $u(x)$ is in L^2 and $\Phi(x) + W(x)$ is in L^2_{loc}, $H_0 u(x)$ is in L^1_{loc}. Hence we get from Theorem 4.1 in [8] the distribution inequality

$$\operatorname{Re}[(\operatorname{sgn} u(x)) H_0 u(x)] \geqq H_0 |u(x)| \quad \text{in } \mathscr{D}', \tag{4.3}$$

where sgn $u(x)$ is a bounded measurable function defined by $\overline{u(x)}/|u(x)|$ for a point x such that $u(x) \neq 0$ and zero for a point x such that $u(x) = 0$. $\overline{u(x)}$ is the complex conjugate of $u(x)$. (4.3) means that

$$(\operatorname{Re}[(\operatorname{sgn} u(x)) H_0 u(x)], f(x)) \geqq (H_0 |u(x)|, f(x))$$

hold for all $f(x) \in \mathscr{E}$ with $f(x) \geqq 0$. Inserting $H_0 u(x) = -(\lambda + \Phi + W) u(x)$ into (4.3),

$$\begin{aligned}(\lambda + H_0)|u(x)| &\leqq -(\Phi + W)|u(x)| \\ &\leqq -\Phi(x)|u(x)| \quad \text{in } \mathscr{D}'\end{aligned} \tag{4.4}$$

is obtained. Here we used $W(x) \in L^2_{loc}$ and $W(x) \geqq 0$ a.e. for the last inequality.

Now

$$
\begin{aligned}
|(\Phi(x)\,|u(x)|, f(x))| &\leqq \|u(x)\|\,\|\Phi(x) f(x)\| \\
&\leqq C_1 \|u(x)\|\,\|(H_0+1) f(x)\|
\end{aligned}
$$

follow from H_0-boundedness of $\Phi(x)$ for all $f(x) \in \mathscr{E}$, where C_1 is a constant. It is easy to see that the same inequalities remain valid for all $f(x) \in \mathscr{S}$. So $\Phi(x)\,|u(x)|$ belongs to $\mathscr{S}'$. $\mathscr{S}'$ is the dual space of $\mathscr{S}$. It is also easy to see $(\lambda + H_0)\,|u(x)| \in \mathscr{S}'$. Hence we obtain by (4.4)

$$
-(\Phi(x)\,|u(x)|, f(x)) \geqq ((\lambda + H_0)\,|u(x)|, f(x)) \tag{4.5}
$$

for all $f(x) \in \mathscr{S}$ with $f(x) \geqq 0$. Let $\psi(x) \geqq 0$ on R^n be an arbitrary function in $\mathscr{S}$ and set $\varphi(x) = (\lambda + H_0)^{-1} \psi(x)$. Then $\varphi(x)$ belongs to $\mathscr{S}$. $\varphi(x) \geqq 0$ on R^{2n} follows from (3.3) and (3.4) in [8] or Theorems XIII.52, 54 and the example on page 220 in [14]. So inserting this $\varphi(x)$ into (4.5) as $f(x)$, we get

$$
-(\Phi(x)\,|u(x)|, (\lambda + H_0)^{-1} \psi(x)) \geqq (|u(x)|, \psi(x)). \tag{4.6}
$$

Now $\Phi(x)$ is assumed to be H_0-bounded with relative bound less than one. So there exist constants $0 \leqq a' < 1$ and $0 \leqq b'$ such that

$$
\begin{aligned}
\|\Phi(x) f(x)\| &< a' \|H_0 f(x)\| + b' \|f(x)\| \\
&< a' \|(\lambda + H_0) f(x)\| + b' \|f(x)\|
\end{aligned}
$$

are valid for all $f(x) \in \mathscr{E}$. We can easily see that these inequalities remain valid for all $f(x) \in \mathscr{S}$. Consequently we get for all $g(x) \in \mathscr{S}$

$$
\begin{aligned}
\|\Phi(x)(\lambda + H_0)^{-1} g(x)\| &< a' \|g(x)\| + b' \|(\lambda + H_0)^{-1} g(x)\| \\
&< \left(a' + \frac{b'}{\lambda}\right) \|g(x)\|,
\end{aligned}
$$

which also remain valid for all $g(x) \in L^2$. Hence $\Phi(x)(\lambda + H_0)^{-1}$ is a bounded operator from L^2 to L^2 and its operator norm is bounded by a less constant than $\left(a' + \frac{b'}{\lambda}\right)$. Therefore we see that $\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)|$ belongs to L^2 and

$$
\|\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)|\| < \left(a' + \frac{b'}{\lambda}\right) \|u(x)\| \tag{4.7}
$$

is valid, because $u(x)$ belongs to L^2. Moreover (4.6) indicates

$$
-\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)| \geqq |u(x)| \quad \text{a.e.} \tag{4.8}
$$

as the inequality between functions, because $\psi(x) \geqq 0$ is arbitrary. Hence we get by (4.7) and (4.8)

$$\| u(x) \| < \left(a' + \frac{b'}{\lambda} \right) \| u(x) \|. \tag{4.9}$$

This shows $u(x) = 0$ a.e. when $\lambda > 0$ is large. Thus we see that if $\lambda > 0$ is large, the range of $\lambda + H_0 + \Phi(x) + W(x)$ is dense in L^2. This completes the proof of Theorem 2.3.

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